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Localization of Unbounded Operators on Guichardet Spaces

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Abstract

As stochastic gradient and Skorohod integral operators, (∇,δ) is an adjoint pair of unbounded operators on Guichardet Spaces. In this paper, we define an adjoint pair of operator (ℓ_s,ℓ_s^*), where $\ell_s=\nabla_s E_s[C]$ with $E_s[C]$ being the conditional expectation operator. We show that ℓ_s (resp. ℓ_s^*) is essentially a kind of localization of the stochastic gradient operators (resp. Skorohod integral operators δ). We examine that ℓ_s and ℓ_s^* satisfy a local CAR (canonical ani-communication relation) and $(\ell_s^*)_{s\geq 0}$ forms a mutually orthogonal operator sequence although each ℓ_s is not a projection operator. We find that ℓ_s is s-adapted operator if and only if ∇_s is s-adapted operator. Finally we show application exponential vector formulation of QS calculus.

Keywords

Stochastic Gradient Operator, Skorohod Integral Operator, Localization, Ex-Ponential Vector, Guichardet Spaces

1. Introduction

The quantum stochastic calculus [4] [6] developed by Hudson and Parthasarathy is essentially a noncommutative extension of classical Ito stochastic calculus. In this theory, annihilation, creation, and number operator processes in boson Fock space play the role of "quantum noises", [2] which are in continuous time. On the other hand, the quantum stochastic calculus has been extended by Hitsuda is by means of the Hitsuda-Skorohod integral of anticipative process [3] [9] and the related gradient operator of Malliavin calculus. In this noncausal formulation the action of each QS integral is defined explicitly on Fock space vectors, and the essential quantum Ito formula is seen in terms of the Skorohod isometry.

In 2002, Attal [1] unify and extend both of the above approaches on Guichardet spaces. In this note, explicitly definitions of QS integrals provided and introduced no unnatural domain limitations. Moreover, maximality of operator domains is demon-strated for these QS integrals on Guichardet spaces.

In this argument, we define an adjoint pair of operator (ℓ_s,ℓ_s^*), where $\ell_s=\nabla_s E_s[C]$ with $E_s[C]$ being

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the conditional expectation (operator). The motivation for this study comes from the following observations. It is known that $E_s[C]$ is a projection operator on Guichardet Spaces. Hence, restricted to the range of $E_s[C]$, ℓ_s coincides with the stochastic gradient operator ∇_s. We show that ℓ_s (resp. ℓ_s^*) is essentially a kind of localization of the stochastic gradient operators (resp. Skorohod integral operators δ). We examine that ℓ_s and ℓ_s^* can be called a local stochastic gradient operators (resp. local Skorohod integral operators δ). Then, it is necessary and important to study a pair of operator (ℓ_s, ℓ_s^*).

This paper is organized as follows. In Section 2, we fix some necessary notations and recall main notions and facts about unbounded operators on Guichardet spaces. In Section 3, Section 4 and Section 5, we state our main results. We first examined that ℓ_s and ℓ_s^* satisfy a local CAR (canonical anti-communication relation) and $(\ell_s^*)_{s\ge 0}$ forms a mutually orthogonal operator sequence although each's is not a projection operator. We find that ℓ_s is s-adapted operator if and only if ∇_s is s-adapted operator. Finally we show application exponential vector formulation of QS calculus.

2. Unbounded Operators on Guichardet Spaces

In this section, we fix some necessary notations and recall main notions and facts about unbounded operators on Guichardet spaces. For detail formulation of unbounded operators, we refer reader to [1].

Let $\mathbb{R}_+$ be the set of all nonnegative real numbers and Γ the finite power set of $\mathbb{R}_+$, namely

$$\Gamma \doteq \{\sigma | \sigma \subset \mathbb{R}_+, \#\sigma < \infty\}$$

where $\#\sigma$ denotes the cardinality of σ as a set, with $\Gamma^{(n)}$ denoting the collection of n element subsets. Obviously, $\Gamma = \bigcup_{n>0} \Gamma^{(n)}$. Particularly, let $\varnothing \in \Gamma^{(0)}$ be an atom of measure 1. We denote by $L^2(\Gamma)$ the usual space of square integral real-valued functions on Γ.

Fixing a complex separable Hilbert space η, Guichardet space tensor product $\eta \otimes L^2(\Gamma)$, which we identify with the space of square-integrable functions $L^2(\Gamma;\eta)$, and is denoted by F. Guichardet space enjoys a continuous tensor product structure: for each $s \ge 0$ the map

$$f \otimes g(\omega) = f(\omega_{s)}) g(\omega_{(s})$$

where $\omega_{s)} = \omega \cap [0,s)$, $\omega_{(s} = \omega \cap (s,\infty)$.

For a Hilbert space-valued map $x : \Gamma \times \mathbb{R}_+ \to \eta$, let $\delta(x)$ be the map $\Gamma \to \eta$ given by

$$\delta(x)(\sigma) = \sum_{s\in\sigma} x_s(\sigma \setminus s)$$

when $\delta(x) \in F$, we call x is Skorohod integrable, $\delta(x)$ is Skorohod integral operator on F and

$$Dom\delta \doteq \{x \in L^2(\Gamma \times \mathbb{R}_+, \eta) : \delta(x) \in F\}$$

For a map $f : \Gamma \to \eta$, let ∇f and Df be the maps $\Gamma \times \mathbb{R}_+ \to \eta$ given by

$$\nabla f(\omega, s) = f(\omega \cup s), \quad Df(\omega, s) = 1_{\{\omega<s\}} f(\omega \cup s)$$

when $f \in F$, we call ∇f and Df the stochastic gradient of f and the adapted gradient of f, respectively. Moreover,

$$Dom\nabla \doteq \{f \in F : \nabla f \in L^2(\Gamma \times \mathbb{R}_+, \eta)\}$$

$$DomD \doteq \{f \in F : Df \in L^2(\Gamma_s \times \mathbb{R}_+, \eta)\}$$

where $\Gamma_s = \{\omega \in \Gamma : \omega \subset [0,s)\}$. Obviously, if $f \in F_s$, $\nabla f = Df$, where $F_s = L^2(\Gamma_s, \eta)$.

Let $\Gamma_{ad} = \{(\omega, s) \in \Gamma \times \mathbb{R}_+ : \omega < s\}$, the adapted projection on $L^2(\Gamma \times \mathbb{R}_+, \eta)$ is the orthogonal projection onto the closed subspace $L^2(\Gamma_{ad} \times \mathbb{R}_+, \eta)$:

$$P_{ad} : (\omega, s) \mapsto 1_{\{\omega<s\}} x_s(\omega)$$

Remark 2.1 As Hilbert space operators δ, ∇ and D are unbounded operators. $(\delta, Dom\delta)$ and $(\nabla, Dom\nabla)$ are closed, densely defined operators. Especially, δ is adjoint operator of ∇ and

$$Dom\delta \supset Dom\sqrt{N\otimes I}; Dom\nabla = Dom\sqrt{N}$$

where N is the number operator, $Nf(\sigma) = \#\sigma f(\sigma)$ with maximal domain and I is identical operator.

Lemma 2.1 [1] Let $f \in F$ and $x:\Gamma\times\mathbb{R}_+ \to \eta$ be Skorohod integrable, if the map

$$(\omega, s) \mapsto \langle x_s(\omega), f(\omega \cup s)\rangle$$

is integrable, then

$$\langle \delta(x), f\rangle = \iint \langle x_s(\omega), \nabla_s f(\omega)\rangle d\omega ds . \quad (1)$$

Lemma 2.2 [1] Let $x:\Gamma\times\mathbb{R}_+ \to \eta$ be measurable. If $P_t x_t \in F$ for almost every $t \geq 0$, then

$$D_t\delta(x) = \delta_0^t(\nabla_t x_.) + P_t x_t , \quad (2)$$

where (1) may call the canonical-commutation relations.

3. Local Skorohod Integral and Stochastic Gradient Operators

In the present section we state and prove our main results. We first make some preparations.

Let C be an operator on F with domain V, we define an conditioned expectation operator $E_s[C]$ on F by the a.e. prescription

$$(E_s[C]f)(\omega) = (CP_s D_{\omega_{(s}} f)(\omega_{s)}) ,$$

with domain

$$\{ f \in \mathrm{D}_s[V] : \sigma \mapsto 1_{\{\sigma > s\}} P_s C P_s D_\sigma f \ is\ square\ integral\ \Gamma \to F\}$$

where $\mathrm{D}_s[V] \doteq \{ f \in F : P_s D_\sigma f \in V(\subset F)\ for\ a.a.\ \sigma > s\}$, $\omega_{s)} = \omega \cap [0, s)$, $\omega_{(s} = \omega \cap (s, \infty)$.

Clearly, $\mathrm{D}_s[V]$ is a subspce of F, and for any $f \in \mathrm{D}_s[V]$, we have $P_s f \in \mathrm{D}_s[V], D_t f \in \mathrm{D}_s[V]$ for a.a. $t > s$. Thus $\mathrm{D}_s[V]$ is an s-adapted subspace.

Remark 3.1 If C is s-adapted(i.e. for all $f \in DomC$, $P_s f = P_s C P_s f, D_t Cf = CD_t f$ for a.a. $t > s$), then the subspaces $F_s \cap DomC$ and $F_s \cap DomE_s[C]$ coincide, and $E_s[C]g = Cg$ for g in this subspace, it follows that

$$E_s[C]P_s f = CP_s f, E_s[C]D_s f = CD_s f$$

Whenever $P_s f$ belongs to $DomC$. If C is densely defined, s-adapted and $E_s[C]$ is closable, then $E_s[C^*] = E_s[C]^*$.

Remark 3.2 $E_s[C]$ is s-adapted operator and $E_s[C] \in F$.

Definition 3.1 For $s \in \mathbb{R}_+$, we call $\ell_s = \nabla_s E_s[C]$ the local stochastic gradient operator and its adjoint operator $\ell_s^* = \delta E_s[C]$ is the local Skorohod integral operator. And operator domain of ℓ_s is given by

$$Dom\ell_s \doteq Dom\nabla_s \cap DomE_s[C]$$

where C is operator on F.

We note that for $f \in Dom\ell_s$,

$$(\nabla_s E_s[C]\ f)(\omega) = \nabla_s(E_s[C]\ f(\omega)) = CP_s D_{\omega_{(s}} f(\omega_{s)} \cup s)$$

$$(E_s[C]\nabla_s\ f)(\omega) = E_s[C]\,(\nabla_s f(\omega)) = CP_s D_{\omega_{(s}} f(\omega_{s)} \cup s)$$

hence, $\nabla_s E_s[C] = E_s[C]\nabla_s$. Especially, when $C = P_s$, $f(\omega) = P_s P_s D_{\omega_{(s}} f(\omega_{s)})$, we have $\ell_s = \nabla_s$.

Theorem 3.1 By lemma2.2, we can get the following relations

$$\ell_s \ell_s^* = \ell_s^* \ell_s + E_s[C] \quad (3)$$

which we may call the local CAR(canonical anti-commutation relations).

Proof we note that

$$\ell_s\ell_s^* = \nabla_s E_s[C]\,\delta E_s[C] = E_s[C]\nabla_s \delta E_s[C]$$
$$= E_s[C](\delta(\nabla_s E_s[C]) + E_s[C]) = \ell_s^*\ell_s + E_s[C]$$

The next theorem shows that ℓ_s is not a projection operator on F.

Theorem 3.2 $\ell_t\ell_s = 0$, whenever $t \geq s$ and $t,s > 0$.

Proof Let $t,s > 0$ with $t \geq s$. The following algebraic relations are evident for $t \geq s$,

$$P_sP_t f = P_tP_s f = P_s f,$$
$$D_tD_s f = D_tP_s f = 0,$$
$$D_sP_t f = P_tD_s f = D_s f.$$

We show that $E_t[C]\,\nabla_s E_s[C] = \nabla_s E_s[C]$, thus

$$\ell_t\ell_s = \nabla_t E_t[C]\,\nabla_s E_s[C] = \nabla_t(E_t[C]\,\nabla_s E_s[C]) = 0.$$

We note that $\nabla_s\nabla_t^* \neq 0$ for $t,s > 0$ with $t \neq s$, which means that the $(\nabla_s^*)_{s\geq 0}$ is not mutually orthogonal. However, the theorem below shows that the local operator sequence $(\ell_s^*)_{s\geq 0}$ is mutually orthogonal.

Theorem 3.3 $\ell_t\ell_s^* = 0$,whenever $t \neq s$ and $t,s \geq 0$.

Proof Let $t \neq s$ and $t,s \geq 0$. If $t > s$, then we can show that $\nabla_t E_s[C] = 0$, from which it follows that

$$\ell_t\ell_s^* = \nabla_t E_t[C]\,E_s[C]\delta = (\nabla E_s[C])\delta = 0.$$

Now, if $t < s$, then by the result of the first step we have

$$\ell_t\ell_s^* = (\ell_s\ell_t^*)^* = 0$$

This completes the proof.

Theorem 3.4 ℓ_s is s-adapted operator if and only if ∇_s is s-adapted operator.

Proof we know that $\ell_s = \nabla_s E_s[C]$ and $E_s[C]$ is s-adapted operator. we have

$$P_s\ell_s f = P_s\nabla_s E_s[C]\,f,$$

$D_t\ell_s f = D_t\nabla_s E_s[C]\,f$, for a.a. $t \geq s$, obviously, if ∇_s is s-adapted, then $P_s\nabla_s E_s[C]\,f = \nabla_s E_s[C]\,P_s f$ and $D_t\nabla_s E_s[C]\,f = \nabla_s E_s[C]\,D_t f$ for a.a. $t \geq s$. On the other hand, if ℓ_s is s-adapted, ∇_s is also s-adapted.

4. Application to Exponential Vector Formulation of QS Calculus

Recall that in the exponential vector formulation of QS calculus, all processes are defined on a domain of the algebraic tensor product form $V_0 \odot \mathscr{E}(S)$, where V_0 is a dense subspace of η and $\mathscr{E}(S) \doteq Lin\{\varepsilon(\varphi) : \varphi \in S\}$ which S is a subset of $L^2(\mathbb{R}_+)$ and $\varepsilon(\varphi)$ denotes the expential vector of the test function φ which in Guichardet spaces given by $\varepsilon(\varphi) = \prod_{s\in\sigma}\varphi(s)$.

For all s and a.a. t, we have

$$P_s v\varepsilon(\varphi) = v\varepsilon(\varphi_{[0,s)}), \text{ and } D_t v\varepsilon(\varphi) = \varphi(t) v\varepsilon(\varphi_{[0,t)}), \tag{4}$$

since, the domain of the form $V_0 \odot \mathscr{E}(S)$ are s-adapted. Note the a.e. identity

$$D_\tau v\varepsilon(\varphi) = v\varepsilon(\varphi_{[0,t)})\varepsilon(\varphi_{[t,\infty)})(\tau), \text{ where } t = \wedge\tau. \tag{5}$$

Theorem 4.1 ℓ_s be an operator on F with domain of the form $V_0 \odot \mathscr{E}(S)$. Then ℓ_s is s-adapted if and only if, for all $v \in V_0$ and $\varphi \in S$:

$$\ell_s v\varepsilon(\varphi_{[0,s)}) \in F_s;\ \ell_s v\varepsilon(\varphi) = \ell_s v\varepsilon(\varphi_{[0,s)}) \otimes \varepsilon(\varphi_{[s,\infty)}), \tag{6}$$

where $F_s = \eta \otimes L^2(\Gamma_s)$.

Proof By definition of ℓ_s, ℓ_s be an operator on F with domain of the form $V_0 \odot \mathscr{E}(S)$. We note that if ℓ_s is s-adapted, then $\ell_s v\varepsilon(\varphi_{[0,s)}) \in F_s$. Let $v \in V_0$ and $\varphi \in S$, by (4), for a.a. $\omega > s$,

$$P_s D_\omega \nu\varepsilon(\varphi) = \nu\varepsilon(\varphi_{[0,s)})\varepsilon(\varphi_{[s,\infty)})(\omega),$$

and so, for a.a. ω,

$$\begin{aligned}(\ell_s \nu\varepsilon(\varphi))(\omega) &= \ell_s \nu\varepsilon(\varphi_{[0,s)})(\omega_{s)}) \otimes \varepsilon(\varphi_{[s,\infty)})(\omega_{(s}) \\ &= (\ell_s \nu\varepsilon(\varphi_{[0,s)})) \otimes (\varepsilon(\varphi_{[s,\infty)}))(\omega).\end{aligned}$$

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