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A Strong Law of Large Numbers for Set-Valued Random Variables in G_α Space

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Abstract

In this paper, we shall represent a strong law of large numbers (SLLN) for weighted sums of set-valued random variables in the sense of the Hausdorff metric d_H, based on the result of single-valued random variable obtained by Taylor [1].

Keywords

Set-Valued Random Variable, the Laws of Large Numbers, Hausdorff Metric

1. Introduction

We all know that the limit theories are important in probability and statistics. For single-valued case, many beautiful results for limit theory have been obtained. In [1], there are many results of laws of large numbers at different kinds of conditions and different kinds of spaces. With the development of set-valued random theory, the theory of set-valued random variables and their applications have become one of new and active branches in probability theory. And the theory of set-valued random variables has been developed quite extensively (cf. [2]-[7] etc.). In [1], Artstein and Vitale used an embedding theorem to prove a strong law of large numbers for independent and identically distributed set-valued random variables whose basic space is $\mathbb{R}^d$, and Hiai extended it to separable Banach space $\mathfrak{X}$ in [8]. Taylor and Inoue proved SLLN's for only independent case in Banach space in [7]. Many other authors such as Giné, Hahn and Zinn [9], Puri and Ralescu [10] discussed SLLN's under different settings for set-valued random variables where the underlying space is a separable Banach space.

In this paper, what we concerned is the SLLN of set-valued independent random variables in G_α space. Here the geometric conditions are imposed on the Banach spaces to obtain SLLN for set-valued random variables. The results are both the extension of the single-valued's case and the extension of the set-valued's case.

This paper is organized as follows. In Section 2, we shall briefly introduce some definitions and basic results of set-valued random variables. In Section 3, we shall prove a strong law of large numbers for set-valued independent random variables in G_α space.

2. Preliminaries on Set-Valued Random Variables

Throughout this paper, we assume that $(\Omega, \mathcal{A}, \mu)$ is a nonatomic complete probability space, $(\mathfrak{X}, \|\cdot\|)$ is a real separable Banach space, $\mathbb{N}$ is the set of nature numbers, $\mathbf{K}(\mathfrak{X})$ is the family of all nonempty closed subsets

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of $\mathfrak{X}$, and $\mathbf{K}_{bc}(\mathfrak{X})$ is the family of all nonempty bounded closed convex subsets of $\mathfrak{X}$.

Let A and B be two nonempty subsets of $\mathfrak{X}$ and let $\lambda \in \mathbb{R}$, the set of all real numbers. We define addition and scalar multiplication as

$$A + B = \{a + b : a \in A, b \in B\},$$

$$\lambda A = \{\lambda a : a \in A\}.$$

The Hausdorff metric on $\mathbf{K}(\mathfrak{X})$ is defined by

$$d_H(A, B) = \max\{\sup_{a\in A}\inf_{b\in B} \| a - b \|, \sup_{b\in B}\inf_{a\in A} \| a - b \|\},$$

for $A, B \in \mathbf{K}(\mathfrak{X})$. For an A in $\mathbf{K}(\mathfrak{X})$, let $\| A \|_{\mathbf{K}} = d_H(\{0\}, A)$. The metric space $(\mathbf{K}_b(\mathfrak{X}), d_H)$ is complete, and $\mathbf{K}_{bc}(\mathfrak{X})$ is a closed subset of $(\mathbf{K}_b(\mathfrak{X}), d_H)$ (cf. [6], Theorems 1.1.2 and 1.1.3). For more general hyperspaces, more topological properties of hyperspaces, readers may refer to a good book [11].

For each $A \in \mathbf{K}(\mathfrak{X})$, define the support function by

$$s(x^*, A) = \sup_{a\in A} < x^*, a >, \quad x^* \in \mathfrak{X}^*,$$

where $\mathfrak{X}^*$ is the dual space of $\mathfrak{X}$.

Let $\mathbf{S}^*$ denote the unit sphere of $\mathfrak{X}^*$, $C(\mathbf{S}^*)$ the all continuous functions of $\mathbf{S}^*$, and the norm is defined as $\| v \|_C = \sup_{x^* \in S^*}$

The following is the equivalent definition of Hausdorff metric.

For each $A, B \in \mathbf{K}_{bc}(\mathfrak{X})$,

$$d_H(A, B) = \sup\{| s(x^*, A) - s(x^*, A) | : x^* \in S^*\}.$$

A set-valued mapping $F : \Omega \to \mathbf{K}(\mathfrak{X})$ is called *a set-valued random variable* (*or a random set, or a multifunction*) if, for each open subset O of $\mathfrak{X}$, $F^{-1}(O) = \{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in A$.

For each set-valued random variable F, *the expectation of* F, denoted by $E[F]$, is defined as

$$E[F] = \{\int_\Omega f d\mu : f \in S_F\},$$

where $\int_\Omega f d\mu$ is the usual Bochner integral in $L^1[\Omega, \mathfrak{X}]$, the family of integrable $\mathfrak{X}$-valued random variables, and $S_F = \{ f \in L^1[\Omega; X] : f(\omega) \in F(\omega), a.e.(\mu)\}$. This integral was first introduced by Aumann [3], called Aumann integral in literature.

3. Main Results

In this section, we will give the limit theorems for independent set-valued random variables in G_α space. The following definition and lemma are from [1], which will be used later.

Definition 3.1 A Banach space $\mathfrak{X}$ is said to satisfy the condition G_α for some $\alpha, 0 < \alpha \le 1$, if there exists a mapping $G : \mathfrak{X} \to \mathfrak{X}^*$ such that

(i) $\| G(x) \| = \| x \|^\alpha$;

(ii) $G(x)x = \| x \|^{1+\alpha}$;

(iii) $\| G(x) - G(y) \| \le A \| x - y \|^\alpha$ for all $x, y \in \mathfrak{X}$ and some positive constant A.

Note that Hilbert spaces are G_1 with constant $A = 1$ and identity mapping G.

Lemma 3.1 Let $\mathfrak{X}$ be a separable Banach space which is G_α for some $0 < \alpha \le 1$ and let $\{V_1, V_2, \cdots, V_n\}$ be single-valued independent random elements in $\mathfrak{X}$ such that $E[V_k] = 0$ and $E[\| V_k \|^{1+\alpha}] < \infty$ for each $k = 1, 2, \cdots, n.$ then

$$E[\| V_1 + \cdots + V_n \|^{1+\alpha}] \le A \sum_{k=1}^{n} E[\| V_k \|^{1+\alpha}]$$

where A is the positive constant in (iii).

Theorem 3.1 Let $\mathfrak{X}$ be a separable Banach space which is G_α for some $0<\alpha\le 1$. Let $\{F_n : n\ge 1\}$ be a sequence of independent set-valued random variables in $\mathbf{K}_{bc}(\mathfrak{X})$, such that $E[F_n]=\{0\}$ for each n. If

$$\sum_{j=1}^{\infty} E[\| F_j \|_{\mathbf{K}}]^{1+\alpha} < \infty$$

where $\phi_0(t)=t^{1+\alpha}$ for $0\le t\le 1$ and $\phi_0(t)=t$ for $t\ge 1$, then $\sum_{j=1}^{\infty} F_j$ converges with probability 1 in the sense of d_H.

Proof. Define

$$U_j = F_j I_{\{\|F_j\|_{\mathbf{K}}\le 1\}} \quad and \quad W_j = F_j I_{\{\|F_j\|_{\mathbf{K}}>1\}}.$$

Note that $F_j = U_j + W_j$ for each j and that both $\{U_j : j\ge 1\}$ and $\{W_j : j\ge 1\}$ are independent sequences of set-valued random variables. Next, for each m and n

$$E[\| \sum_{j=n}^{m} W_j \|_{\mathbf{K}}] \le \sum_{j=n}^{m} E[\| W_j \|_{\mathbf{K}}] \le \sum_{j=n}^{m} E[\phi_0(\| F_j \|_{\mathbf{K}})].$$

That means $\{E[\| \sum_{j=1}^{m} W_j \|_{\mathbf{K}}] : m\ge 1\}$ is a Cauchy sequence and hence

$$E[\| \sum_{j=1}^{m} W_j \|_{\mathbf{K}}] \ \ converges$$

as $m\to\infty$. Since convergence in the mean implied convergence in probability, Ito and Nisio's result in [12] for independent random elements(rf. Section 4.5) provides that

$$\| \sum_{j=1}^{\infty} W_j \|_{\mathbf{K}} \ \ converges\ in\ probability\ 1\ as \ \ n\to\infty.$$

Then for $n,m\ge 1, m>n$, by triangular inequality we have

$$\begin{aligned} d_H\Big(\sum_{j=1}^{n} W_j, \sum_{j=1}^{m} W_j\Big) &= d_H\Big(\sum_{j=1}^{n} W_j, \sum_{j=1}^{n} W_j + \sum_{j=n+1}^{m} W_j\Big) \\ &\le d_H\Big(\{0\}, \sum_{j=n+1}^{m} W_j\Big) = \| \sum_{j=n+1}^{m} W_j \|_{\mathbf{K}} \to 0,\ a.e.\ as\ n,m\to\infty. \end{aligned}$$

By the completeness of $(\mathbf{K}_b(\mathfrak{X}), d_H)$, we can have $\sum_{j=1}^{n} W_j$ converges almost everywhere in the sense of d_H.

Since by equivalent definition of Hausdorff metric, we have

$$\begin{aligned} E[\| \sum_{j=n}^{m} U_j \|_{\mathbf{K}}^{1+\alpha}] &= E[d_H(\sum_{j=n}^{m} U_j, \{0\})]^{1+\alpha} \\ &= E[\sup_{x^*\in S^*} | s(x^*, \sum_{j=n}^{m} U_j) |]^{1+\alpha}. \end{aligned}$$

For any fixed n,m, there exists a sequence $x_k^* \in S^*$, such that

$$\lim_{k\to\infty} | s(x_k^*, \sum_{j=n}^{m} U_j) | = \sup_{x^*\in S^*} | s(x^*, \sum_{j=n}^{m} U_j) |.$$

Then by dominated convergence theorem, Minkowski inequality and Lemma 3.1, we have

$$E[\|\sum_{j=n}^{m}U_j\|_{\mathbf{K}}^{1+\alpha}]=E[\lim_{k\to\infty}|s(x_k^*,\sum_{j=n}^{m}U_j)|^{1+\alpha}]=\lim_{k\to\infty}E[|s(x_k^*,\sum_{j=n}^{m}U_j)|^{1+\alpha}]$$
$$\le\lim_{k\to\infty}E[|s(x_k^*,\sum_{j=n}^{m}U_j)-E[s(x_k^*,\sum_{j=n}^{m}U_j)]|+|E[s(x_k^*,\sum_{j=n}^{m}U_j)]|]^{1+\alpha}$$
$$=\lim_{k\to\infty}E[|s(x_k^*,\sum_{j=n}^{m}U_j)-E[s(x_k^*,\sum_{j=n}^{m}U_j)]|+|E[s(x_k^*,\sum_{j=n}^{m}U_j)]|]^{1+\alpha}$$
$$\le 2^{1+\alpha}\left\{A\lim_{k\to\infty}\sum_{j=n}^{m}E[|s(x_k^*,U_j)-s(x_k^*,E[U_j])|^{1+\alpha}]+\lim_{k\to\infty}[\sum_{j=n}^{m}E[|s(x_k^*,W_j)|]]^{1+\alpha}\right\}$$
$$\le 2^{1+\alpha}\left\{A2^{2+\alpha}\sum_{j=n}^{m}E[\sup_{x^*\in S^*}|s(x^*,U_j))|^{1+\alpha}]+[\sum_{j=n}^{m}E[\sup_{x^*\in S^*}|s(x^*,W_j)|]]^{1+\alpha}\right\}$$
$$\le 2^{1+\alpha}\left\{A2^{2+\alpha}\sum_{j=n}^{m}E[\phi_0(\|F_j\|_{\mathbf{K}})]+[\sum_{j=n}^{m}E[\phi_0(\|F_j\|_{\mathbf{K}})]]^{1+\alpha}\right\}$$

for each n and m. Thus, $\{E[\|\sum_{j=1}^{m}U_j\|_{\mathbf{K}}]:m\ge 1\}$ is a Cauchy sequence, and hence converges. Hence, by the similar way as above to prove $\sum_{j=1}^{\infty}W_j$ converges with probability one in the sense of d_H. We also can prove that

$$\sum_{j=1}^{\infty}U_j \quad \textit{converges}$$

with probability one in the sense of d_H. The result was proved. □

From theorem 3.1, we can easily obtain the following corollary.

Corollary 3.2 Let $\mathfrak{X}$ be a separable Banach space which is G_α for some $0<\alpha\le 1$. Let $\{F_n\}$ be a sequence of independent set-valued random variables in $\mathbf{K}_b(\mathfrak{X})$ such that $E[F_n]=\{0\}$ for each n. If $\phi_n:R^+\to R^+, n=1,2,\cdots,$ are continuous and such that $\frac{\phi_n(t)}{t}$ and $\frac{t^{1+\alpha}}{\phi_n(t)}$ are non-decreasing, then for each $\alpha_n\subset R^+$ the convergence of

$$\sum_{n=1}^{\infty}\frac{E[\phi_n(\|F_n\|_{\mathbf{K}})]}{\phi_n(\alpha_n)}$$

implies that

$$\sum_{n=1}^{\infty}\frac{F_n}{\alpha_n}$$

converges with probability one in the sense of d_H.

Proof. Let

$$U_j=\frac{F_j}{\alpha_j}I_{\{\|F_j\|_{\mathbf{K}}\le\alpha_j\}} \quad \textit{and} \quad W_j=\frac{F_j}{\alpha_j}I_{\{\|F_j\|_{\mathbf{K}}>\alpha_j\}}.$$

If $\|F_n\|_{\mathbf{K}}>\alpha_n$, by the non-decreasing property of $\frac{\phi_n(t)}{t}$, we have

$$\frac{\phi_n(\alpha_n)}{\alpha_n}\le\frac{\phi_n(\|F_n\|_{\mathbf{K}})}{\|F_n\|_{\mathbf{K}}}.$$

That is

$$\frac{\| F_n \|_{\mathbf{K}}}{\alpha_n} \le \frac{\phi_n(\| F_n \|_{\mathbf{K}})}{\phi_n(\alpha_n)}. \tag{4.1}$$

If $\| F_n \|_{\mathbf{K}} \le \alpha_n$, by the non-decreasing property of $\dfrac{t^{1+\alpha}}{\phi_n(t)}$, we have

$$\frac{\| F_n \|_{\mathbf{K}}^{1+\alpha}}{\phi_n(\| F_n \|_{\mathbf{K}})} \le \frac{\alpha_n^{1+\alpha}}{\phi_n(\alpha_n)}.$$

That is

$$\frac{\| F_n \|_{\mathbf{K}}^{1+\alpha}}{\alpha_n^{1+\alpha}} \le \frac{\phi_n(\| F_n \|_{\mathbf{K}})}{\phi_n(\alpha_n)}. \tag{4.2}$$

Then as the similar proof of theorem 3.1, we can prove both $\sum_{j=1}^{\infty} U_j$ and $\sum_{j=1}^{\infty} W_j$ converges with probability one, and the result was obtained. □

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