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(*f*, *p*)-Asymptotically Lacunary Equivalent Sequences with Respect to the Ideal *I*

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Abstract

In this study, we define (*f*, *p*)-Asymptotically Lacunary Equivalent Sequences with respect to the ideal *I* using a non-trivial ideal $I \subset P(N)$, a lacunary sequence $\theta = (k_r)$, a strictly positive sequence $p = (p_k)$, and a modulus function *f*, and obtain some revelent connections between these notions.

Keywords

Asymptotically Equivalence, Ideal Convergence, Lacunary Sequence, Modulus Function, Statistically Limit

1. Introduction

Let s, ℓ_∞, c denote the spaces of all real sequences, bounded, and convergent sequences,respectively. Any subspace of *s* is called a sequence space.

Following Freedman *et al*. [1], we call the sequence $\theta = (k_r)$ lacunary if it is an increasing sequence of integers such that $k_0 = 0, h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = k_r / k_{r-1}$. These notations will be used troughout the paper. The sequence space of lacunary strongly convergent sequences N_θ was defined by Freedman *et al*. [1], as follows:

$$N_\theta = \left\{ x = (x_i) \in s : \lim_r h_r^{-1} \sum_{i \in I_r} |x_i - s| = 0 \text{ for some } s \right\}.$$

The notion of modulus function was introduced by Nakano [2]. We recall that a modulus *f* is a function from $[0, \infty)$ to $[0, \infty)$ such that 1) $f(x) = 0$ if and only if $x = 0$, 2) $f(x+y) \le f(x) + f(y)$, for $x, y \ge 0$, 3) *f* is increasing and 4) *f* is continuous from the right at 0. Hence *f* must be continuous everywhere on $[0, \infty)$.

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Connor [3], Kolk [4], Maddox [5], Öztürk and Bilgin [6], Pehlivan and Fisher [7], Ruckle [8] and others used a modulus function to construct sequence spaces.

Marouf presented definitions for asymptotically equivalent sequences and asymptotic regular matrices in [9]. Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices in [10]. Subsequently, many authors have shown their interest to solve different problems arising in this area (see [11]-[13]).

The concept of I-convergence was introduced by Kostyrko *et al*. in a metric space [14]. Later it was further studied by Dass *et al*. [15], Dems [16], Savas and Gumus [17], Kumar and Sharma [18], Kumar and Mursaleen [19] and many others.

Recently, Bilgin [20] used modulus function to define some notions of asymptotically equivalent sequences and studied some of their connections. Kumar and Sharma extended these concepts by presenting a non-trivial ideal *I*

This paper presents introduce some new notions, (*f*, *p*)-asymptotically equivalent of multiple *L*, strong (*f*, *p*)-asymptotically equivalent of multiple *L*, and strong (*f*, *p*)-asymptotically lacunary equivalent of multiple *L* with respect to the ideal *I* which is a natural combination of the definition for asymptotically equivalent, a non-trivial ideal *I*, Lacunary sequence, a strictly positive sequence $p=(p_k)$, and Modulus function. In addition to these definitions, we obtain some revelent connections between these notions.

2. Definitions and Notations

Now we recall some definitions of sequence spaces (see [2] [4]-[6] [15], and [20]-[25]).

Definition 2.1. A sequence $[x]$ is statistically convergent to L if

$\lim_n \frac{1}{n}\left|\left\{k \le n : |x_k - L| \ge \varepsilon\right\}\right| = 0$ for every $\varepsilon > 0$, (denoted by $st-\lim x = L$), where the vertical bars denote the cardinality of the

Enclosed set.

Definition 2.2. A sequence $[x]$ is strongly(Cesaro) summable to L if $\lim_n \frac{1}{n}\sum_{k=1}^{n}|x_k - L| = 0$, (denoted by $w-\lim x = L$).

Definition 2.3. Let *f* be any modulus; the sequence $[x]$ is strongly (Cesaro) summable to L with respect to a modulus if $\lim_n \frac{1}{n}\sum_{k=1}^{n} f\left(|x_k - L|\right) = 0$, (denoted by $w_f - \lim x = L$).

Definition 2.4. Two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically equivalent if $\lim_k \frac{x_k}{y_k} = 1$, (denoted by $x \sim y$).

Definition 2.5. Two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$, $\lim_n \frac{1}{n}\left|\left\{k \le n : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| = 0$, (denoted by $x \overset{S}{\sim} y$) and simply asymptotically statistical equivalent, if $L = 1$.

Definition 2.6. Two nonnegative sequences $[x]$ and $[y]$ are said to be strong asymptotically equivalent of multiple L provided that

$\lim_n \frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k} - L\right| = 0$ (denoted by $x \overset{w}{\sim} y$) and simply strong asymptotically equivalent, if $L = 1$.

Definition 2.7. Let θ be a lacunary sequence; the two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$,

$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| = 0$, (denoted by $x \overset{S_\theta}{\sim} y$) and simply asymptotically lacunary statistical equivalent, if $L = 1$.

Definition 2.8. Let θ be a lacunary sequence; the two nonnegative sequences $[x]$ and $[y]$ are said to be strong asymptotically lacunary equivalent of multiple L provided that $\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_k}{y_k}-L\right|=0$ (denoted by $x \overset{N_\theta}{\sim} y$) and simply strong asymptotically lacunary equivalent, if $L=1$.

Definition 2.9. Let f be any modulus; the two nonnegative sequences $[x]$ and $[y]$ are said to be f-asymptotically equivalent of multiple L provided that,

$\lim_k f\left(\left|\frac{x_k}{y_k}-L\right|\right)=0$ (denoted by $x \overset{f}{\sim} y$) and simply strong f-asymptotically equivalent, if $L=1$.

Definition 2.10. Let f be any modulus; the two nonnegative sequences $[x]$ and $[y]$ are said to be strong f-asymptotically equivalent of multiple L provided that, $\lim_n \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_k}{y_k}-L\right|\right)=0$ (denoted by $x \overset{w_f}{\sim} y$) and simply strong f-asymptotically equivalent, if $L=1$.

Definition 2.11. Let f be any modulus and θ be a lacunary sequence; the two nonnegative sequences $[x]$ and $[y]$ are said to be strong f-asymptotically lacunary equivalent of multiple L provided that $\lim_r \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_k}{y_k}-L\right|\right)=0$ (denoted by $\overset{N_{\theta,f}}{\sim} y$) and simply strong f-asymptotically lacunary equivalent, if $L=1$.

For any non-empty set X, let $P(X)$ denote the power set of X.

Definition 2.12. A family $I \subseteq P(X)$ is said to be an ideal in X if

1) $\varnothing \in I$;
2) $A, B \in I$ imply $A \bigcup B \in I$ and
3) $A \in I, B \subset A$ imply $B \in I$.

Definition 2.13. A non-empty family $F \subseteq P(X)$ is said to be a filter in X if

1) $\varnothing \notin F$;
2) $A, B \in F$ imply $A \bigcap B \in F$ and
3) $A \in F, B \supset A$ imply $B \in F$.

An ideal I is said to be non-trivial if $I \neq \{\varnothing\}$ and $X \notin I$. A non-trivial ideal I is called admissible if it contains all the singleton sets. Moreover, if I is a non-trivial ideal on X, then $F = F(I) = \{X - A : A \in I\}$ is a filter on X and conversely. The filter $F(I)$ is called the filter associated with the ideal I.

Definition 2.14. Let $I \subset P(N)$ be a non-trivial ideal in N and (X, ρ) be a metric space. A sequence $[x]$ in X is said to be I-convergent to ξ if for each $\varepsilon > 0$, the set $\{k \in N : \rho(x_k, \xi) \geq \varepsilon\} \in I$.

In this case, we write $I-\lim_{k\to\infty} x_k = \xi$.

Definition 2.15. A sequence $[x]$ of numbers is said to be I-statistical convergent or $S(I)$-convergent to L, if for every $\varepsilon > 0$ and $\delta > 0$, we have

$$\left\{n \in N; \frac{1}{n}\left|\left\{k \leq n : |x_k - L| \geq \varepsilon\right\}\right| \geq \delta\right\} \in I.$$

In this case, we write $x_k \to L(S(I))$ or $S(I)-\lim_{k\to\infty} x_k = L$.

Definition 2.16 Let $I \subset P(N)$ be a non-trivial ideal in N. The two non-negative sequences $[x]$ and $[y]$ are said to be strongly asymptotically equivalent of multiple L with respect to the ideal I provided that for each $\varepsilon > 0$

$$\left\{n \in N; \frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k}-L\right| \geq \varepsilon\right\} \in I.$$

denoted by $x \overset{I(w)}{\sim} y$ and simply strongly asymptotically equivalent with respect to the ideal I, if $L=1$.

Definition 2.17. Let $I \subset P(N)$ be a non-trivial ideal in N and $\theta = (k_r)$ be a lacunary sequence. The two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically lacunary statistical equivalent of multiple L with respect to the ideal I provided that for each $\varepsilon > 0$

and $\gamma > 0$, $\left\{ r \in N; \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \in I$

denoted by $x \overset{I(S_\theta)}{\sim} y$ and simply asymptotically lacunary statistical equivalent with respect to the ideal *I*, if $L = 1$.

Definition 2.18. Let $I \subset P(N)$ be a non-trivial ideal in *N* and $\theta = (k_r)$ be a lacunary sequence. The two non-negative sequences $[x]$ and $[y]$ are said to be strongly asymptotically lacunary equivalent of multiple *L* with respect to the ideal *I* provided that for $\varepsilon > 0$

$$\left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in I$$

denoted by $x \overset{I(N_\theta)}{\sim} y$ and simply strongly asymptotically lacunary equivalent with respect to the ideal *I*, if $L = 1$.

Definition 2.19. Let $I \subset P(N)$ be a non-trivial ideal in *N* and *f* be a modulus function. The two non-negative sequences $[x]$ and $[y]$ are said to be f-asymptotically equivalent ofmultiple *L* with respect to the ideal *I* provided that for each $\varepsilon > 0$

$$\left\{ k \in N; f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in I$$

denoted by $x \overset{I(f)}{\sim} y$ and simply *f*-asymptotically equivalent with respect to the ideal *I*, if $L = 1$.

Definition 2.20. Let $I \subset P(N)$ be a non-trivial ideal in *N* and *f* be a modulus function. The two non-negative sequences $[x]$ and $[y]$ are said to be strongly
f-asymptotically equivalent of multiple *L* with respect to the ideal *I* provided that for each $\varepsilon > 0$,

$$\left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in I$$

denoted by $x \overset{I(w_f)}{\sim} y$ and simply strongly *f*-asymptotically equivalent with respect to the ideal *I*, if $L = 1$.

Definition 2.21. Let $I \subset P(N)$ be a non-trivial ideal in *N*, *f* be a modulus function and $\theta = (k_r)$ be a lacunary sequence. The two non-negative sequences $[x]$ and $[y]$ are said to be strongly *f*-asymptotically lacunary equivalent of multiple *L* with respect to the ideal *I* provided that for each $\varepsilon > 0$,

$$\left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in I$$

denoted by $x \overset{I(N_\theta^f)}{\sim} y$ and simply strongly *f*-asymptotically lacunary equivalent with respect to the ideal *I*, if $L = 1$.

3. Main Results

We now consider our main results. We begin with the following definitions.

Definition 3.1. Let $I \subset P(N)$ be a non-trivial ideal in *N*, *f* be a modulus function, and $p = (p_k)$ be a sequence of positive real numbers. Two number sequences $[x]$ and $[y]$ are said to be strongly (*f*, *p*)-asymptotically equivalent of multiple *L* with respect to the ideal *I* provided that for each $\varepsilon > 0$,

$$\left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{p_k} \geq \varepsilon \right\} \in I$$

denoted by $x \overset{I(w^{(f,p)})}{\sim} y$ and simply strongly (*f*, *p*)-asymptotically equivalent with respect to the ideal *I*, if $L=1$.

If we take $f(x)=x$ for $x \geq 0$, we write $x \overset{I(w^{p})}{\sim} y$ instead of $x \overset{I(w^{(f,p)})}{\sim} y$ and simply strongly *p*-asymptotically equivalent with respect to the ideal *I*, if $L=1$.

If we take $p_k = p$ for all $k \in N$, we write $x \overset{I(w^{fp})}{\sim} y$ instead of $x \overset{I(w^{(f,p)})}{\sim} y$

Definition 3.2. Let $I \subset P(N)$ be a non-trivial ideal in *N*, *f* be a modulus function, $\theta=(k_r)$ be a lacunary sequence, and $p=(p_k)$ be a sequence of positive real numbers. Two number sequences $[x]$ and $[y]$ are said to be strongly (*f*, *p*)-asymptotically lacunary equivalent of multiple *L* with respect to the ideal *I* provided that for each $\varepsilon > 0$,

$\left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{p_k} \geq \varepsilon \right\} \in I$ denoted by $x \overset{I(N_\theta^{(f,p)})}{\sim} y$ and simply strongly (*f*, *p*)-asymptotically lacunary equivalent with respect to the ideal *I*, if $L=1$.

If we take $p_k = p$ for all $k \in N$, we write $x \overset{I(N_\theta^{fp})}{\sim} y$ instead of $x \overset{I(N_\theta^{(f,p)})}{\sim} y$

Note that,we put $p=1$, we write $x \overset{I(N_\theta^{f})}{\sim} y$ instead of $x \overset{I(N_\theta^{fp})}{\sim} y$. Hence $x \overset{I(N_\theta^{f})}{\sim} y$ is the same as the $x \overset{I(N_\theta^{f})}{\sim} y$ of Kumar and Sharma [15]. Also if we put $f(x)=x$ for $x \geq 0$, we write $x \overset{I(N_\theta^{p})}{\sim} y$ instead of $x \overset{I(N_\theta^{(f,p)})}{\sim} y$. Hence $x \overset{I(N_\theta^{p})}{\sim} y$ is the same as the $x \overset{N_\theta^{L(p)}(I)}{\sim} y$ of Savas and Gumus [25]

We start this section with the following theorem to show that the relation between (*f*, *p*)-asymptotically equivalence and strong *p*-asymptotically equivalence with respect to the ideal *I*

Theorem 3.1. Let $I \subset P(N)$ be a non-trivial ideal in *N*, *f* be a modulus function, $\theta=(k_r)$ be a lacunary sequence and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$, then

1) if $x \overset{I(w^{p})}{\sim} y$ then $x \overset{I(w^{(f,p)})}{\sim} y$, and

2) if $\lim_{t\to\infty} \frac{f(t)}{t} = \beta > 0$, then $x \overset{I(w^{p})}{\sim} y \Leftrightarrow x \overset{I(w^{(f,p)})}{\sim} y$.

Proof. Part 1): Let $x \overset{I(w^{p})}{\sim} y$ and ε > 0. We choose $0 < \delta < 1$ such that $f(u) < \varepsilon$ for every u with $0 \leq u \leq \delta$. We can write

$$\begin{aligned} &\frac{1}{n} \sum_{k=1}^{n} \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{p_k} \\ &= \frac{1}{n} \sum_{1} \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{p_k} + \frac{1}{n} \sum_{2} \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{p_k} \end{aligned}$$

where the first summation is over $\left| \frac{x_k}{y_k} - L \right| \leq \delta$ and the second summation over $\left| \frac{x_k}{y_k} - L \right| > \delta$. By definition of *f*, we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{p_k} \leq \max\left\{ \varepsilon^h, \varepsilon^H \right\} + \max\left\{ 1, \left(2f(1)\delta^{-1} \right)^H \right\} \frac{1}{n} \sum_{k=1}^{n} \left[\left| \frac{x_k}{y_k} - L \right| \right]^{p_k}$$

Thus,

$$\left\{n\in N;\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k}\geq\varepsilon\right\}\subseteq\left\{n\in N;\frac{1}{n}\sum_{k=1}^{n}\left[\left|\frac{x_k}{y_k}-L\right|\right]^{p_k}\geq\frac{\varepsilon-\max\left\{\varepsilon^h,\varepsilon^H\right\}}{\max\left\{1,\left(2f(1)\delta^{-1}\right)^H\right\}}\right\}.$$

Since $x\overset{I(w^p)}{\sim}y$, it follows the later set, and hence, the first set in above expression belongs to *I*. This proves that $x\overset{I(w^{(f,p)})}{\sim}y$

Part 2): If $\lim_{t\to\infty}\frac{f(t)}{t}=\beta>0$, then $f(t)\geq\beta t$ for all $t>0$. Let $x\overset{I(w^{(f,p)})}{\sim}y$, clearly

$$\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k}$$
$$\geq\frac{1}{n}\sum_{k=1}^{n}\left[\beta\left|\frac{x_k}{y_k}-L\right|\right]^{p_k}\geq\min\left\{\beta^h,\beta^H\right\}\frac{1}{n}\sum_{k=1}^{n}\left[\left|\frac{x_k}{y_k}-L\right|\right]^{p_k}$$

it follows that for each $\varepsilon>0$, we have

$$\left\{n\in N;\frac{1}{n}\sum_{k=1}^{n}\left[\left|\frac{x_k}{y_k}-L\right|\right]^{p_k}\geq\varepsilon\right\}\subseteq\left\{n\in N;\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k}\geq\varepsilon\min\left\{\beta^h,\beta^H\right\}\right\}$$

Since $x\overset{I(w^{(f,p)})}{\sim}y$, it follows that the later set belongs to *I*, and therefore, the theorem is proved.

Theorem 3.2. Let $I\subset P(N)$ be a non-trivial ideal in *N*, *f* be a modulus function, $\theta=(k_r)$ be a lacunary sequence and $0<h=\inf_k p_k\leq p_k\leq\sup_k p_k=H<\infty$, then

1) if $x\overset{I(N_\theta^p)}{\sim}y$ then $x\overset{I(N_\theta^{(f,p)})}{\sim}y$, and

2) if $\lim_{t\to\infty}\frac{f(t)}{t}=\beta>0$, then $x\overset{I(N_\theta^p)}{\sim}y\Leftrightarrow x\overset{I(N_\theta^{(f,p)})}{\sim}y$.

Proof. The proof of Theorem 3.2 is very similar to the Theorem 3.1. Then, we omit it.

The next theorem shows the relationship between the strong (*f*, *p*)-asymptotically equivalence and the strong (*f*, *p*)-asymptotically lacunary equivalence with respect to the ideal *I*.

Theorem 3.3. Let $I\subset P(N)$ be a non-trivial ideal in *N*, *f* be a modulus function, $\theta=(k_r)$ be a lacunary sequence and $p=(p_k)$ be a sequence of positive real numbers. Then

1) if $\limsup_r q_r<\infty$ then $x\overset{I(N_\theta^{(f,p)})}{\sim}y$ implies $x\overset{I(w^{(f,p)})}{\sim}y$;

2) if $\liminf_r q_r>1$ then $x\overset{I(w^{(f,p)})}{\sim}y$ implies $x\overset{I(N_\theta^{(f,p)})}{\sim}y$;

3) if $1<\liminf_r q_r\leq\limsup_r q_r<\infty$, then $x\overset{I(w^{(f,p)})}{\sim}y\Leftrightarrow x\overset{I(N_\theta^{(f,p)})}{\sim}y$.

Proof. Part (i): If $\limsup_r q_r<\infty$ then there exists $K>0$ such that $q_r<K$ for every *r*. Now suppose that $x\overset{I(N_\theta^{(f,p)})}{\sim}y$ and $\varepsilon>0$. Let

$$A=\left\{r\in N;\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k}<\varepsilon\right\}$$

Hence, for all $j \in A$ we have $H_j = \frac{1}{h_j}\sum_{k\in I_j}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} < \varepsilon$. Let n be any integer with $k_r \ge n > k_{r-1}$. Now write

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k}\\
&\le \frac{1}{k_{r-1}}\sum_{k=1}^{k_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} = \frac{1}{k_{r-1}}\sum_{m=1}^{r}\sum_{k\in I_m}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k}\\
&= \frac{1}{k_{r-1}}\sum_{m=1}^{r}\frac{k_m-k_{m-1}}{h_m}\sum_{k\in I_m}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} = \frac{1}{k_{r-1}}\sum_{m=1}^{r}\left(k_m-k_{m-1}\right)\sup_{j\in A}H_j\\
&= \frac{k_r}{k_{r-1}}\sup_{j\in A}H_j = q_r\sup_{j\in A}H_j < K\varepsilon = \varepsilon^{|}
\end{aligned}$$

it follows that for any $\varepsilon^{|} > 0$, $\left\{n\in N; \frac{1}{n}\sum_{k=1}^{n}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} < \varepsilon^{|}\right\} \in F(I)$

which yields that $x \overset{I\left(w^{(f,p)}\right)}{\sim} y$. Because for any set $A \in F(I)$, $\bigcup\left\{n : k_{r-1} < n < k_r, r \in A\right\} \in F(I)$.

Part (ii): Let $x \overset{I\left(w^{(f,p)}\right)}{\sim} y$ and $\liminf_r q_r > 1$. There exists $\delta > 0$ such that $q_r = \left(k_r/k_{r-1}\right) \ge 1+\delta$ for all $r \ge 1$. We have, for sufficiently large r, that $\left(k_r/h_r\right) \le \frac{1+\delta}{\delta}$ and $\left(k_{r-1}/h_r\right) \le \frac{1}{\delta}$. Let $\varepsilon > 0$ and define the set

$$A = \left\{k_r \in N; \frac{1}{k_r}\sum_{k=1}^{k_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} < \varepsilon\right\}.$$

We have $A \in F(I)$, which is the filter of the ideal I, For each $k_r \in A$, we have

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} &= \frac{1}{h_r}\sum_{k=1}^{k_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} - \frac{1}{h_r}\sum_{k=1}^{k_{r-1}}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k}\\
&= \frac{k_r}{k_r h_r}\sum_{k=1}^{k_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} - \frac{k_{r-1}}{h_r k_{r-1}}\sum_{k=1}^{k_{r-1}}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k}\\
&\le \frac{k_r}{k_r h_r}\sum_{k=1}^{k_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} < \left(\frac{1+\delta}{\delta}\right)\varepsilon = \varepsilon^{|}
\end{aligned}$$

it follows that for any $\varepsilon^{|} > 0$,

$\left\{r\in N; \frac{1}{h_r}\sum_{k\in I_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} < \varepsilon^{|}\right\} \in F(I)$ which yields that $x \overset{I\left(N_\theta^{(f,p)}\right)}{\sim} y$

Part (iii): This immediately follows from (i) and (ii).

Now we give relation between asymptotically statistical equivalence and strong (*f*, *p*)-asymptotically equivalence with respect to the ideal *I*. Also we give relation between asymptotically lacunary statistical equivalence and strong (*f*, *p*)-asymptotically lacunary equivalence with respect to the ideal *I*.

Theorem 3.4. Let $I \subset P(N)$ be a non-trivial ideal in N, f be a modulus function, $\theta = (k_r)$ be a lacunary sequence and $0 < h = \inf_k p_k \le p_k \le \sup_k p_k = H < \infty$, then

1) if $x \overset{I\left(w^{(f,p)}\right)}{\sim} y$ then $x \overset{I(S)}{\sim} y$,

2) if f is bounded then $x \overset{I(S)}{\sim} y \Leftrightarrow x \overset{I\left(w^{(f,p)}\right)}{\sim} y$.

Proof. Part 1): Suppose $x \overset{I\left(w^{(f,p)}\right)}{\sim} y$, and let $\varepsilon > 0$ and $\sum_1$ denote the sum over $k \le n$ with $\left|\frac{x_k}{y_k} - L\right| \ge \varepsilon$ then we can write

$$\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} \ge \frac{1}{n}\sum_{1}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} \ge \min\left\{f(\varepsilon)^h, f(\varepsilon)^H\right\}\frac{1}{n}\left|\left\{k \le n : \left|\frac{x_k}{y_k}-L\right| \ge \varepsilon\right\}\right|$$

Consequently, for any $\gamma > 0$, we have

$$\left\{n \in N; \frac{1}{n}\left|\left\{k \le n : \left|\frac{x_k}{y_k}-L\right| \ge \varepsilon\right\}\right| \ge \gamma\right\}$$

$$\subseteq \left\{n \in N; \frac{1}{n}\sum_{k=1}^{n}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} \ge \gamma \min\left\{f(\varepsilon)^h, f(\varepsilon)^H\right\}\right\} \in I.$$

Therefore we have $x \overset{I(S)}{\sim} y$

2) Suppose f is bounded and $x \overset{I(S)}{\sim} y$. Since f is bounded, there exists an integer T such that $|f(x)| \le T$ for all $x \ge 0$.

Moreover, for $\varepsilon > 0$, We split the sum for $k \le n$ into sums over $\left|\frac{x_k}{y_k} - L\right| \ge \varepsilon$ and $\left|\frac{x_k}{y_k} - L\right| < \varepsilon$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} \le \max\left\{T^{h,}, T^H\right\}\frac{1}{n}\left|\left\{k \le n : \left|\frac{x_k}{y_k}-L\right| \ge \varepsilon\right\}\right| + \max\left\{f(\varepsilon)^h, f(\varepsilon)^H\right\}$$

Consequently, we have

$$\left\{n \in N; \frac{1}{n}\sum_{k=1}^{n}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} \ge \varepsilon\right\}$$

$$\subseteq \left\{n \in N; \frac{1}{n}\left|\left\{k \le n : \left|\frac{x_k}{y_k}-L\right| \ge \varepsilon\right\}\right| \ge \frac{\varepsilon - \max\left\{f(\varepsilon)^h, f(\varepsilon)^H\right\}}{\max\left\{T^{h,}, T^H\right\}}\right\} \in I.$$

Therefore we have $x \overset{I\left(w^{(f,p)}\right)}{\sim} y$

Theorem 3.5. Let $I \subset P(N)$ be a non-trivial ideal in N, f be a modulus function, $\theta = (k_r)$ be a lacunary sequence and $0 < h = \inf_k p_k \le p_k \le \sup_k p_k = H < \infty$, then

1) if $x \overset{I\left(N_\theta^{(f,p)}\right)}{\sim} y$ then $x \overset{I(S_\theta)}{\sim} y$,

2) if f is bounded then $x \overset{I\left(N_\theta^{(f,p)}\right)}{\sim} y \Leftrightarrow x \overset{I(S_\theta)}{\sim} y$.

Proof. Part 1): Take $\varepsilon > 0$ and let $\sum_1$ denote the sum over $k \in I_r$ with $\left|\frac{x_k}{y_k} - L\right| \ge \varepsilon$. Then

$$\frac{1}{h_r}\sum_{k \in I_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} \ge \frac{1}{h_r}\sum_{1}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} \ge \min\left\{f(\varepsilon)^h, f(\varepsilon)^H\right\}\frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k}-L\right| \ge \varepsilon\right\}\right|,$$

and $\left\{ r \in N; \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\}$ $\subseteq \left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{p_k} \geq \gamma \min\left\{ f(\varepsilon)^h, f(\varepsilon)^H \right\} \right\} \in I.$ But then, by definition of an ideal, later set belongs to I, and therefore $x \overset{I(S_\theta)}{\sim} y$

Part 2): Suppose that f is bounded and $x \overset{I(S_\theta)}{\sim} y$. Since f is bounded, there exists an integer T such that $|f(x)| \leq T$ for all $x \geq 0$. We see that

$\frac{1}{h_r} \sum_{k \in I_r} \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{p_k} \leq \max\left\{ T^{h}, T^H \right\} \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| + \max\left\{ f(\varepsilon)^h, f(\varepsilon)^H \right\}$, so we have

$$\left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{p_k} \geq \varepsilon \right\}$$
$$\subseteq \left\{ r \in N; \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \frac{\varepsilon - \max\left\{ f(\varepsilon)^h, f(\varepsilon)^H \right\}}{\max\left\{ T^{h}, T^H \right\}} \right\} \in I$$

Therefore we have $x \overset{I\left(N_\theta^{(f,p)}\right)}{\sim} y$

Let $p_k = p$ for all k, $t_k = t$ for all k and $0 < p \leq t$. Then it follows following Theorem.

Theorem 3.6. Let $I \subset P(N)$ be a non-trivial ideal in N, f be a modulus function, and $\theta = (k_r)$ be a lacunary sequence, then

$x \overset{I\left(N_\theta^{ft}\right)}{\sim} y$ implies $x \overset{I\left(N_\theta^{fp}\right)}{\sim} y$,

Proof. Let $x \overset{I\left(N_\theta^{ft}\right)}{\sim} y$ It follows from Holder's inequality

$$\frac{1}{h_r} \sum_{k \in I_r} \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{p} \leq \left(\frac{1}{h_r} \sum_{k \in I_r} \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{t} \right)^{p/t}$$

and $\left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{p} \geq \varepsilon \right\} \subseteq \left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{t} \geq \varepsilon^{t/p} \right\} \in I$. Thus we have $x \overset{I\left(N_\theta^{fp}\right)}{\sim} y$

We now consider that (p_k) and (t_k) are not constant sequences.

Theorem 3.7. Let $I \subset P(N)$ be a non-trivial ideal in N, f be a modulus function, $\theta = (k_r)$ be a lacunary sequence, $0 < p_k \leq t_k$ for all k and (t_k / p_k) be bounded, then $x \overset{I\left(N_\theta^{(f,t)}\right)}{\sim} y$ implies $x \overset{I\left(N_\theta^{(f,p)}\right)}{\sim} y$

Proof. Let $x \overset{I\left(N_\theta^{(f,t)}\right)}{\sim} y$. $z_k = \left[f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]^{t_k}$ and $\lambda_k = (p_k / t_k)$, so that $0 < \lambda \leq \lambda_k \leq 1$: We define the sequences (u_k) and (v_k) as follows: For $z_k \geq 1$; let $u_k = z_k$ and $v_k = 0$ and for $z_k < 1$; let $v_k = z_k$ and $u_k = 0$. Then we have $z_k = u_k + v_k$; $z_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that $u_k^{\lambda_k} \leq u_k \leq z_k$ and $v_k^{\lambda_k} \leq v_k^{\lambda}$. Therefore

$$\frac{1}{h_r}\sum_{k\in I_r} z_k^{\lambda_k} = \frac{1}{h_r}\sum_{k\in I_r}\left(u_k^{\lambda_k}+v_k^{\lambda_k}\right) \le \frac{1}{h_r}\sum_{k\in I_r} z_k + \frac{1}{h_r}\sum_{k\in I_r} v_k^{\lambda}$$

Now for each r;

$$\frac{1}{h_r}\sum_{k\in I_r} v_k^{\lambda} = \sum_{k\in I_r}\left(\frac{1}{h_r}v_k\right)^{\lambda}\left(\frac{1}{h_r}\right)^{1-\lambda} \le \left(\sum_{k\in I_r}\left[\left(\frac{1}{h_r}v_k\right)^{\lambda}\right]^{1/\lambda}\right)^{\lambda}\left(\sum_{k\in I_r}\left[\left(\frac{1}{h_r}\right)^{1-\lambda}\right]^{1/1-\lambda}\right)^{1-\lambda} < \left(\frac{1}{h_r}\sum_{k\in I_r} v_k\right)^{\lambda} \quad \text{and so}$$

$$\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} = \frac{1}{h_r}\sum_{k\in I_r} z_k^{\lambda_k} \le \frac{1}{h_r}\sum_{k\in I_r} z_k + \left(\frac{1}{h_r}\sum_{k\in I_r} v_k\right)^{\lambda}$$

$$= \begin{cases} \frac{1}{h_r}\sum_{k\in I_r} z_k, & z_k \ge 1 \\ \frac{1}{h_r}\sum_{k\in I_r} z_k + \left(\frac{1}{h_r}\sum_{k\in I_r} z_k\right)^{\lambda}, & z_k < 1 \end{cases} \le \begin{cases} \frac{1}{h_r}\sum_{k\in I_r} z_k, & z_k \ge 1 \\ 2\left(\frac{1}{h_r}\sum_{k\in I_r} z_k\right)^{\lambda}, & z_k < 1 \end{cases}$$

If $\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} \ge \varepsilon$ then

$$\begin{cases} \frac{1}{h_r}\sum_{k\in I_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{t_k} \ge \varepsilon, & z_k \ge 1 \\ \frac{1}{h_r}\sum_{k\in I_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{t_k} \ge \left(\frac{\varepsilon}{2}\right)^{1/\lambda}, & z_k < 1 \end{cases}$$

Hence $\left\{r \in N; \frac{1}{h_r}\sum_{k\in I_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{p_k} \ge \varepsilon\right\} \subseteq \left\{r \in N; \frac{1}{h_r}\sum_{k\in I_r}\left[f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]^{t_k} \ge \min\left\{\varepsilon, \left(\frac{\varepsilon}{2}\right)^{1/\lambda}\right\}\right\} \in I.$

Thus we have $x \overset{I\left(N_\theta^{(f,p)}\right)}{\sim} y$

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