

A New Iterative Method for Multi-Moving Boundary Problems Based Boundary Integral Method

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Abstract

The present paper deals with very important practical problems of wide range of applications. The main target of the present paper is to track all moving boundaries that appear throughout the whole process when dealing with multi-moving boundary problems continuously with time up to the end of the process with high accuracy and minimum number of iterations. A new numerical iterative scheme based the boundary integral equation method is developed to track the moving boundaries as well as compute all unknowns in the problem. Three practical applications, one for vaporization and two for ablation were solved and their results were compared with finite element, heat balance integral and the source and sink results and a good agreement were obtained.

Keywords

Multi-Moving Boundary Problems, Vaporization Problem, Ablation Problem, Source and Sink Method, Finite Element Method, Heat Balance Integral Method, Boundary Integral Method

1. Introduction

The differential equation in a certain domain satisfying some given conditions is referred to as boundary-value problem. If one or more of the boundaries are not known and moving with time, the problem then is referred to as moving boundary problem [1] [2]. Furthermore, if the governing equation is time independent as well as the boundary condition, then the problem is referred to as free boundary problem [3]. Phase change problems are practical examples of free and moving boundary problems, frequently appear in industrial process and other problems of technological interests such as the determination of the depth of the frost or thaw penetration, de-

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signing of the roadway or other engineering works in cold regions and the so-called re-entry problem of hypersonic missiles in aeronautical science [4] [5]. When a body is exposed to heat flux the surface of the body changes phase, the changed phase is immediately removed upon formation. This is referred to ablation problem [6]. A major difference between Stefan and ablation problems is that the overall domain in Stefan problem remains fixed in space while the domain in the ablation problem is variable and diminishes in size with time. Numerical methods become more popular rather than analytical methods. Some of famous numerical methods are finite differences [7], finite elements [8], boundary elements [9] and more recent mesh-less (mesh-free) numerical methods [10]. In many aspects, the boundary integral equation method for solving boundary value problems proves to be advantageous over the conventional numerical methods. The present paper deals with very important practical problems of wide range of applications. The main target of the present paper is to track all moving boundaries that appear throughout the whole process when dealing with multi-moving boundary problems continuously with time up to the end of the process with high accuracy and minimum number of iterations. A new numerical iterative scheme based the boundary integral equation method is developed to track the moving boundaries as well as compute all unknowns in the problem.

2. Mathematical Model

A semi-infinite solid initially at uniform temperature with the following constraints, there is no sub cooling, no mushy zone, solid and liquid phases have equal and constant properties and finally no convection. The mathematical formulation consists of three different stages as follow:

Heating stage

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial u(x,t)}{\partial t}, \quad 0 < x < \infty \quad (1)$$

$$u(x,0) = u_i \quad (2)$$

$$\frac{\partial u(0,t)}{\partial x} = -\frac{q(t)}{K} \quad (3)$$

$$u(x,t)_{x \rightarrow \infty} = u(\ell, t) = u_i \quad (4)$$

As $x = 0$ reaches the melting temperature, the second stage starts appearing.

Liquid-solid stage

$$\frac{\partial^2 u_\ell(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial u_\ell(x,t)}{\partial t}, \quad 0 < x < s_1(t) \quad (5)$$

$$\frac{\partial u_\ell(0,t)}{\partial x} = -\frac{q(t)}{K} \quad (6)$$

$$\frac{\partial^2 u_s(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial u_s(x,t)}{\partial t}, \quad s_1(t) < x < \infty \quad (7)$$

$$u_s(x,t)_{x \rightarrow \infty} = u_s(\ell, t) = u_i \quad (8)$$

$$u_s(x,t)_{x=s_1(t)} = u_\ell(x,t)_{x=s_1(t)} = u_m \quad (9)$$

$$K_s \frac{\partial u_s(x,t)}{\partial x} - K_\ell \frac{\partial u_\ell(x,t)}{\partial x} = \rho L_m \frac{ds_1(t)}{dt}, \quad x = s_1(t) \quad (10)$$

Gas-liquid-solid stage

$$\frac{\partial^2 u_v(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial u_v(x,t)}{\partial t}, \quad 0 < x < s_2(t) \quad (11)$$

$$\frac{\partial^2 u_\ell(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial u_\ell(x,t)}{\partial t}, \quad s_2(t) < x < s_1(t) \quad (12)$$

$$\frac{\partial^2 u_s(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial u_s(x,t)}{\partial t}, \quad s_1(t) < x < \infty \quad (13)$$

$$u_s(x,t)_{x=s_1(t)} = u_\ell(x,t)_{x=s_1(t)} = u_m \quad (14)$$

$$u_\ell(x,t) = u_v(x,t) = u_v \quad (15)$$

$$K_v \frac{\partial u_v}{\partial x} - K_\ell \frac{\partial u_\ell}{\partial x} = \rho L_v \frac{ds_2(t)}{dt} \quad (16)$$

$$u_\ell(x,t) = u_s(x,t) = u_m \quad (17)$$

3. Boundary Integral Formulation

Starting by the weighted residual statement for diffusion equation as follows:

$$\int_0^\tau \int_\Omega \left(k \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} \right) u^*(\xi, x; \tau, t) d\Omega dt = 0 \quad (18)$$

In Equation (18) $u^*(\xi, x; \tau, t)$ is the fundamental solution for diffusion equation and it is defined as:

$$u^*(\xi, x; \tau, t) = \frac{1}{\sqrt{4\pi k(\tau-t)}} \exp \left[-\frac{r^2(\xi, x)}{4k(\tau-t)} \right] \quad (19)$$

In which, $r^2(\xi, x) = X(\xi) - X(x)$ and $X(\xi)$ is the source point while $X(x)$ is the field point. Integrating equation (18) twice by parts leads to:

$$\begin{aligned} & \int_0^\tau \int_\Omega \left(k \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} \right) u^*(\xi, x; \tau, t) d\Omega dt \\ &= \int_0^\tau \int_\Omega \left(k \frac{\partial^2 u^*(\xi, x; \tau, t)}{\partial x^2} + \frac{\partial u^*(\xi, x; \tau, t)}{\partial t} \right) u d\Omega dt - k \int_0^\tau \int_\Gamma q^*(\xi, x; \tau, t) u d\Gamma dt = 0 \end{aligned} \quad (20)$$

For one-dimensional problems, Equation (20) takes the following form:

$$\int_0^\tau \int_0^\ell \left(k \frac{\partial^2 u^*(\xi, x; \tau, t)}{\partial x^2} + \frac{\partial u^*(\xi, x; \tau, t)}{\partial t} \right) u dx dt - k \int_0^\tau \int_0^\ell q^*(\xi, x; \tau, t) u dx dt = 0 \quad (21)$$

In Equation (21)

$$k \frac{\partial^2 u^*(\xi, x; \tau, t)}{\partial x^2} + \frac{\partial u^*(\xi, x; \tau, t)}{\partial t} = -\delta_i \quad (22)$$

$$-\delta_i u^* = -u^* \quad (23)$$

Making use of Equations (22) and (23) into Equation (20), the last one takes the following form:

$$-k \int_0^\tau \int_0^\ell u^*(\xi, x; \tau, t) q dx dt - \int_0^\tau \int_0^\ell u(x, t) u^*(\xi, x; \tau, t) dx = 0 \quad (24)$$

For any point the integral equation takes the following form:

$$u(\xi, \tau) = \int_0^\ell u(x, 0) u^*(\xi, x; \tau, 0) dx + k \int_0^\tau \int_0^\ell u^*(\xi, x; \tau, t) q(x, t) dx dt - \int_0^\tau \int_0^\ell q^*(\xi, x; \tau, t) u(x, t) dx dt \quad (25)$$

Discretization of Time

Equation (25) after the discretization of time takes the following form:

$$u(\xi, t_n) = \int_0^\ell u(x, 0) u^*(\xi, x; t_n, 0) dx + k \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left[\frac{\partial u(x, t_i)}{\partial x} u^*(\xi, x; t_n, t) \right]_0^\ell - k \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left[u(x, t_i) \frac{\partial u^*(\xi, x; t_n, t)}{\partial n} \right]_0^\ell dt \quad (26)$$

Assume that the potential u and the flux q are constant within each time step therefore Equation (26) can be written as:

$$u(\xi, t_n) = \int_0^\ell u(x, 0) u^*(\xi, x; t_n, 0) dx + \int_0^{t_F} [q(x, t) u^*(\xi, x; t_n, t) - u(x, t) q^*(\xi, x; t_n, t)] dt \quad (27)$$

In Equation (27), we have two different time integrals, they are:

$$I_1 = \int_0^{t_F} u^*(\xi, x; t_n, t) dt = \frac{r}{2k\sqrt{\pi}} \left(a_{i-1}^{-\frac{1}{2}} e^{-a_{i-1}} - a_i^{-\frac{1}{2}} e^{-a_i} \right) + \sqrt{\pi} \left\{ \operatorname{erf}(\sqrt{a_{i-1}}) - \operatorname{erf}(\sqrt{a_i}) \right\} \quad (28)$$

$$I_2 = \int_0^{t_F} q^*(\xi, x; t_n, t) dt = \operatorname{sign}(r) \frac{1}{2k} \left[\operatorname{erf}\left(a_i^{\frac{1}{2}}\right) - \operatorname{erf}\left(a_{i-1}^{\frac{1}{2}}\right) \right] \quad (29)$$

$$a_{i-1} = \frac{r^2}{4k(t_n - t_{i-1})} \quad \& \quad a_i = \frac{r^2}{4k(t_n - t_i)} \quad (30)$$

The domain integral

$$\int_0^\ell u(x, 0) u^*(\xi, x; t_n, 0) dx = (-) \frac{u(x, 0)}{2} \left[\operatorname{erf}\left(\frac{\xi - \ell}{2\sqrt{kt_n}}\right) - \operatorname{erf}\left(\frac{\xi}{2\sqrt{kt_n}}\right) \right] \quad (31)$$

Now, the final form for the integral equation corresponding to the diffusion equation defined over fixed domain will be:

$$u(\xi, t_n) = \frac{u(x, 0)}{2} \left[\operatorname{erf}\left(\frac{\xi}{2\sqrt{kt_n}}\right) - \operatorname{erf}\left(\frac{\xi - \ell}{2\sqrt{kt_n}}\right) \right] + \sum_{i=1}^n \left[q_{x=\ell} \int_{t_{i-1}}^{t_i} u^* dt - u_{x=\ell} \int_{t_{i-1}}^{t_i} q^* dt \right]_{x=\ell} - \sum_{i=1}^n \left[q_{x=0} \int_{t_{i-1}}^{t_i} u^* dt - u_{x=0} \int_{t_{i-1}}^{t_i} q^* dt \right]_{x=0} \quad (32)$$

Similar procedure can be carried out taking into consideration the moving boundaries and their normal velocities, therefore and according to [11] [12], and for the second and third stages, the integral equation corresponding to the diffusion equation with moving boundaries in a general final form will be:

$$c(\xi) u(\xi, t_k) = \alpha_i \int_{t_j}^{t_k} \left[u^*(\xi, x; t_k, t) \frac{\partial u(x, t)}{\partial x} - u(x, t) \frac{\partial u^*(\xi, x; t_k, t)}{\partial x} + \frac{1}{\alpha_i} u(x, t) u^*(\xi, x; t_k, t) V_n \right]_{x_1}^{x_2} dt \quad (33)$$

In Equation (33), V_n represents the outward normal velocity to the surface. In one-dimensional problem, as in our case study, this velocity may be $\frac{ds_1(t)}{dt}$ or $\frac{ds_2(t)}{dt}$ according to the phase underhand.

4. New Fixed-Moving-Fixed Algorithm

In the present paper, a generalized numerical algorithm and code for multi-moving boundary problem are developed, using visual Fortran 6.6. The main code consists of main program and three subroutines. The flow chart describing the main parts of the proposed algorithm is shown in **Figure 1**.

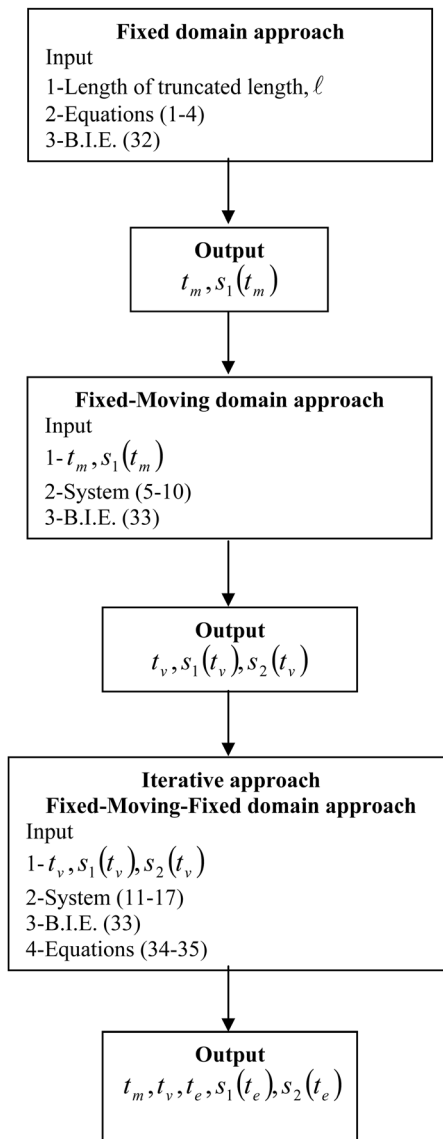


Figure 1. General fixed-moving-fixed integral algorithm.

4.1. Algorithm and Subroutine ONEPHASE

In this subroutine, a single phase bounded by two fixed boundaries are solved, using Equation (32) and the output will be the time at which melting starts and the corresponding location of the first moving boundary, separating the liquid and the solid, $s_1(t)$.

Suggested Algorithm

- 1) Input data, mold length ℓ , spatial grid size Δx and time step Δt .
- 2) Apply the boundary integral equation, given by Equation (32) at the end points of the domain of interest to estimate $u(x = s_1(t^j), t^j)$.
- 3) Check $u(x = s_1(t^j), t^j) \geq t_m, i = 1, 2, \dots, N$, if yes, then go to the next two-phase subroutine, with outputs, time of melting and the corresponding position for the moving boundary separating liquid and solid. If no update both the spatial and time variables $x_{i+1} = x_{i+1} + \Delta x, t^{j+1} = t^j + \Delta t$ and repeat steps 2 - 3 till the re-

quired output of this subroutine achieved.

4.2. Algorithm and Subroutine TWOPHASE

This subroutine concerns mainly with two phases each phase will be bounded by one fixed and the second is moving are solved and the output will be the time at which vapor starts and the corresponding location of the first and second moving boundaries $s_1(t)$ and $s_2(t)$.

Suggested Algorithm

- 1) The input unit here is the output from one-phase subroutine.
- 2) Solve solid phase subjected to melting temperature at the moving boundary and initial temperature at the fixed end to get $\left(\frac{\partial u_s}{\partial x}\right)_{s_1(t)}$.
- 3) By knowing $\frac{ds_1(t)}{dt}$ and $\left(\frac{\partial u_s}{\partial x}\right)_{s_1(t)}$ to estimate $\left(\frac{\partial u_\ell}{\partial x}\right)_{s_1(t)}$.
- 4) Check $u(x = s_1(t^j), t^j) \geq u_v$, if yes, then the output of this subroutine will be the time at which vapor starts appearing with the new position of the moving boundary separating liquid/solid and the starting position of the second moving boundary separating vapor(gas)/liquid, if no update the moving boundary separating liquid/solid.

4.3. Algorithm and Subroutine THREEPHASE

This subroutine is for three phases, the first one is bounded by two boundaries, one fixed and one moving, the second phase is bounded by two moving boundaries, and the third phase is also bounded by two boundaries one fixed and one moving. The output of this subroutine is to track all moving boundaries, determination all unknowns in all phases up to the end of the process with minimum number of iterations and high prescribed accuracy. The flow chart of this subroutine is shown in **Figure 2**. In this figure, the absolute errors $|E_1|$ and $|E_2|$ are defined as follow:

$$|E_1| = \left| \left(K_s \frac{\partial u_s(x,t)}{\partial x} - K_\ell \frac{\partial u_\ell(x,t)}{\partial x} \right) - \rho L_m \frac{ds_1(t)}{dt} \right| \quad (34)$$

$$|E_2| = \left| \left(K_v \frac{\partial u_v}{\partial x} - K_\ell \frac{\partial u_\ell}{\partial x} \right) - \rho L_v \frac{ds_2(t)}{dt} \right| \quad (35)$$

5. Numerical Results

5.1. Vaporization Test Problem

In this section, three test problems are solved to check the validity of the proposed algorithm and the high accuracy expected. The first example is a multi-moving boundary problem [13]. The thermo-physical properties are listed below in **Table 1**.

The solid material herein is of a low thermal conductivity and so the vaporization occurs before the moving boundary separating liquid and solid reaches the adiabatic boundary. The result due to the present algorithm is shown in **Figure 3**. In this figure, both vapor/liquid and liquid/solid moving boundaries are plotted both on the same figure. The liquid/solid moving boundary is in the left side, while the other one is on the right. From this figure, one can clearly determine the melting, vapor and the time at which the process ends. These times are determined and summarized in **Table 2**, at two different time steps. The absolute errors between the results due to the present method compared with the finite elements method are also presented in the same table. As we see, the absolute errors decrease by decreasing the time step. On the other hand by decreasing the time step, the number of iterations increases. The number of iterations are shown in **Table 2**, corresponding to the two time steps used. It is clear that the increase in the number of iterations is not so much but the accuracy improved to

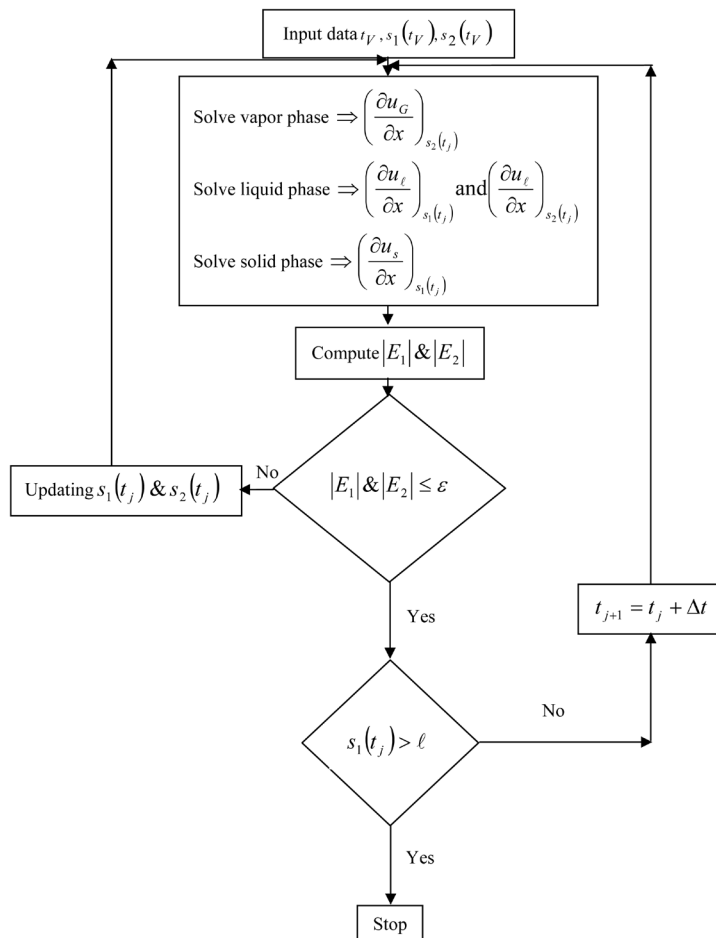


Figure 2. Flow chart for THREEPHASE subroutine.

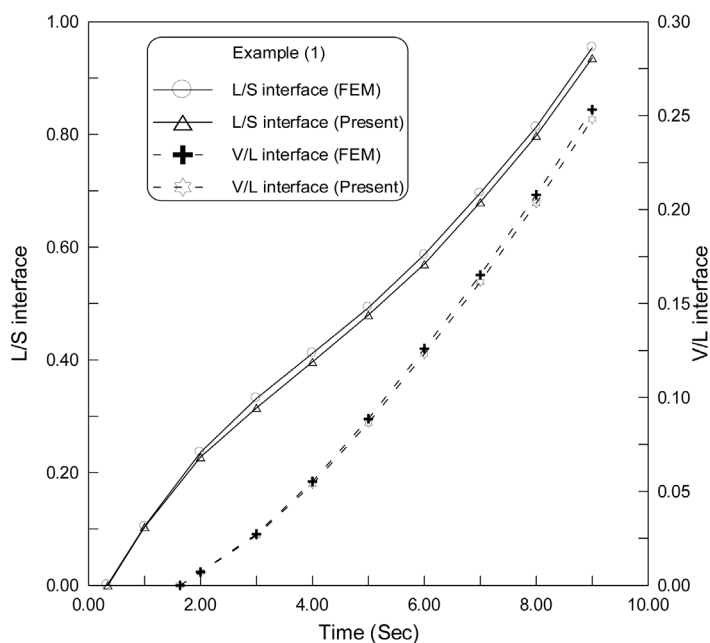


Figure 3. Moving boundaries location for vaporization test problem.

Table 1. Numerical data for vaporization test problem.

Parameter	Definition	Numerical data
c_s	Specific heat for solid	4.944
c_ℓ	Specific heat for liquid	4.944
K_s	Thermal conductivity for solid	0.259
K_ℓ	Thermal conductivity for liquid	0.259
u_m	Melting temperature	1454
u_v	Vaporization temperature	3000
L_m	Latent heat for melting	2160
L_v	Latent heat for vaporization	37,200
$u(x,0)$	Initial temperature	27
$q(t)$	Input heat flux at the boundary $x=0$	2500
ρ	Density	1

Table 2. Comparison between melting, vaporization and end time.

Time step	Type of time	FE	Present	Absolute error	Average number of iterations
$\Delta t = 0.25$	t_m	0.36150	0.36158	8.0×10^{-5}	12 - 16
	t_v	1.63034	1.63041	7.0×10^{-5}	
	t_e	9.75464	9.75468	3.4×10^{-5}	
$\Delta t = 0.0625$	t_m	0.32767	0.32771	4.0×10^{-5}	20 - 24
	t_v	1.63446	1.63449	3.25×10^{-5}	
	t_e	9.38719	9.38721	2.0×10^{-5}	

nearly the half.

5.2. Ablation Test Problem-1

A solid medium initially at uniform temperature, $u_i = 300^\circ \text{K}$, the boundary $x=0$ exposed to two different cases of input heat flux, constant, $Q(t) = 5 \times 10^6$ and linear, $Q(t) = 3 \times 10^4 t$ respectively. The domain in the present problem is still fixed while appearing two moving interfaces, solid-liquid and vapor (gas)-liquid (Ablation surface). The results due to the present algorithm are shown in **Figures 4-6**, respectively and the results are compared with the results due to the source and sink method [14]. **Figure 4** shows the movement of solid-liquid due to constant input heat flux, and the resulting ablation surface due to the same input heat flux is shown in **Figure 5**. The same results due to linear case are plotted on the same plot as shown in **Figure 6**. From the computations and figures, it is found that the solid-liquid interface has the same behavior in both constant and linear cases of input heat flux that is concave upward. In case of linear heat flux input this concavity becomes more apparent than the constant case. In the contrary, the vapor (gas)/liquid interface behaves concave downward but in linear case this concavity increases.

5.3. Ablation Test Problem-2

This problem is for a long enough solid mold initially at a uniform temperature. The surface $x=0$ exposed to two different high input heat flux, constant and linear and the vapor is removed as soon as it formed-ablation problem-. Two specific heat flux boundary condition are chosen in the present computations, namely, $q(t) = 2.0, 10 (t/t_c)$ and the following values used in the calculation [15] (**Table 3**).

The ablation thickness due to the present method compared with the corresponding from the heat balance integral method is shown in **Figure 7**. It is clear that the ablation thickness is concave downward in both case of

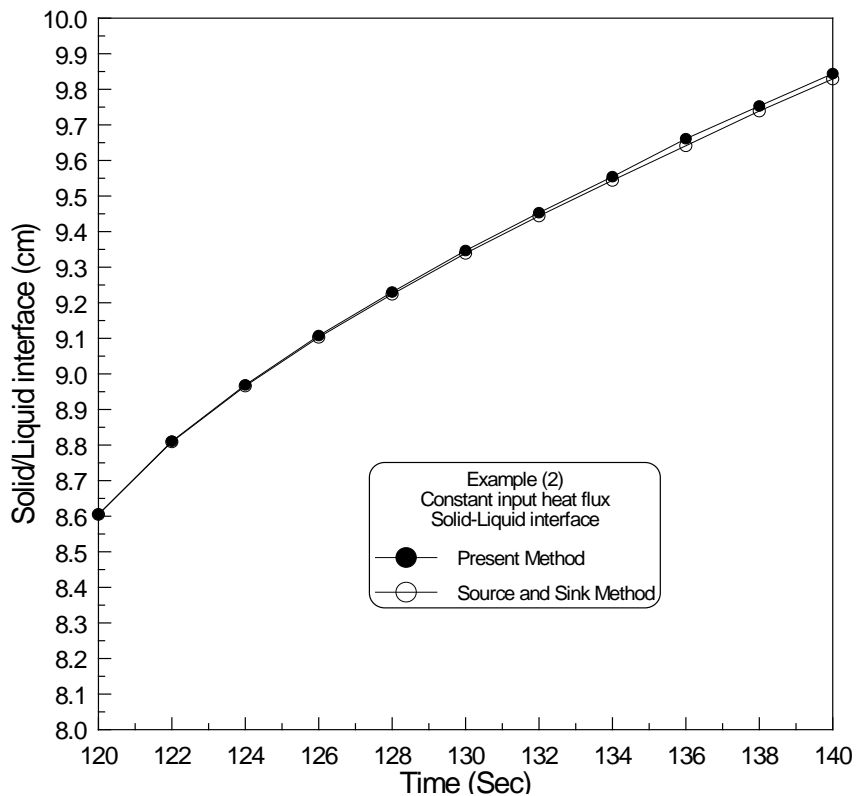


Figure 4. Solid/liquid due to Constant heat flux-problem-1.

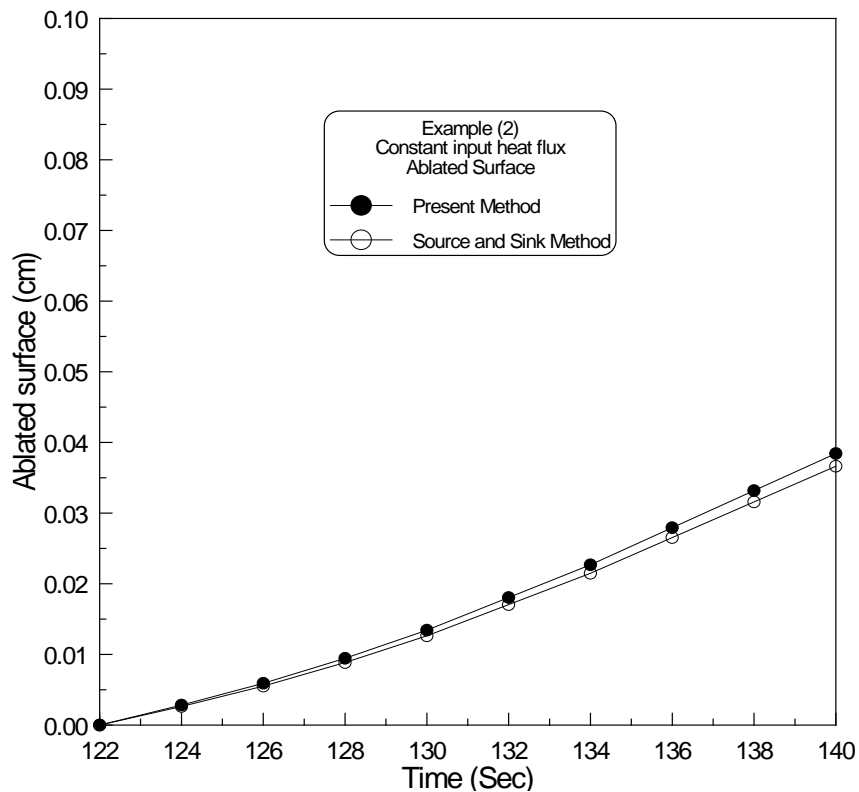


Figure 5. Ablated surface due to constant heat flux-problem-1.

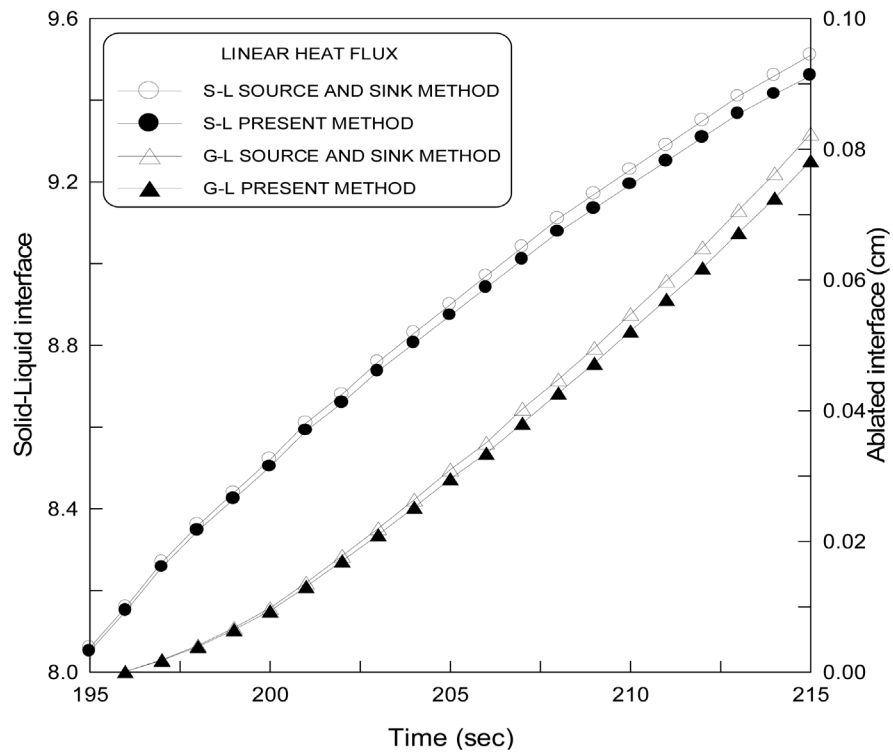


Figure 6. Solid/liquid and Ablated surface due to linear heat flux-problem-1.

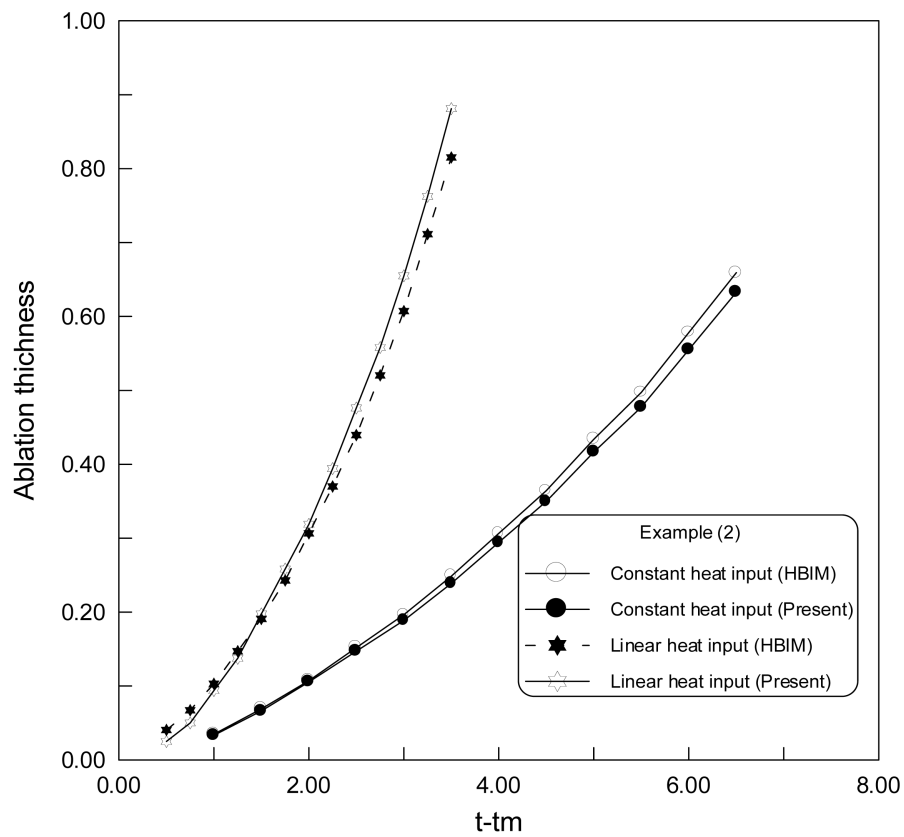


Figure 7. Ablation thickness for ablation test problem-2.

Table 3. Thermo-physical parameters.

Symbol	Physical meaning	Numerical value
α	Thermal diffusivity	$0.1 \text{ ft}^2 / \text{sec}$
t_c	Characteristic time	10sec
L	characteristic length	1.0st
q_r	Reference heat flux	$10 \text{ Btu} / \text{sec} - \text{ft}^2$
$T_m - T_0$	Temperature difference	100° F
ν	Inverse Stefan number	1
T	Reference time	100 sec

input heat flux and at the same time, the ablation thickness in case of linear heat flux is higher than that for constant case. There is a good agreement between the two methods in both cases.

6. Conclusion

The importance of the present paper comes from its dealing with practical applications of wide range of our daily life. The boundary integral equation method is not so new but it is used as a mathematical tool due to its simplicity in use. Based on this method, a generalized numerical algorithm and computer code are developed to solve such applications. It is found from the computations that the developed algorithm and the code are so simple to handle and that an acceptable accuracy is obtained. Also by decreasing the time step, and a little bit increase of iterations, the absolute errors are decreased to the half nearly. Finally, the algorithm and subsequently the code can be easily modified to cover higher dimensional problems with acceptable accuracy, which can be improved by decreasing the time step, but on the other hand, stability should be achieved.

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Nomenclature

u : Temperature
 u^* : Fundamental solution
 α : Thermal diffusivity
 K : Conductivity
 ρ : Density
 $q(t)$: Input heat flux
 $s_1(t)$: Liquid-solid interface
 $s_2(t)$: Vapor-liquid interface
 ℓ : Truncated long enough boundary length
 L : Latent heat

Subscript

s : Solid
 ℓ : Liquid
 i : Initial
 v : Vapor
 m : Melting