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Qualitative Properties of Solutions of a Doubly Nonlinear Reaction-Diffusion System with a Source

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Abstract

In this paper, we study properties of solutions to doubly nonlinear reaction-diffusion systems with variable density and source. We demonstrate the possibilities of the self-similar approach to studying the qualitative properties of solutions of such reaction-diffusion systems. We also study the finite speed of propagation (FSP) properties of solutions, an asymptotic behavior of the compactly supported solutions and free boundary asymptotic solutions in quick diffusive and critical cases.

Keywords

Double Nonlinear Reaction-Diffusion Equation, Self-Similar Solution, Asymptotics

1. Introduction

Let's consider properties of the Cauchy problem for the following system of nonlinear reaction-diffusion equations in the domain $Q=\{(t,x):t>0,x\in R^N\}$

$$\begin{aligned}\frac{\partial u}{\partial t}&=div\left(|x|^k u^{m_1-1}|\nabla u|^{p-2}\nabla u\right)+\gamma(t)v^{\beta_1},\\ \frac{\partial v}{\partial t}&=div\left(|x|^k v^{m_2-1}|\nabla v|^{p-2}\nabla v\right)+\gamma(t)u^{\beta_2},\end{aligned}\tag{1}$$

$$\begin{aligned}&u(0,x)=u_0(x)\ge 0,\\ &v(0,x)=v_0(x)\ge 0,\ x\in R^N,\end{aligned}\tag{2}$$

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where $k \in R, m_1, m_2 > 1, \beta_1, \beta_2 \ge 1, p \ge 2$ are given positive numbers, $\nabla(\cdot) = grad(\cdot)$ and $u_0(x) \ge 0$, $v_0(x) \ge 0$, $0 < \gamma(t) \in C(0, +\infty)$. System (1) describes different physical process in two cŏmponential inhomogeneous nonlinear environments. For example, the processes of the reaction-diffusion, heat conductivity, polytrophic filtration of liquids and gas with a source power which is equal to v^{β_1}, u^{β_2}. Cases, when $k = l, p = 2, m_1 = m_2 = 0$, were considered in [1]-[7].

The system (1) in the domain, where $u = v = 0$ is degenerated, and in the domain of degeneration it may not have the classical solution. Therefore, we study the weak solutions of system (1) which also have physical sense: $0 \le u, v \in C(Q)$ and $|x|^k u^{m_1-1}|\nabla u|^{p-2}\nabla u$, $|x|^k v^{m_2-1}|\nabla v|^{p-2}\nabla v \in C(Q)$ satisfy some integral identity in the sense of distribution [1]. For the solution of system (1) there are phenomena of the *finite speed of a propagation* (FSP). That is, there are functions $l_1(t), l_2(t)$ that satisfy $u(t,x) \equiv 0$ and $v(t,x) \equiv 0$ at $|x| \ge l_1(t)$ and $|x| \ge l_2(t)$. In the case of $l_1(t), l_2(t) < \infty$ for $\forall t > 0$, a solution of problems (1), (2) is called *space localization* of a disturbance. The surfaces $|x| = l_1(t)$ and $|x| = l_2(t)$ are called a free boundary or a front, respectively.

The process of the reaction-diffusion with double nonlinearity in the case of one equation has been investigated by many authors (see [8]-[15] and the references therein). FSP and blow-up property for equations with variable density

$$\rho(x)\frac{\partial u}{\partial t} = div\left(u^{m-1}|\nabla u|^{p-2}\nabla u\right),\ (x,t) \in R^{N+1},\ \rho(x) = |x|^{-l},\ l \ge 0$$

was established in [8] [9]. An asymptotic of self-similar solutions was studied in [15]. Martynenko and Tedeev [10] [11] studied the Cauchy problem for the following two equations with variable coefficients:

$$\rho(x)u_t = div\left(u^{m-1}|\nabla u|^{p-2}\nabla u\right) + u^{\beta},\ x \in R^N,\ t > 0,$$

and

$$\rho(x)u_t = div\left(u^{m-1}|\nabla u|^{p-2}\nabla u\right) + \rho(x)u^{\beta},\ x \in R^N,\ t > 0,$$

where $p > 1, m + p - 3 > 0, \beta > m + p - 2, \rho(x) = |x|^{-n}$, or $\rho(x) = (1 + |x|)^{-n}$.

They showed that under some restrictions to the parameters and initial data, any nontrivial solution to the Cauchy problem blows up in finite time. Moreover, the authors established a sharp universal estimate of the solution near the blow-up point.

It is well know that qualitative properties of solutions of the equation similar to (1) have not been investigated thoroughly. There are some results in [1]-[6] corresponding to the case $p = 2$.

In the present work, the qualitative properties of solutions of system (1) are studied based on the self-similar and approximately self-similar approach. We establish one way of construction of the critical exponent and property finite speed of perturbation (FSP) for system (1). An asymptotic property of compactly supported solutions (c.s.s.) of the considered problem and the behavior of the free boundary for the case $m_i + p - 3 > 0, i = 1,2$ are obtained. We prove the existence of solution with finite property. An asymptotic of a self-similar solution for the fast diffusion case $(m_i + p - 3 < 0, i = 1,2)$ and a critical case $m_i + p - 3 = 0, i = 1,2$ are also studied.

2. Approximate Self-Similar and Self-Similar Equations

Below we provide a method of nonlinear splitting for construction of self-similar and approximately self-similar equation. For construction of the self-similar and approximately self-similar solutions of system (1) we search the solutions $u(t,x), v(t,x)$ in the form

$$\begin{cases} u(t,x) = \bar{u}(t)\, w\left(\tau(t), \varphi(|x|)\right), \\ v(t,x) = \bar{v}(t)\, z\left(\tau(t), \varphi(|x|)\right). \end{cases} \quad (3)$$

Here, we obtain $\bar{u}(t), \bar{v}(t)$ as

$$\bar{u}(t) = A\left[T + \int_0^t \gamma(y)\,\mathrm{d}y\right]^{-(\beta_1+1)/(\beta_1\beta_2-1)},$$

$$\bar{v}(t) = B\left[T + \int_0^t \gamma(y)\,\mathrm{d}y\right]^{-(\beta_2+1)/(\beta_1\beta_2-1)},$$

Which are the solutions of following equations

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}t}=\gamma(t)\bar{v}^{\beta_1},\ \frac{\mathrm{d}\bar{v}}{\mathrm{d}t}=\gamma(t)\bar{u}^{\beta_2}.$$

Substituting (3), the system (1) is reduced to the following system of equations

$$\begin{cases}\dfrac{\partial w}{\partial \tau}=\varphi^{1-s}\dfrac{\partial}{\partial\varphi}\left(\varphi^{s-1}w^{m_1-1}\left|\dfrac{\partial w}{\partial\varphi}\right|^{p-2}\dfrac{\partial w}{\partial\varphi}\right)+\gamma(t)\bar{v}^{-(m_2+p-3)}\bar{u}^{\beta_1}\left(w+z^{\beta_1}\right),\\ \dfrac{\partial z}{\partial \tau}=\varphi^{1-s}\dfrac{\partial}{\partial\varphi}\left(\varphi^{s-1}z^{m_2-1}\left|\dfrac{\partial z}{\partial\varphi}\right|^{p-2}\dfrac{\partial z}{\partial\varphi}\right)+\gamma(t)\bar{u}^{-(m_1+p-3)}\bar{v}^{\beta_2}\left(z+w^{\beta_2}\right),\end{cases}\tag{4}$$

where the functions $\tau(t),\varphi(x)$ are chosen as following

$$\begin{gathered}\tau(t)=\int_0^t\bar{u}^{m_1+p-3}(\eta)\mathrm{d}\eta=\int_0^t\bar{v}^{m_2+p-3}(\eta)\mathrm{d}\eta,\ \text{ if }\ m_i+p-3\neq0,\ i=1,2,\\ \tau(t)=T+t,\ \text{if}\ m_i+p-3=0,\ i=1,2,\end{gathered}\tag{5}$$

$$\varphi(r)=\frac{1}{p_1}|r|^{p_1},\ |r|=\sqrt{\sum_{i=1}^{N}x_i^2},\ p_1=\frac{p-k}{p},\ s=p\frac{N}{p-k},\ k<p.$$

It is easy to establish that the system (4) has approximately self-similar solution of kind

$$w(\tau,\varphi)=f(\xi),\ z(\tau,\varphi)=\psi(\xi),\tag{6}$$

where $\xi=\dfrac{\varphi(r)}{\tau^{1/p}}$ and the functions $f(\xi),\psi(\xi)$ satisfies the approximately self-similar system equations

$$\begin{gathered}\xi^{1-s}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{s-1}f^{m_1-1}\left|\frac{\mathrm{d}f}{\mathrm{d}\xi}\right|^{p-2}\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)+\frac{\xi}{p}\frac{\mathrm{d}f}{\mathrm{d}\xi}+\gamma(t)\tau\bar{u}^{-(m_1+p-2)}\bar{v}^{\beta_1}\left(-f+\psi^{\beta_1}\right)=0,\\ \xi^{1-s}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{s-1}\psi^{m_2-1}\left|\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right|^{p-2}\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right)+\frac{\xi}{p}\frac{\mathrm{d}\psi}{\mathrm{d}\xi}+\gamma(t)\tau\bar{v}^{-(m_2+p-2)}\bar{u}^{\beta_2}\left(-\psi+f^{\beta_2}\right)=0.\end{gathered}\tag{7}$$

It is easy to prove that as $t\to\infty$

$$\begin{gathered}\gamma(t)\tau\bar{u}^{-(m_1+p-2)}\bar{v}^{\beta_1}\to c_1,\\ \gamma(t)\tau\bar{v}^{-(m_2+p-2)}\bar{u}^{\beta_2}\to c_2,\end{gathered}\tag{8}$$

for $0<\gamma(t)\in H$, where H -Hardy's body [2], c_1,c_2 are constants. In this case, it is easy to show that system (1) becomes a self-similar for a sufficient large *t*. Therefore it is possible to consider the system (7) as an asymptotically self-similar system of equation corresponding to system (1). In particular case, when $\gamma(t)=\text{const}$ approximately self-similar systems (7) will be as self-similar if

$$(\beta_1+1)(m_1+p-3)=(\beta_2+1)(m_2+p-3).\tag{9}$$

In this case for the functions $f(\xi),\psi(\xi)$ we have the following self-similar system of equation in "radial" form

$$\begin{gathered}\xi^{1-s}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{s-1}f^{m_1-1}\left|\frac{\mathrm{d}f}{\mathrm{d}\xi}\right|^{p-2}\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)+\frac{\xi}{p}\frac{\mathrm{d}f}{\mathrm{d}\xi}+a_1\left[-f+\psi^{\beta_1}\right]=0,\\ \xi^{1-s}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{s-1}\psi^{m_2-1}\left|\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right|^{p-2}\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right)+\frac{\xi}{p}\frac{\mathrm{d}\psi}{\mathrm{d}\xi}+a_2\left[-\psi+f^{\beta_2}\right]=0,\end{gathered}\tag{10}$$

where

$$a_1 = \left[(\beta_1\beta_2 - 1) - (m_1 + p - 3)(\beta_1 + 1)\right] / (\beta_1\beta_2 - 1),$$
$$a_2 = \left[(\beta_1\beta_2 - 1) - (m_2 + p - 3)(\beta_2 + 1)\right] / (\beta_1\beta_2 - 1).$$

In the case $p = 2$ or $m = 1$ in (10), the properties of the different solutions as computing aspects of the system Equation (10) were studied by many authors [8]-[15]. In singular, one equation case, when $\beta = m + p - 2$ the existence of positive solutions of the Equation (10) was studied in [14].

3. Slowly Diffusion Case: $m_i + p - 3 > 0, i = 1, 2$

3.1. A Global Solvability of Solutions

We prove properties of a global solvability of weak solutions of the system (1) using a comparison principle (see [1]). For this goal we construct a new system of equation using the standard equation method as in [3]:

$$u_+(t,x) = (T+t)^{\alpha_1} \bar{f}(\xi),$$
$$v_+(t,x) = (T+t)^{\alpha_2} \bar{\psi}(\xi), \tag{11}$$

where $\alpha_i = -\dfrac{\beta_i + 1}{\beta_i\beta_{3-i} - 1}$, $\xi = \dfrac{\varphi(|x|)}{[\tau(t)]^{1/p}}$, $\tau(t) = \dfrac{1}{\lambda_1}(T+t)^{\lambda_1}$, $\lambda_i = 1 - \alpha_i(m_i + p - 3)$, $i = 1, 2.$

In the case, $\alpha_1(m_1 + p - 3) = \alpha_2(m_2 + p - 3)$

$$\bar{f}(\xi) = \left(a - \xi^{p/(p-1)}\right)_+^{q_2}, \ \bar{\psi}(\xi) = \left(a - \xi^{p/(p-1)}\right)_+^{q_1},$$

where $q_1 = \dfrac{p-1}{m_1 + p - 3}, q_2 = \dfrac{p-1}{m_2 + p - 3}, a > 0, (b)_+ = \max(0, b).$

Fujita type critical exponent for the system (1) is numerical parameters for which the following equality holds:

$$(\beta_i + 1)/(\beta_i\beta_{3-i} - 1) = N/\left[p + (p + m_i - 3)N\right], \quad i = 1, 2. \tag{12}$$

This result consists of the result of Escobedo, Herero [15] for the case when $k = 0, p = 2, p + m_i - 3 = 0, i = 1, 2$ in (1).

Theorem 1. (*A global solvability*). *Assume* $k < p$, $m_i + p - 3 > 0$,

$$\beta_i > \frac{p + m_{3-i} - 3}{p + m_i - 3}, \ i = 1, 2,$$

$$-\frac{N}{p} + \frac{a_1(\beta_1 + 1)}{\beta_1\beta_2 - 1} + a_1 a^{q_2\beta_1 - q_1} \le 0,$$

$$-\frac{N}{p} + \frac{a_2(\beta_2 + 1)}{\beta_1\beta_2 - 1} + a_2 a^{q_1\beta_2 - q_2} \le 0,$$

$$u_+(0,x) \ge u_0(x), \ v_+(0,x) \ge v_0(x), \ x \in R^N.$$

Then for sufficiently small $u_0(x), v_0(x)$ *the followings holds*

$$u(t,x) \le A_1 u_+(t,x), \ v(t,x) \le A_2 v_+(t,x) \text{ in } Q, \tag{13}$$

where the functions $u_+(t,x), v_+(t,x)$ *defined as above,* $A_1 > 0, A_2 > 0$ *are constants.*

Proof. For proving theorem 1 we use a comparison principle. As a comparison solution we take the functions $A_1 u_+(t,x), A_2 v_+(t,x)$, where $A_1 > 0, A_2 > 0$.

It is easy to check that

$$\bar{u}^{-(m_1+p-2)}A(A_1u_+,A_2v_+)=\left[-A_1^{m_1+p-3}(\gamma\gamma_1)^{p-1}N+a_1\frac{\beta_1+1}{\beta_1\beta_2-1}\right]\bar{f}+a_1A_2^{\beta_1}\bar{\psi}^{\beta_1},$$

$$\bar{v}^{-(m_2+p-2)}B(A_1u_+,A_2v_+)=\left[-A_2^{m_2+p-3}(\gamma\gamma_2)^{p-1}N+a_2\frac{\beta_2+1}{\beta_1\beta_2-1}\right]\bar{\psi}+a_2A_1^{\beta_2}\bar{f}^{\beta_2},$$

If $A_1^{m_1+p-3}(\gamma q_1)^{p-1}=1/p,\ A_2^{m_2+p-3}(\gamma q_2)^{p-1}=1/p$.
Then we have

$$\bar{u}^{-(m_1+p-2)}A(A_1u_+,A_2v_+)=\left[-\frac{N}{p}+a_1\frac{\beta_1+1}{\beta_1\beta_2-1}\right]\bar{f}+a_1A_2^{\beta_1}\bar{\psi}^{\beta_1},$$

$$\bar{v}^{-(m_2+p-2)}B(A_1u_+,A_2v_+)=\left[-\frac{N}{p}+a_2\frac{\beta_2+1}{\beta_1\beta_2-1}\right]\bar{\psi}+a_2A_1^{\beta_2}\bar{f}^{\beta_2}.$$

In order to apply a comparison principle we note that $A(u_+,v_+)\le 0,\quad B(u_+,v_+)\le 0$ in $D=\left\{(t,x):t>0,|x|\le l(t)=a^{(p-1)/p}\left[\tau(t)\right]^{1/p}\right\}$. Since

$$\bar{\psi}^{\beta_1}(\xi)\bar{f}(\xi)=\left(a-\xi^{p/(p-1)}\right)^{\beta_1q_2-q_1}=\exp\left(-\left[(p-1)\frac{\beta_1}{m_2+p-3}-\frac{1}{m_1+p-3}\right]\eta\right),$$

$$\bar{f}^{\beta_2}(\xi)\bar{\psi}(\xi)=\left(a-\xi^{p/(p-1)}\right)^{\beta_2q_1-q_2}=\exp\left(-\left[(p-1)\frac{\beta_2}{m_1+p-3}-\frac{1}{m_2+p-3}\right]\eta\right).$$

Therefore,

$$\max\left(\bar{\psi}^{\beta_1}(\xi)\bar{f}(\xi)\right)=a^{\beta_1q_2-q_1},\ \max\left(\bar{f}^{\beta_2}(\xi)\bar{\psi}(\xi)\right)=a^{\beta_2q_1-q_2}.$$

Then according to the hypotheses of Theorem 1 and comparison principle we have

$$u(t,x)\le A_1u_+(t,x),\ v(t,x)\le A_2v_+(t,x)\ \text{in}\ Q,$$

if

$$A_1u_+(0,x)\ge u_0(x),\ A_2v_+(0,x)\ge v_0(x),\ x\in R^N.$$

The proof of the theorem is complete.
We notice that if

$$\frac{\beta_i+1}{\beta_i\beta_{3-i}-1}=\frac{N}{p+(p+m_i-3)N},\ i=1,2,$$

then

$$\bar{u}^{-(m_1+p-2)}A(A_1u_+,A_2v_+)=A_2^{\beta_1}\bar{\psi}^{\beta_1}\ge 0,$$

$$\bar{v}^{-(m_2+p-2)}B(A_1u_+,A_2v_+)=A_1^{\beta_2}\bar{f}^{\beta_2}\ge 0.$$

It means that

$$u(t,x)\ge A_1u_+(t,x),\ v(t,x)\ge A_2v_+(t,x)\ \text{in}\ Q,$$

if $u_0(x)\ge A_1u_+(0,x),v_0(x)\ge A_2v_+(0,x),x\in R^N$.

3.2. Property of Finite Speed of a Perturbation

Corollary 1. *Suppose that the hypotheses of Theorem* 1 *holds. Then a solution of the problems* (1), (2) *has FSP property.*

Indeed, for a weak solution of the problems (1), (2) we have

$$u(t,x)\le A_1u_+(t,x),\ v(t,x)\le A_2v_+(t,x)\ \text{in}\ Q,$$

It follows that

$$u(t,x) \equiv 0,\ v(t,x) \equiv 0,\ \text{in } Q/D,$$

where $D = \left\{(x,t): t > 0, |x| < a^{(p-1)/p}\left[\tau(t)\right]^{1/(p-k)}\right\}$. It means that the solution of the problems (1), (2) have FSP property.

Critical case. The case $m_i + p - 3 < 0, i = 1,2$ will be called a critical case.

Theorem 2. *Let* $k < p, m_i + p - 3 < 0, i = 1,2,\ \frac{\beta_1 + 1}{\beta_1\beta_2 - 1} < \frac{N}{p}, \frac{\beta_2 + 1}{\beta_1\beta_2 - 1} < \frac{N}{p}$. *Then for sufficiently small* $u_0(x), v_0(x)$ *the problems* (1), (2) *have global solution and the following inequalities in Q hold*

$$u(t,x) \le (T+t)^{-N/p} \exp\left(-\left(\frac{\xi^p}{p}\right)\right),$$
$$v(t,x) \le (T+t)^{-N/p} \exp\left(-\left(\frac{\xi^p}{p}\right)\right), \tag{14}$$

here $\xi = \frac{\varphi(|x|)}{\left[\tau(t)\right]^{1/p}}$,

Proof. Proof of the theorem is based on the comparison principle. We take for comparison the functions

$$u_+(t,x) = B_1 (T+t)^{-N/p}\, \overline{f}(\xi),$$
$$v_+(t,x) = B_2 (T+t)^{-N/p}\, \overline{\psi}(\xi),$$

where $\overline{f}(\xi) = \exp\left(-\left(\frac{\xi^p}{p}\right)\right), \overline{\psi}(\xi) = \exp\left(-\left(\frac{\xi^p}{p}\right)\right), \xi = \frac{|\eta|}{(T+t)^{1/p}}, B_i > 0, i = 1,2.$

It is easy to check that

$$\overline{u}^{-(m_1+p-2)} A(B_1u_+, B_2v_+) = \left[-\frac{N}{p} + \frac{\beta_1 + 1}{\beta_1\beta_2 - 1}\right] + B_2^{\beta_1}\overline{\psi}^{\beta_2 - 1}\psi^{\beta_1},$$
$$\overline{v}^{-(m_2+p-2)} B(B_1u_+, B_2v_+) = \left[-\frac{N}{p} + \frac{\beta_1 + 1}{\beta_1\beta_2 - 1}\right] + B_1^{\beta_2}\overline{f}^{\beta_1 - 1} f^{\beta_2}.$$

From the hypothesis of Theorem 2 and last expressions we have

$$A(B_1u_+, B_2v_+) \le 0,$$
$$B(B_1u_+, B_2v_+) \le 0 \text{ in } Q,$$

if the constants B_1, B_2 such that

$$B_1^{\beta_2} \le -\frac{N}{p} + \frac{\beta_1 + 1}{\beta_1\beta_2 - 1},\quad B_2^{\beta_1} \le -\frac{N}{p} + \frac{\beta_2 + 1}{\beta_1\beta_2 - 1}.$$

This inequality due to the comparison principle completes the proof of the theorem.

Value β_1, β_2 for which

$$\frac{N}{p} = \frac{\beta_1 + 1}{\beta_1\beta_2 - 1},$$
$$\frac{N}{p} = \frac{\beta_2 + 1}{\beta_1\beta_2 - 1},$$

corresponds to Fujita type critical exponent proved earlier by Escobedo, Herrero [15] for the case $p = 2$.

4. Asymptotic of the Self-Similar Solutions

Now we study asymptotic of the weak compact supported solutions (c.s.s.) of the system (10) when $\gamma(t) = \text{const}$. Consider this system equation with boundary condition

$$\begin{aligned} &f(0) = c_1 > 0, \ f(d) = 0, \\ &\psi(0) = c_2 > 0, \ \psi(d) = 0, \end{aligned} \tag{15}$$

where $0 < d < +\infty$.

The existence of a self-similar weak c.s. solution for the problems (10), (15) in the case $k = 0, p = 2$ was studied in [6] where the authors obtained conditions for existence of the c.s. solution.

We seek solution of the system (10) in the form

$$\begin{aligned} &f(\xi) = \bar{f}(\xi) y_1(\eta), \ \ \eta = -\ln\left(a - \xi^{\frac{p}{p-1}}\right), \\ &\psi(\xi) = \bar{\psi}(\xi) y_2(\eta), \ \ \eta = -\ln\left(a - \xi^{\frac{p}{p-1}}\right), \end{aligned} \tag{16}$$

where

$$\begin{aligned} &\bar{f}(\xi) = \left(a - \xi^{\gamma}\right)_+^{q_1}, \ \bar{\psi}(\xi) = \left(a - \xi^{\gamma}\right)_+^{q_2}, \ a > 0, \\ &q_1 = \frac{p-1}{p+m_1-3}, \ q_2 = \frac{p-1}{p+m_2-3}, \ \gamma = \frac{p}{p-1}. \end{aligned} \tag{17}$$

Theorem 3. *Assume that* $q_1 > 0, q_2 > 0, \beta_1 q_2 > q_1, \beta_2 q_1 > q_2$. *Then the weak compactly support solutions* (*c.s.s*) $f(\xi), \psi(\xi)$ *of the system* (10) *as* $\eta \to \infty$ $\left(\eta = -\ln\left(a - \xi^{p/(p-1)}\right)\right)$ *has asymptotic*

$$\begin{aligned} &f(\xi) = c_1 \bar{f}(\xi), \\ &\psi(\xi) = c_2 \bar{\psi}(\xi), \end{aligned}$$

where the coefficients c_1, c_2 *satisfied to system of the algebraic equations*

$$\begin{aligned} &-q_1^{p-1} c_1^{m_1+p-3} + a_1 \gamma^{-p} = 0, \\ &-q_2^{p-1} c_2^{m_2+p-3} + a_2 \gamma^{-p} = 0. \end{aligned}$$

Proof. It is easy to check that

$$\xi^{N-1} \bar{f}^{m_1-1} \left|\frac{d\bar{f}}{d\xi}\right|^{p-2} \frac{d\bar{f}}{d\xi} + \xi^N (\gamma q_1)^{p-1} \quad \bar{f} \in C(0,\infty),$$

$$\xi^{N-1} \bar{\psi}^{m_2-1} \left|\frac{d\bar{\psi}}{d\xi}\right|^{p-2} \frac{d\bar{\psi}}{d\xi} + (\gamma q_2)^{p-1} \xi^N \quad \bar{\psi} \in C(0,\infty)$$

and

$$\begin{aligned} &\xi^{1-N} \frac{d}{d\xi}\left(\xi^{N-1} \bar{f}^{m_1-1} \left|\frac{d\bar{f}}{d\xi}\right|^{p-2} \frac{d\bar{f}}{d\xi}\right) = -(\gamma q_1)^{p-1} \left(N\bar{f} + \xi \frac{d\bar{f}}{d\xi}\right), \\ &\xi^{1-N} \frac{d}{d\xi}\left(\xi^{N-1} \bar{\psi}^{m_2-1} \left|\frac{d\bar{\psi}}{d\xi}\right|^{p-2} \frac{d\bar{\psi}}{d\xi}\right) = -(\gamma q_2)^{p-1} \left(N\bar{\psi} + \xi \frac{d\bar{\psi}}{d\xi}\right), \ \ \gamma = \frac{p}{p-1}. \end{aligned}$$

We will show that the functions $\bar{f}(\xi), \bar{\psi}(\xi)$ should be main member of asymptotic of solution of the system (10). For this goal we search the solution of system (10) in the form

$$f(\xi)=\overline{f}(\xi)w(\eta),$$
$$\psi(\xi)=\overline{\psi}(\xi)z(\eta),\quad \eta=-\ln\left(a-\xi^{p/(p-1)}\right).$$

By using expression (10) it is easy to cheek that

$$\xi^{s-1}f^{m_1-1}\left|\frac{\mathrm{d}f}{\mathrm{d}\xi}\right|^{p-2}\frac{\mathrm{d}f}{\mathrm{d}\xi}=\gamma^{p-1}\xi^{s}\overline{f}L_1(w),$$
$$L_1(w)=w^{m_1-1}\left|\frac{\mathrm{d}w}{\mathrm{d}\eta}-q_1w\right|^{p-2}\left(\frac{\mathrm{d}w}{\mathrm{d}\eta}-q_1w\right),$$
$$\xi^{s-1}\psi^{m_2-1}\left|\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right|^{p-2}\frac{\mathrm{d}\psi}{\mathrm{d}\xi}=\gamma^{p-1}\xi^{s}\overline{\psi}L_2(z),$$
$$L_2(z)=z^{m_2-1}\left|\frac{\mathrm{d}z}{\mathrm{d}\eta}-q_2z\right|^{p-2}\left(\frac{\mathrm{d}z}{\mathrm{d}\eta}-q_2z\right),$$
$$\xi^{1-s}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{s-1}f^{m_1-1}\left|\frac{\mathrm{d}f}{\mathrm{d}\xi}\right|^{p-2}\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)=\gamma^{p-1}\overline{f}\left[\left(s-q_1\gamma\frac{\xi^{\gamma}}{a-\xi^{\gamma}}\right)L_1w+\gamma\frac{\xi^{\gamma}}{a-\xi^{\gamma}}\frac{\mathrm{d}}{\mathrm{d}\eta}L_1w\right],$$
$$\xi^{1-s}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{s-1}\psi^{m_2-1}\left|\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right|^{p-2}\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right)=\gamma^{p-1}\overline{\psi}\left[\left(s-q_2\gamma\frac{\xi^{\gamma}}{a-\xi^{\gamma}}\right)L_2z+\gamma\frac{\xi^{\gamma}}{a-\xi^{\gamma}}\overline{\psi}\frac{\mathrm{d}}{\mathrm{d}\eta}L_2z\right].$$

Therefore according transformation (16) the system (10) reduced to the system

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\eta}L_1(w)+\left(\frac{s}{\gamma}\varphi_1(\eta)-q_1\right)L_1(w)+\frac{1}{p}\gamma^{-p}\varphi_1(\eta)\left(\frac{\mathrm{d}w}{\mathrm{d}\eta}-q_1w\right)\\&-\frac{\beta_1+1}{\beta_1\beta_2-1}\gamma^{-p}\varphi_1(\eta)w+\gamma^{-p}\frac{\mathrm{e}^{(-q_1+\beta_1q_2-1)\eta}}{a-\mathrm{e}^{-\eta}}z^{\beta_1}=0,\\&\frac{\mathrm{d}}{\mathrm{d}\eta}L_2(z)+\left(\frac{s}{\gamma}\varphi_1(\eta)-q_2\right)L_2(z)+\frac{1}{p}\gamma^{-p}\varphi_1(\eta)\left(\frac{\mathrm{d}z}{\mathrm{d}\eta}-q_2z\right)\\&-\frac{\beta_2+1}{\beta_1\beta_2-1}\gamma^{-p}\varphi_1(\eta)z+\gamma^{-p}\frac{\mathrm{e}^{(-q_2+\beta_2q_1-1)\eta}}{a-\mathrm{e}^{-\eta}}w^{\beta_2}=0,\end{aligned}\tag{18}$$

where $\varphi_1(\eta)=\dfrac{\mathrm{e}^{-\eta}}{a-\mathrm{e}^{-\eta}}$.

Analysis of solution of last system shows that $w\to c_1, z\to c_2$, as $\eta\to\infty$, where constants c_1,c_2 are the solutions of the algebraic system equations

$$-q_1^{p-1}c_1^{m_1+p-3}+a_1\gamma^{-p}=0,$$
$$-q_2^{p-1}c_2^{m_2+p-3}+a_2\gamma^{-p}=0.$$

The proof of the theorem is complete.

5. Quick Diffusion Case: $m_i+p-3<0, i=1,2$

Theorem 4. *Let* $m_i+p-3<0, i=1,2$. *Then regular (quenching) solution of the system* (10) *as* $\xi\to\infty$ *has asymptotic*

$$f(\xi)=c_3\left(a+\xi^{p/(p-1)}\right)^{q_1},$$
$$\psi(\xi)=c_4\left(a+\xi^{p/(p-1)}\right)^{q_2}.$$

Here

1) *if* $-q_1+\beta_1 q_2=0, -q_2+\beta_2 q_1=0,$ *then the coefficients* c_3, c_4 *are the roots of the nonlinear system of the algebraic equations*

$$\begin{aligned}&\gamma^{p-1}\left(\frac{pN}{(p-k)+\gamma q_1}\right)c_3^{m_1+p-2}+\left(a_1 q_1+\frac{\beta_1+1}{\beta_1\beta_2-1}\right)c_3+c_4^{\beta_1}=0,\\&\gamma^{p-1}\left(\frac{pN}{(p-k)+\gamma q_2}\right)c_4^{m_2+p-2}+\left(a_2 q_2+\frac{\beta_2+1}{\beta_1\beta_2-1}\right)c_4+c_3^{\beta_2}=0.\end{aligned}\tag{19}$$

2) *if* $\beta_1 q_2+1<q_1, \beta_2 q_1+1<q_2,$ *then the coefficients* c_3, c_4 *are the roots of the nonlinear system of the algebraic equations*

$$\begin{aligned}&\gamma^{p-1}\left(\frac{pN}{(p-k)+\gamma q_1}\right)c_3^{m_1+p-3}+\left(a_1 q_1+\frac{\beta_1+1}{\beta_1\beta_2-1}\right)=0,\\&\gamma^{p-1}\left(\frac{pN}{(p-k)+\gamma q_2}\right)c_4^{m_2+p-3}+\left(a_2 q_2+\frac{\beta_2+1}{\beta_1\beta_2-1}\right)=0.\end{aligned}\tag{20}$$

Proof. We will seek a solution of system (10) in following form

$$\begin{aligned}&f(\xi)=\tilde{f}(\xi)w(\eta),\\&\psi(\xi)=\tilde{\psi}(\xi)z(\eta),\ \eta=\ln\left(a+\xi^{p/(p-1)}\right).\end{aligned}\tag{21}$$

Since

$$\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}f^{m_1-1}\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right)=\gamma^{p-1}\bar{f}\left(s+q_1\gamma\frac{\xi^\gamma}{a+\xi^\gamma}\right)L_1 w+\gamma\frac{\xi^\gamma}{a+\xi^\gamma}\bar{f}\frac{d}{d\eta}L_1 w,$$

$$\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}\psi^{m_2-1}\left|\frac{d\psi}{d\xi}\right|^{p-2}\frac{d\psi}{d\xi}\right)=\gamma^{p-1}\bar{\psi}\left(s+q_2\gamma\frac{\xi^\gamma}{a+\xi^\gamma}\right)L_2 z+\gamma\frac{\xi^\gamma}{a+\xi^\gamma}\bar{\psi}\frac{d}{d\eta}L_2 z.$$

By substituting (21) into (10) we get

$$\begin{aligned}&\frac{d}{d\eta}L_1(w)+\left(\frac{s}{\gamma}\varphi_2(\eta)+q_1\right)L_1(w)+a_1\gamma^{-p}\left(\frac{dw}{d\eta}+q_1 w\right)\\&+\frac{\beta_1+1}{\beta_1\beta_2-1}\gamma^{-p}\varphi_2(\eta)w+\gamma^{-p}\varphi_2(\eta)e^{(-q_1+\beta_1 q_2)\eta}z^{\beta_1}=0,\\&\frac{d}{d\eta}L_2(z)+\left(\frac{s}{\gamma}\varphi_2(\eta)+q_2\right)L_2(z)+a_2\gamma^{-p}\left(\frac{dz}{d\eta}+q_2 z\right)\\&+\frac{\beta_2+1}{\beta_1\beta_2-1}\gamma^{-p}\varphi_2(\eta)z+\gamma^{-p}\varphi_2(\eta)e^{(-q_2+\beta_2 q_1)\eta}w^{\beta_2}=0,\end{aligned}\tag{22}$$

where $\varphi_2(\eta)=\dfrac{e^\eta}{a+e^\eta}.$

Analyzing of solutions system (22) when $\eta\to\infty$ we conclude that the solutions of this system $w\to c_3, z\to c_4,$ where constants c_3, c_4 are solutions of the algebraic system (19), (20).

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