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Hopf Bifurcation Analysis for a Modified Time-Delay Predator-Prey System with Harvesting

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Abstract

In this paper, we consider the direction and stability of time-delay induced Hopf bifurcation in a delayed predator-prey system with harvesting. We show that the positive equilibrium point is asymptotically stable in the absence of time delay, but loses its stability via the Hopf bifurcation when the time delay increases beyond a threshold. Furthermore, using the norm form and the center manifold theory, we investigate the stability and direction of the Hopf bifurcation.

Keywords

Hopf Bifurcation, Time-Delay, Predator-Prey Model

1. Introduction

Due to its universal existence and importance, the study on the dynamics of predator-prey systems is one of the dominant subjects in ecology and mathematical ecology since Lotka [1] and Volterra [2] proposed the well-known predator-prey model [3]-[6]. Recently, a new method of central manifold has been developed to study the stability of delay induced bifurcation. In this paper, we study the following system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = rx\left[1-\dfrac{x(t-\tau)}{K}\right]-\dfrac{axy}{\beta+x}, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -(E+\delta y)+\dfrac{bx(t-\tau)y}{\beta+x(t-\tau)}, \end{cases} \tag{1}$$

with

$$x(0)=x_0 \geq 0, y(0)=y_0 \geq 0, \delta < b, \tag{2}$$

where dot means differentiation with respect to time t, $x(t)$ and $y(t)$ are the prey and predator population densities, respectively. Parameter $r>0$ is the specific growth rate of prey in the absence of predation and without environment limitation. K is environmental carrying capacity. The functional response of the predator

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is of Holling's type with $a, b, \beta > 0$. And all parameters involved with the model are positive.

The purpose of this paper is to investigate the effect of time-delay on a modified predator-prey model with harvesting. We discussed the existence of Hopf bifurcation of system (1) and the direction of Hopf bifurcation and the stability of bifurcated periodic solutions are given.

2. Positive Equilibrium and Locally Asymptotically Stabiliy

After some calculations, we note system (1) has no boundary equilibria. However, it is more interesting to study the dynamical behaviors of the interior equilibrium points $E_1^*(x_1^*, y_1^*)$ and $E_2^*(x_2^*, y_2^*)$, where

$$x_1^* = \frac{r\left[K(b-\delta)+\delta\beta\right]+\Delta}{2r(b-\delta)}, x_2^* = \frac{r\left[K(b-\delta)+\delta\beta\right]-\Delta}{2r(b-\delta)}, \Delta = \sqrt{r^2\left[K(b-\delta)-\delta\beta\right]^2 - 4(b-\delta)KaE},$$

$$y_1^* = \frac{brx_1^*(K-x_1^*)-KaE}{Ka\delta}, y_2^* = \frac{brx_2^*(K-x_2^*)-KaE}{Ka\delta}.$$

The two distinct interior equilibrium points E_1^*, E_2^* exist whenever

$$(H1) \quad 0 < E \le \frac{r^2\left[K(b-\delta)-\delta\beta\right]^2}{4(b-\delta)Ka}, \delta\beta < K(b-\delta)$$

holds.

We transform the interior equilibrium $E^*(x^*, y^*)$ to the origin by the transformation $\bar{x} = x - x^*$, $\bar{y} = y - y^*$. Respectively, we still denote $\bar{x}$ and $\bar{y}$ by x and y. Thus, system (1) is transformed into

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = r(x+x^*)\left[1-\dfrac{x^*}{K}-\dfrac{x(t-\tau)}{K}\right] - \dfrac{a(x+x^*)(y+y^*)}{\beta+x+x^*}, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -(E+\delta y+\delta y^*) + \dfrac{b\left[x(t-\tau)+x^*\right](y+y^*)}{\beta+x(t-\tau)+x^*}. \end{cases} \tag{3}$$

First, we give the condition such that $E^*(x^*, y^*)$ is locally stable. For simplicity, we denote

$$G(\lambda) = \begin{vmatrix} \lambda + \left[-r\left(1-\dfrac{x^*}{K}\right)\right] + \dfrac{rx^*}{K}e^{-\lambda\tau} + \dfrac{a\beta y^*}{\left(\beta+x^*\right)^2} & -\dfrac{ax^*}{\beta+x^*} \\ \dfrac{b\beta e^{-\lambda\tau} y^*}{\left(\beta+x^*\right)^2} & \lambda+\delta-\dfrac{bx^*}{\beta+x^*} \end{vmatrix}. \tag{4}$$

The characteristic polynomial of $G(\lambda)$ is

$$\varphi(\lambda) = \lambda^2 + b_1\lambda + b_2 + \left(b_3\lambda + b_4\right)e^{-\lambda\tau}, \tag{5}$$

where

$$b_1 = \frac{rx^*}{K} + \frac{a\beta y^*}{\left(\beta+x^*\right)^2} + \delta - \frac{bx^*}{\beta+x^*} - r, \quad b_2 = \left(\frac{rx^*}{K} - r + \frac{a\beta y^*}{\left(\beta+x^*\right)^2}\right)\left(\delta - \frac{bx^*}{\beta+x^*}\right),$$

$$b_3 = \frac{rx^*}{K}, \qquad b_4 = \frac{rx^*}{K}\left(\delta - \frac{bx^*}{\beta+x^*}\right) + \frac{ab\beta x^* y^*}{\left(\beta+x^*\right)^3}.$$

Now we consider the locally asymptotically stabiliy of the system without time-delay. Then we have

$$\lambda^2 + (b_1+b_3)\lambda + b_2 + b_4 = 0. \tag{6}$$

If

$$(H2) \quad b_1 + b_3 > 0, b_2 + b_4 > 0$$

holds, then it follows from the Routh-Hurwitz criterion that two roots of (6) have negative real parts.

Theorem 1. If $(H1)$ and $(H2)$ hold, the interior equilibrium point E^* of system (1) is locally asymptotically stable.

3. Hopf Bifurcaion

In the section, we study whether there exists periodic solutions of system (1) about the interior equilibrium point E^*. Now we have the following results.

Theorem 2. If the system (1) satisfies the hypothesis $(H1)$ $(H2)$ and $(H3)\, bx^* - \delta x^* < \delta\beta < 2bx^* - \delta x^*$ holds, then there exists a critical point τ^* such that the positive equilibrium point E^* is locally asymptotically stable for $\tau \in [0, \tau^*]$ and unstable for $\tau \in (\tau^*, \infty)$, where τ^* is defined in Equation (14).

By the use of the instability result for the delayed differential Equations, in order to prove the instability of the equilibrium point, it is sufficient to show that there exists a purely imaginary $i\omega$ and a positive real τ such that

$$\varphi(\lambda, \tau) = 0 \tag{7}$$

where $\varphi(\lambda, \tau)$ is defined in Equation (5).

If $i\omega$ is a root of Equation (7), then we have

$$\begin{cases} b_3\omega \sin\omega\tau + b_4 \cos\omega\tau = \omega^2 - b_2, \\ b_3\omega\cos\omega\tau - b_4 \sin\omega\tau = -b_1\omega, \end{cases} \tag{8}$$

which leads to

$$R(\omega) = \omega^4 - \left(2b_2 + b_3{}^2 - b_1{}^2\right)\omega^2 + b_2{}^2 - b_4{}^2 = 0. \tag{9}$$

Let $z = \omega^2$, then Equation (9) takes the form

$$z^2 - \left(2b_2 + b_3{}^2 - b_1{}^2\right)z + b_2{}^2 - b_4{}^2 = 0. \tag{10}$$

Since $(H3)$ holds, we have $b_2 - b_4 < 0$, which leads to $b_2{}^2 - b_4{}^2 < 0$. Thus Equation (10) has at least one positive root, which leads to

$$\omega_j = \sqrt{\frac{2b_2 + b_3^2 - b_1^2 + \sqrt{(2b_2 + b_3^2 - b_1^2)^2 - 4(b_2^2 - b_4^2)}}{2}}. \tag{11}$$

Set $\dfrac{\arccos(h(\omega_j))}{\omega_j}$ as the root of Equation (8) with $\omega = \omega_j$, we have

$$\tau_j^{(n)} = \frac{\arccos(h(\omega_j))}{\omega_j} + \frac{2n\pi}{\omega_j^+}, n = 0, 1, \ldots \tag{12}$$

where

$$h(\omega_j) = \frac{b_4\omega_j{}^2 - b_2b_4 - b_1b_3\omega_j{}^2}{b_3{}^2\omega_j{}^2 + b_4{}^2}. \tag{13}$$

Then $\pm i\omega_j$ are a pair of simple purely imaginary roots of Equation (8) with $\tau = \tau_j^{(n)}$, and we have

$$\tau^* = \tau_{j0}^{(0)} = \min_{j=1,2}\{\tau_j^{(0)}\}, \omega^* = \omega_{j0}. \tag{14}$$

Then by the Butler's Lemma, E^* is unstable for $\tau > \tau^*$. On the other hand, if $\tau \in [0, \tau^*)$, then Equation (7) have no roots on the imaginary axis. Then Equation (7) for $\tau \in [0, \tau^*)$, only has negative real part roots, which implies that E^* is locally asymptotically stable for $\tau < \tau^*$.

Theorem 3. If the system (1) satisfies the hypothesis $(H1)$ $(H2)$ and $(H3)$, then the system (1) undergoes Hopf bifurcation at E^* when $\tau = \tau^*$.

Proof. The Hopf bifurcation will be proved if we can show that

$$\operatorname{sgn}\left[\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\right]_{\tau=\tau^*} > 0. \tag{15}$$

From Equation (7), we have

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{e^{\lambda\tau}}{\lambda}\frac{b_3\lambda^2 + 2b_4\lambda + b_1b_4 - b_2b_3}{(b_3\lambda + b_4)^2} - \frac{\tau}{\lambda}. \tag{16}$$

Substituting Equation (8) into Equation (16), we have

$$\begin{aligned}&\operatorname{sgn}\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau^*} = \operatorname{sgn}\operatorname{Re}\left(\frac{e^{\lambda\tau}}{\lambda}\frac{b_3\lambda^2 + 2b_4\lambda + b_1b_4 - b_2b_3}{(b_3\lambda + b_4)^2} - \frac{\tau}{\lambda}\right)_{\tau=\tau^*}\\ &= \operatorname{sgn}\left\{b_3^4\omega^{*6} + 3b_3^2b_4^2\omega^{*4} + \left[2(b_4^2 - b_2b_3^2)b_4^2 + (b_1^2b_4^2 - b_2^2b_3^2)b_3^2\right]\omega^{*2} + \left[(b_1^2b_4^2 - b_2^2b_3^2) - 2b_2b_4^2\right]b_4^2\right\}.\end{aligned} \tag{17}$$

Substituting Equation (14) into the above equation, we have

$$\operatorname{sgn}\left[\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\right]_{\tau=\tau^*} > 0.$$

Therefore, the transversality condition is satisfied. Therefore Hopf bifurcation occurs at $\tau = \tau^*$.

4. The Direction and Stability of the Hopf Bifurcation

In this section, we analyze the direction and stability of the Hopf bifurcation of (3) obtained in Theorem 3 by taking τ as the bifurcation parameter.

Let $w_1 = x, w_2 = y, \gamma = \tau - \tau^*$, then $\gamma = 0$ is the Hopf bifurcation value of system (3). Rescale the time by $t = t/\tau$ to normalize the delay. The periodic solution of system (3) is equivalent to the solution of the following system

$$\begin{cases}\dfrac{dw_1}{dt} = (\gamma + \tau^*)[r(w_1 + x^*)(1 - \dfrac{x^*}{K} - \dfrac{w_1(t-1)}{K}) - \dfrac{a(w_1 + x^*)(w_2 + y^*)}{\beta + w_1 + x^*}]\\ \dfrac{dw_2}{dt} = (\gamma + \tau^*)[-(E + \delta w_2 + \delta y^*) + \dfrac{b(w_1(t-1) + x^*)(w_2 + y^*)}{\beta + w_1(t-1) + x^*}]\end{cases}. \tag{18}$$

We define i, j, l as nonnegative integer, define $f^{(1)}(w_1, w_2, h), f^{(2)}(w_2, h), f_{ijl}^{(1)}(i + j + l \ge 1), f_{jl}^{(2)}(j + l \ge 1)$ as follows

$$\begin{aligned}&f^{(1)}(w_1, w_2, h) = r(w_1 + x^*)(1 - \frac{x^*}{K} - \frac{h}{K}) - \frac{a(w_1 + x^*)(w_2 + y^*)}{\beta + w_1 + x^*}, \quad f_{ijl}^{(1)} = \frac{\partial^{i+j+l} f^{(1)}(0,0,0)}{\partial w_1^{\,i} w_2^{\,j} h^l},\\ &f^{(2)}(w_2, h) = -(E + \delta w_2 + \delta y^*) + \frac{b(h + x^*)(w_2 + y^*)}{\beta + h + x^*}, \qquad f_{jl}^{(2)} = \frac{\partial^{j+l} f^{(2)}(0,0)}{\partial w_2^{\,j} h^l}.\end{aligned}$$

Rewrite system (18) to

$$\begin{cases}\dfrac{dw_1}{dt} = (\gamma + \tau^*)[\, f_{100}^{(1)} w_1 + f_{010}^{(1)} w_2 + f_{001}^{(1)} w_1(t-1) + \displaystyle\sum_{i+j+l\ge 2} \frac{1}{i!\,j!\,l!} w_1^{\,i} w_2^{\,j} w_1^{\,l}(t-1)],\\ \dfrac{dw_2}{dt} = (\gamma + \tau^*)[\, f_{01}^{(2)} w_2 + f_{10}^{(2)} w_1(t-1) + \displaystyle\sum_{j+l\ge 2} \frac{1}{j!\,l!} w_2^{\,j} w_1^{\,l}(t-1)],\end{cases} \tag{19}$$

where

$$f_{001}^{(1)}=-\frac{rx^*}{K},\ f_{010}^{(1)}=-\frac{rx^*}{\beta+x^*},\ f_{101}^{(1)}=-\frac{r}{k},\ f_{100}^{(1)}=r\left(1-\frac{x^*}{K}\right)-\frac{a\beta y^*}{\left(\beta+x^*\right)^2},\ f_{03}^{(2)}=\frac{6b\beta y^*}{\left(\beta+x^*\right)^4},$$

$$f_{110}^{(1)}=-\frac{a\beta}{\left(\beta+x^*\right)^2},\ f_{210}^{(1)}=\frac{2a\beta}{\left(\beta+x^*\right)^3},\ f_{200}^{(1)}=\frac{2a\beta y^*}{\left(\beta+x^*\right)^3},\ f_{300}^{(1)}=-\frac{6a\beta y^*}{\left(\beta+x^*\right)^4},\ f_{01}^{(2)}=\frac{b\beta y^*}{\left(\beta+x^*\right)^2},\tag{20}$$

$$f_{10}^{(2)}=-\delta+\frac{bx^*}{\beta+x^*},\ f_{11}^{(2)}=\frac{b\beta}{\left(\beta+x^*\right)^2},\ f_{02}^{(2)}=\frac{-2b\beta y^*}{\left(\beta+x^*\right)^3},\ f_{12}^{(2)}=\frac{-2b\beta}{\left(\beta+x^*\right)^3}.$$

We use the method which is based on the center manifold and normal form theory, and define $\mathbf{C}=C([0,1],\mathbb{R}^2)$. Then the system (19) is transformed into a functional differential equation as

$$\dot{w}(t)=L_\gamma(w_t)+f(\gamma,w_t),\tag{21}$$

where $w(t)=(w_1(t),w_2(t))^T\in\mathbb{R}^2$ and $L_\gamma:\mathbf{C}\to\mathbb{R}^2, f:\mathbb{R}\times\mathbf{C}\to\mathbb{R}^2$ are respectively represented by

$$L_\gamma(\phi)=(\tau^*+\gamma)\begin{pmatrix} f_{100}^{(1)} & f_{010}^{(1)} \\ 0 & f_{10}^{(2)}\end{pmatrix}\begin{pmatrix}\phi_1(0)\\ \phi_2(0)\end{pmatrix}+(\tau^*+\gamma)\begin{pmatrix} f_{001}^{(1)} & 0 \\ f_{01}^{(2)} & 0\end{pmatrix}\begin{pmatrix}\phi_1(-1)\\ \phi_2(-1)\end{pmatrix},\tag{22}$$

and

$$f(\gamma,\phi)=(\tau^*+\gamma)\begin{pmatrix}\sum\limits_{i+j+l\ge 2}\frac{1}{i!j!l!}f_{ijl}^{(1)}\phi_1^i(0)\phi_2^j(0)\phi_1^l(-1)\\ \sum\limits_{j+l\ge 2}\frac{1}{j!l!}f_{jl}^{(2)}\phi_2^j(0)\phi_1^l(-1)\end{pmatrix},\tag{23}$$

where $\phi(\theta)=(\phi_1(\theta),\phi_2(\theta))\in\mathbf{C}$. By the Riesz representation theorem, there exist a 2×2 matrix $\eta(\theta,\gamma)$, whose elements are of bounded variation functions such that

$$L_\gamma\phi=\int_{-1}^{0}[d\eta(\theta,\gamma)]\phi(\theta),\ for\ \phi\in\mathbf{C}.\tag{24}$$

In fact, we can choose

$$\eta(\theta,\gamma)=(\tau^*+\gamma)\begin{pmatrix} f_{100}^{(1)} & f_{010}^{(1)} \\ 0 & f_{10}^{(2)}\end{pmatrix}\delta(\theta)+(\tau^*+\gamma)\begin{pmatrix} f_{001}^{(1)} & 0 \\ f_{01}^{(2)} & 0\end{pmatrix}\delta(\theta+1),\tag{25}$$

where δ is the Dirac delta function. For $\phi\in C^1([-1,0],\mathbb{R}^2)$, we define

$$A(\gamma)\phi=\begin{cases}\frac{d\phi(\theta)}{d\theta}, & for\ \theta\in[-1,0),\\ \int_{-1}^{0}[d\eta(s,\gamma)]\phi(s), & for\ \theta=0,\end{cases}\tag{26}$$

and

$$R(\gamma)\phi=\begin{cases}0, & for\ \theta\in[-1,0),\\ f(\gamma,\phi) & for\ \theta=0.\end{cases}\tag{27}$$

Thus system (21) is equivalent to

$$\dot{w}(t)=A(\gamma)(w_t)+R(\gamma)(w_t),\tag{28}$$

where $w_t(\theta)=w(t+\theta)$ for $\theta\in[-1,0]$.

For $\psi\in C^1([-1,0],(\mathbb{R}^2)^*)$, define

$$A^*\psi(s)=\begin{cases}-\frac{d\psi(s)}{ds}, & for\ s\in[-1,0),\\ \int_{-1}^{0}\psi(-t)d\eta^T(t,0), & for\ s=0,\end{cases}\tag{29}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{0}^{\theta} \bar{\psi}(\xi-\theta) d\eta(\theta)\phi(\xi) d\xi, \tag{30}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A(0)$ and A^* are adjoint operators. From the discussion in Theorem 2, we know that $\pm i\omega^*\tau^*$ are eigenvalues of $A(0)$ and therefore they are also eigenvalues of A^*.

Suppose $q(\theta) = (q_1, q_2)^T e^{i\omega^*\tau^*\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega^*\tau^*$. Thus, $A(0)q(\theta) = i\omega^*\tau^* q(\theta)$. From the definition of $A(0)$ we have

$$\begin{pmatrix} f_{100}^{(1)} + f_{001}^{(1)} e^{-i\omega^*\tau^*} - i\omega^* & f_{010}^{(1)} \\ f_{01}^{(2)} e^{-i\omega^*\tau^*} & f_{10}^{(2)} - i\omega^* \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0.$$

Then we have

$$q(\theta) = (1, -\frac{f_{100}^{(1)} + f_{001}^{(1)} e^{-i\omega^*\tau^*} - i\omega^*}{f_{010}^{(1)}})^T e^{i\omega^*\tau^*\theta}, \tag{31}$$

Similarly, let $q^*(s) = M(q_1^*, q_2^*) e^{i\omega^*\tau^* s}$ be the eigenvector of A^* corresponding to $-i\omega^*\tau^*$. Then by $A^* q^*(s) = -i\omega^*\tau^* q^*(s)$ and the definition of A^*, we obtain

$$\begin{pmatrix} f_{100}^{(1)} + i\omega^* + f_{001}^{(1)} e^{-i\omega^*\tau^*} & f_{01}^{(2)} e^{-i\omega^*\tau^*} \\ f_{010}^{(1)} & f_{10}^{(2)} + i\omega^* \end{pmatrix} \begin{pmatrix} q_1^* \\ q_2^* \end{pmatrix} = 0.$$

Therefore

$$q^*(s) = M(1, -\frac{f_{010}^{(1)}}{f_{10}^{(2)} + i\omega^*}) e^{i\omega^*\tau^* s}, \tag{32}$$

In order to ensure, we need to determine the value of M, from Equation (29) we have

$$\begin{aligned} \langle q^*(s), q(\theta) \rangle &= \bar{q}^*(0)q(0) - \int_{-1}^{0}\int_{0}^{\theta} \bar{q}^*(\xi-\theta) d\eta(\theta) q(\xi) d\xi \\ &= \bar{M}\bar{q}_1^*(q_1 + \tau^* e^{-i\omega^*\tau^*}(f_{001}^{(1)} q_1 + 0q_2)) + \bar{M}\bar{q}_2^*(q_2 + \tau^* e^{-i\omega^*\tau^*}(f_{01}^{(2)} q_1 + 0q_2)) \\ &= \bar{M}(\bar{q}_1^* q_1 + \bar{q}_2^* q_2 + q_1\bar{q}_1^* f_{001}^{(1)} \tau^* e^{-i\omega^*\tau^*} + q_1\bar{q}_2^* f_{01}^{(2)} \tau^* e^{-i\omega^*\tau^*}). \end{aligned} \tag{33}$$

Then we can choose M such as

$$\bar{M} = \frac{1}{\bar{q}_1^* q_1 + \bar{q}_2^* q_2 + (\bar{q}_1^* f_{001}^{(1)} + \bar{q}_2^* f_{01}^{(2)}) q_1 \tau^* e^{-i\omega^*\tau^*}}, \tag{34}$$

where $\bar{M}$ is the conjugate complex number of M.

Next we will compute the coordinate to describe the center manifold C_0 at $\gamma = 0$. Let w_t be the solution of Equation (27) when $\gamma = 0$. Define

$$z(t) = \langle q^*, w_t \rangle, \quad W(t, \theta) = w_t(\theta) - 2\operatorname{Re}\{z(t)q(\theta)\}. \tag{35}$$

On the center manifold C_0, we have $W(t, \theta) = W(z(t), \bar{z}(t), \theta)$, where

$$W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta) z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots, \tag{36}$$

z and $\bar{z}$ are local coordinates for the center manifold C_0 in the direction of q and $\overline{q^*}$. Note that W is real if w_t is real. We only concern with the real solutions. For solution $w_t \in C_0$ of Equation (27), since $\gamma = 0$ and Equation (35), we have

$$\dot{z}(t) = \langle \bar{q}^*, \dot{w}_t \rangle = i\omega^*\tau^* z + \overline{q^*}(0) f(0, W(z, \bar{z}, \theta) + 2\operatorname{Re}\{zq(\theta)\}) \triangleq i\omega^*\tau^* z + \overline{q^*}(0) f(z, \bar{z}). \tag{37}$$

We rewrite above equation as

$$\dot{z}(t) = i\omega^*\tau^* z + g(z,\overline{z}), \tag{38}$$

where

$$g(z,\overline{z}) = \overline{q^*}(0) f_0(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{\overline{z}^2\overline{z}}{2} + \cdots. \tag{39}$$

From Equation (35) and Equation (36), we obtain that

$$\begin{aligned}\omega_t(\theta) &= W(t,\theta) + 2\operatorname{Re}\{z(t)q(\theta)\} = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + zq + \overline{zq} + \cdots \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + (q_1,q_2)^T e^{i\omega^*\tau^*\theta}z \quad + (\overline{q}_1,\overline{q}_2)^T e^{-i\omega^*\tau^*\theta}\overline{z} + \cdots.\end{aligned} \tag{40}$$

Substituting Equation (23) and Equation (40) into Equation (39), we have

$$\begin{aligned} g(z,\overline{z}) &= \overline{q^*}(0) f_0(z,\overline{z}) = \overline{q^*}(0) f_0(z,\omega_t) = \overline{M}\tau^*\left(\overline{q}_1^*,\overline{q}_2^*\right)\begin{pmatrix}\sum_{i+j+l\ge 2}\frac{1}{i!j!l!}f_{ijl}^{(1)}w_1^{\,i}w_2^{\,j}w_1^{\,l}(t-1) \\ \sum_{j+l\ge 2}\frac{1}{j!l!}f_{jl}^{(2)}w_2^{\,j}w_1^{\,l}(t-1)\end{pmatrix} \\ &= p_1z^2 + p_2z\overline{z} + p_3\overline{z}^2 + p_4z^2\overline{z} + h.o.t.,\end{aligned} \tag{41}$$

where $h.o.t.$ stands for higher order terms, and

$$P_1 = \overline{M}\tau^*[(f_{110}^{(1)}q_2 + f_{101}^{(1)}e^{-i\omega^*\tau^*} + \frac{1}{2}f_{200}^{(1)}) + \overline{q}_2^*(f_{11}^{(2)}q_2e^{-i\omega^*\tau^*} + \frac{1}{2}f_{02}^{(2)}e^{-2i\omega^*\tau^*})],$$

$$P_2 = \overline{M}\tau^*[f_{110}^{(1)}(q_2+\overline{q}_2) + f_{101}^{(1)}(e^{-i\omega^*\tau^*} + e^{i\omega^*\tau^*}) + f_{200}^{(1)}] + \overline{M}\tau^*\overline{q}_2^*[f_{11}^{(2)}(\overline{q}_2e^{-i\omega^*\tau^*} + q_2e^{i\omega^*\tau^*}) + f_{02}^{(2)}],$$

$$P_3 = \overline{M}\tau^*(f_{110}^{(1)}\overline{q}_2 + f_{101}^{(1)}e^{i\omega^*\tau^*} + \frac{1}{2}f_{200}^{(1)}) + \overline{M}\tau^*\overline{q}_2^*(f_{11}^{(2)}\overline{q}_2e^{i\omega^*\tau^*} + \frac{1}{2}f_{02}^{(2)}e^{2i\omega^*\tau^*}),$$

$$\begin{aligned}P_4 = {}&\overline{M}\tau^*[f_{110}^{(1)}(\frac{1}{2}W_{20}^{(1)}(0)\overline{q}_2 + W_{11}^{(1)}(0)q_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{11}(\theta)) + \frac{1}{2}f_{200}^{(1)}(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)) + \frac{1}{2}f_{300}^{(1)} \\ &+ f_{101}^{(1)}(\frac{1}{2}W_{20}^{(1)}(0)e^{i\omega^*\tau^*} + W_{11}^{(1)}(0)e^{-i\omega^*\tau^*} + W_{11}^{(1)}(-1) + \frac{1}{2}W_{20}^{(1)}(-1)) + \frac{1}{2}f_{210}^{(1)}(\overline{q}_2 + 2q_2)] \\ &+ \overline{M}\tau^*\overline{q}_2^*[f_{11}^{(2)}(\frac{1}{2}W_{20}^{(2)}(0)e^{i\omega^*\tau^*} + W_{11}^{(2)}(0)e^{-i\omega^*\tau^*} + W_{11}^{(1)}(-1)q_2 + \frac{1}{2}W_{11}^{(1)}(-1)\overline{q}_2) \\ &+ \frac{1}{2}f_{02}^{(2)}(W_{20}^{(1)}(-1)e^{i\omega^*\tau^*} + 2W_{11}^{(1)}(-1)e^{-i\omega^*\tau^*}) + \frac{1}{2}f_{12}^{(2)}(\overline{q}_2e^{-2i\omega^*\tau^*} + 2q_2) + \frac{1}{2}f_{03}^{(2)}e^{-i\omega^*\tau^*}].\end{aligned}$$

Comparing Equation (39) and Equation (41), we get

$$g_{20} = 2p_1, g_{11} = p_2, g_{02} = 2p_3, g_{21} = 2p_4. \tag{42}$$

Since g_{21} depends on $W_{20}(\theta)$ and $W_{11}(\theta)$, we need to find the values of $W_{20}(\theta)$ and $W_{11}(\theta)$. From Equation (21) and Equation (35), we have

$$\dot{w} = \dot{w}_t - \dot{z}q - \dot{\overline{z}}\overline{q} = \begin{cases} AW - 2\operatorname{Re}\{\overline{q}^*(0)f_0q(\theta)\}, & -1 \le \theta < 0, \\ AW - 2\operatorname{Re}\{\overline{q}^*(0)f_0q(\theta)\} + f_0, & \theta = 0,\end{cases} \triangleq AW + H(z,\overline{z},\theta), \tag{43}$$

where

$$H(z,\overline{z},\theta) = H_{20}\frac{z^2}{2} + H_{11}z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots. \tag{44}$$

From Equation (36), we have

$$\begin{aligned}\dot{w} &= \dot{w}_z \dot{z}(t) + \dot{w}_{\bar{z}} \dot{\bar{z}}(t) \\ &= \left(W_{20}(\theta) z + W_{11}(\theta)\bar{z} + \cdots\right)\left(i\omega^*\tau^* z(t) + g(z,\bar{z})\right) + \left(W_{11}(\theta) z + W_{02}(\theta)\bar{z} + \cdots\right)\left(-i\omega^*\tau^*\bar{z}(t) + \bar{g}(z,\bar{z})\right).\end{aligned} \tag{45}$$

It follows from Equation (39) that

$$\begin{aligned}\dot{w} &= A(0)(W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta) z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots) + H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \\ &= \left(A(0)W_{20}(\theta) + H_{20}(\theta)\right)\frac{z^2}{2} + \left(A(0)W_{11}(\theta) + H_{11}(\theta)\right) z\bar{z} + \left(A(0)W_{02}(\theta) + H_{02}(\theta)\right)\frac{\bar{z}^2}{2} + \cdots.\end{aligned} \tag{46}$$

Comparing the coefficients of z^2 and $z\bar{z}$ from Equation (45) and Equation (46), we get

$$(A(0) - 2i\omega^*\tau^* I)W_{20}(\theta) = -H_{20}(\theta), \quad A(0)W_{11}(\theta) = -H_{11}(\theta). \tag{47}$$

Then for $\theta \in [0,1]$, we have

$$\begin{aligned}H(z,\bar{z},\theta) &= -\bar{q}^*(0) f_0 q(\theta) - q^*(0)\bar{f}_0 \bar{q}(\theta) = -g(z,\bar{z}) q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta) \\ &= -(g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots) q(\theta) - (\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11} z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \cdots)\bar{q}(\theta).\end{aligned} \tag{48}$$

Comparing the coefficients of z^2 and $z\bar{z}$ between Equation (44) and Equation (48), we get

$$H_{20}(\theta) = -g_{20} q(\theta) - \bar{g}_{02}\bar{q}(\theta), \quad H_{11}(\theta) = -g_{11} q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{49}$$

From the definition of $A(\theta)$ and Equation (49), we have

$$\dot{W}_{20}(\theta) = 2i\omega^*\tau^* W_{20}(\theta) + g_{20} q(\theta) + \bar{g}_{02}\bar{q}(\theta). \tag{50}$$

Since $q(\theta) = (q_1, q_2)^T e^{i\omega^*\tau^*\theta}$, we obtain

$$W_{20}(\theta) = \frac{ig_{20}}{\omega^*\tau^*} q(0) e^{i\omega^*\tau^*\theta} + \frac{i\bar{g}_{02}}{3\omega^*\tau^*}\bar{q}(0) e^{-i\omega^*\tau^*\theta} + E_1 e^{2i\omega^*\tau^*\theta}, \tag{51}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})^T$ is a constant vector. Similarly, we have

$$W_{11}(\theta) = \frac{-ig_{11}}{\omega^*\tau^*} q(0) e^{i\omega^*\tau^*\theta} + \frac{i\bar{g}_{11}}{\omega^*\tau^*}\bar{q}(0) e^{-i\omega^*\tau^*\theta} + E_2, \tag{52}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)})^T$ is a constant vector. Now, we shall find the values of E_1 and E_2. From the definition of $A(0)$ and Equation (50), we have

$$\int_{-1}^{0} d\eta(\theta) W_{20}(\theta) = 2i\omega^*\tau^* W_{20}(0) - H_{20}(0), \tag{53}$$

and

$$\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0), \tag{54}$$

where $\eta(\theta) = \eta(\theta, 0)$. In view of Equation (43), we induce that when $\theta = 0$.

$$H(z,\bar{z},0) - 2\,\mathrm{Re}\{\bar{q}^*(0) f_0 q(0)\} + f_0 = -g(z,\bar{z}) q(0) - \bar{g}(z,\bar{z})\bar{q}(0) + f_0. \tag{55}$$

Then we have

$$H_{20}\frac{z^2}{2} + H_{11} z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots = -(g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots) q(0) - (\bar{g}_{20}\frac{z^2}{2} + \bar{g}_{11} z\bar{z} + \bar{g}_{02}\frac{\bar{z}^2}{2} + \cdots)\bar{q}(0) + f_0. \tag{56}$$

Comparing both sides of Equation (56), we obtain

$$H_{20} = -g_{20} q(0) - \bar{g}_{20}\bar{q}(0) + 2\tau^*(H_1, H_2)^T, \; H_{11} = -g_{11} q(0) - \bar{g}_{11}\bar{q}(0) + \tau^*(P_1, P_2)^T, \tag{57}$$

where $H \triangleq (H_1, H_2)^T$ and $P \triangleq (P_1, P_2)^T$ are respectively the coefficients of z^2 and $z\bar{z}$ of $f_0(z,\bar{z})$. Thus we have

$$H=\begin{pmatrix} f_{110}^{(1)}q_2+f_{101}^{(1)}e^{-i\omega^*\tau^*}+\frac{1}{2}f_{200}^{(1)} \\ f_{11}^{(2)}q_2e^{-i\omega^*\tau^*}+\frac{1}{2}f_{02}^{(2)}e^{-2i\omega^*\tau^*} \end{pmatrix}, P=\begin{pmatrix} f_{110}^{(1)}(q_2+\bar{q}_2)+f_{101}^{(1)}(e^{-i\omega^*\tau^*}+e^{i\omega^*\tau^*})+f_{200}^{(1)} \\ f_{11}^{(2)}(\bar{q}_2e^{-i\omega^*\tau^*}+q_2e^{i\omega^*\tau^*})+f_{02}^{(2)} \end{pmatrix}, \tag{58}$$

where $\Phi=e^{-i\omega^*\tau^*}+e^{i\omega^*\tau^*}$.

Since $i\omega^*\tau^*$ is the eigenvalue of $A(0)$ and $q(0)$ is the corresponding eigenvector, we get

$$(i\omega^*\tau^*I-\int_{-1}^{0}e^{i\omega^*\tau^*\theta}d\eta(\theta))q(0)=0, \tag{59}$$

$$(-i\omega^*\tau^*I-\int_{-1}^{0}e^{-i\omega^*\tau^*\theta}d\eta(\theta))\bar{q}(0)=0. \tag{60}$$

Therefore, substituting Equation (53) and Equation (59) into Equation (60), we have

$$(2i\omega^*\tau^*I-\int_{-1}^{0}e^{2i\omega^*\tau^*\theta}d\eta(\theta))E_1=2\tau^*H, \tag{61}$$

that is

$$H^*E_1=2H, \tag{62}$$

where

$$H^*=\begin{pmatrix} 2i\omega^*-f_{100}^{(1)}-f_{001}^{(1)}e^{-2i\omega^*\tau^*} & -f_{01}^{(2)}e^{-2i\omega^*\tau^*} \\ -f_{010}^{(1)} & 2i\omega^*-f_{10}^{(2)} \end{pmatrix}. \tag{63}$$

Thus $E_1^{(i)}=\frac{2\Delta_i}{\Delta}$, $\Delta=Det(H^*)$, and Δ_i is the value of the determinant U_i, where U_i is formed by replacing the i th column vector of H^* by another column vector $(H_1,H_2)^T$ for $i=1,2$. In a similar way, we have

$$P^*E_2=2P, \tag{64}$$

where

$$P^*=\begin{pmatrix} -f_{100}^{(1)}-f_{001}^{(1)} & -f_{01}^{(2)} \\ -f_{010}^{(1)} & -f_{10}^{(2)} \end{pmatrix}. \tag{65}$$

Thus $E_2^{(i)}=\frac{2\tilde{\Delta}_i}{\tilde{\Delta}}$, where $\tilde{\Delta}=Det(P^*)$ and $\tilde{\Delta}_i$ is the value of the determinant V_i that is formed by replacing the i th column vector of P^* by another column vector $(P_1,P_2)^T$ for $i=1,2$. Therefore, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from Equation (51) and Equation (52). Furthermore, we can easily compute g_{21}.

Then the Hopf bifurcating periodic solutions of system (1) at τ^* on the center manifold are determined by the following formulas

$$\begin{aligned} &C_1(0)=\frac{i}{2\tau^*\omega^*}(g_{11}g_{20}-2|g_{11}|^2-\frac{|g_{02}|^2}{3})+\frac{g_{21}}{2}, && v_2=-\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\frac{d\lambda}{d\tau}(\tau^*)\}}, \\ &\beta_2=2\,\mathrm{Re}\{C_1(0)\}, && T_2=\frac{-\mathrm{Im}\{C_1(0)\}+v_2\,\mathrm{Im}\{\frac{d\lambda}{d\tau}(\tau^*)\}}{\tau^*\omega^*}. \end{aligned} \tag{66}$$

Here v_2 determines the direction of Hopf bifurcation. If $v_2>0(v_2<0)$, then the Hopf-bifurcation is forward(backward) and the bifurcating periodic solutions exist for $\tau>\tau^*(\tau<\tau^*)$. Again β_2 determines the stability of the bifurcating periodic solutions. The bifurcating periodic solutions are stable (unstable) if $\beta_2<0(\beta_2>0)$. T_2 determines the period of periodic solutions: the period increases (decreases) if $T_2>0(T_2<0)$. Therefore, we have the following results.

Theorem 4. The Hopf bifurcation of the system (1) occurring at E^* when $\tau = \tau^*$ is forward (backward) if $v_2 > 0 (v_2 < 0)$ and the bifurcating periodic solutions on the center manifold are stable (unstable) if $\text{Re}\{C_1(0)\} < 0 (> 0)$.

5. Conclusion

This paper introduces modified time-delay predator- prey model. Then we study the Hopf bifurcation and the stability of the system. Our results reveal the conditions on the parameters so that the periodic solutions exist surrounding the interior equilibrium point. It shows that τ^* is a critical value for the time delay τ. Furthermore, the direction of Hopf bifurcation and the stability of bifurcated periodic solutions are investigated.

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References

[1] Lotka, A.J. (1925) Elements of Physical Biology. *Nature*, **116**, 461. http://dx.doi.org/10.1038/116461b0

[2] Volterra, V. (1926) Fluctuations in The Abundance of A Species Considered Mathematically. *Nature*, **118**, 558-560. http://dx.doi.org/10.1038/118558a0

[3] Yan, X. and Li, W. (2007) Bifurcation and Global Periodic Solutions in A Delayed Facultative Mutualism System. *Physica D*: *Nonlinear Phenomena*, **227**, 51-69. http://dx.doi.org/10.1016/j.physd.2006.12.007

[4] Tian, C. and Zhang, L. (2013) Hopf Bifurcation Analysis in a Diffusive Food-Chain Model with Time Delay. *Computers & Mathematics with Applications*, **66**, 2139-2153. http://dx.doi.org/10.1016/j.camwa.2013.09.002

[5] Kar, T.K. and Jana, S. (2012) Stability and Bifurcation Analysis of a Stage Structured Predator Prey Model with Time Delay. *Applied Mathematics & Computation*, **219**, 3779-3792. http://dx.doi.org/10.1016/j.amc.2012.10.007

[6] Zhang, J., Li, W. and Yan, X. (2011) Hopf Bifurcation and Turing Instability in Spatial Homogeneous and Inhomogeneous Predator-Prey Models. *Applied Mathematics & Computation*, **218**, 1883-1893. http://dx.doi.org/10.1016/j.amc.2011.06.071