

New Model for L^2 Norm Flow

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Received 17 April 2015; accepted 23 June 2015; published 30 June 2015

Abstract

We introduce a new L^2 norm preserving heat flow in matrix geometry. We show that this flow exists globally and preserves the positivity property of Hermitian matrices.

Keywords

Global Flow, Norm Conservation, Positivity

1. Introduction

In this paper we introduce a new evolution equation in the matrix geometry such that the L^2 norm is preserved. In [1], the author introduced the Ricci flow which exists globally when the initial matrix is a positive definite. The Ricci flow [2] [3] preserves the trace of the initial matrix and the flow converges the scalar matrix with the same trace as the initial matrix. In [4], we have introduced the heat equation, which also preserves the trace of the initial matrix. In [5]-[8], the authors introduce the norm preserving flows which are global flows and converge to eigenfunctions. We know that the fidelity of quantum state is an important subject in quantum computation and quantum information [9] [10], the L^2 norm flow we studied is very closed related to the fidelity. This is the motivation of the study of norm preserving flow in matrix geometry.

To introduce our new L^2 norm flow in matrix geometry, we need to use some language from the book [11] and the papers [1] [4] [12]. Let x, y be two Hermitian matrices on C^n . Define $u = e^{\frac{2\pi i}{n}x}$, $v = e^{\frac{2\pi i}{n}y}$. We use M_n to denote the algebra of all $n \times n$ complex matrices which generated by u and v with the bracket $[u, v] = uv - vu$. Then CI , which is the scalar multiples of the identity matrix I , is the commutant of the operation $[u, v]$. Sometimes we simply use 1 to denote the $n \times n$ identity matrix.

We define two derivations δ_1 and δ_2 on the algebra M_n by the commutators

$$\delta_1 := [y, \cdot], \delta_2 := -[x, \cdot]$$

and define the Laplacian operator on M_n by

$$\hat{\Delta} = \delta_1^* \delta_1 + \delta_2^* \delta_2 = \delta_1^2 + \delta_2^2 = \delta_\mu \delta_\mu,$$

where we have used the Einstein sum convention. We use the Hilbert-Schmidt norm $|\cdot|$ defined by the inner product

$$\langle a, b \rangle := \tau(a^* b)$$

on the algebra M_n and let $\sigma(a) = \langle 1, a \rangle = \tau(a)$. Here a^* is the Hermitian adjoint of the matrix a and τ denotes the usual trace function on M_n . We now state basic properties of δ_1 , δ_2 and $\hat{\Lambda}$ (see also [1]) as follows.

Given a positive definite Hermitian matrix $\Lambda \in M_n$. For any $c \in M_n$, we define the Dirichlet energy

$$D(c) = \sum \langle \delta_\mu c, \delta_\mu c \rangle + \langle \Lambda c, c \rangle$$

and the L^2 mass

$$M(c) = \langle c, c \rangle.$$

Let, for $c \neq 0$,

$$\lambda(c) = \frac{D(c)}{M(c)}.$$

Then the eigenvalues of the operator $\hat{\Lambda} + \Lambda = \hat{L}$ correspond to the critical values of the Dirichlet energy $D(c)$ on the sphere

$$\Sigma = \{c \in M_n; M(c) = 1\}.$$

We consider the evolution flow

$$c_t = -\hat{L}c + \lambda(c)c = -\hat{\Delta}c - \Lambda c + \lambda(c)c \tag{1.1}$$

with its initial matrix $c|_{t=0} = c_0 \in M_n$. Assume $c = c(t)$ is the solution to the flow above. Then

$$\frac{1}{2} \frac{d}{dt} M(c) = \langle c, c_t \rangle = -\langle c, \hat{L}c \rangle + D(c).$$

Since $\langle c, \hat{L}c \rangle = D(c)$, we know that $\frac{d}{dt} M(c) = 0$. Then

$$M(c) = M(c_0).$$

The aim of this paper is to show that there is a global flow to (1.1) with the initial data $c_0 \in M_n$ and the flow preserves the positivity of the initial matrix.

2. Existence of the Global Flow

Firstly, we consider the local existence of the flow (1.1). We prefer to follow the standard notation and we let $L = \hat{\Delta} - \Lambda$, where Λ is a positive definite Hermitian matrix. Let $a = a(t) \in M_n$ be such that

$$a_t = La + \lambda(t)a, \tag{2.1}$$

with the initial matrix $a|_{t=0} = a_0$. Here $a_0 \in M_n$ such that $|a|^2(0) = |a_0|^2 = 1$. Then for $a = a(t)$, we let

$$\lambda(t) = -\frac{\langle La, a \rangle}{\langle a, a \rangle}. \tag{2.2}$$

Formally, if the flow (2.1) exists, then we compute that

$$\frac{d}{dt} |a|^2 = 2 \langle a, a_t \rangle = 2 \langle a, La \rangle + 2\lambda(t) \langle a, a \rangle = 0.$$

Then $|a|^2(t) = |a|^2(0) = 1, \forall t > 0$.

In this section, our aim is to show that there is a global solution to Equation (2.1) for any initial matrix $a_0 \in M_n$ with $|a_0|^2 = 1$.

Assume at first that $\lambda(t) \geq 0$ is any given continuous function and $a = a(t)$ is the corresponding solution of (2.1). Define $b = e^{-\int \lambda(t) dt} a$. Then $b(0) = a(0)$ and we get

$$\begin{aligned}
 b_t &= e^{-[\lambda(t)dt]} (-\lambda(t))a + e^{-[\lambda(t)dt]} a_t = -\lambda(t)ae^{-[\lambda(t)dt]} + e^{-[\lambda(t)dt]} (La + \lambda(t)a) \\
 &= e^{-[\lambda(t)dt]} La = L(e^{-[\lambda(t)dt]} a) = Lb.
 \end{aligned}
 \tag{2.3}$$

The Equation (2.3) can be solved by standard iteration method and we present it in below. Assume (φ_i) and (λ_i) are eigen-matrices and eigenvalues of L as we introduced in [4], such that

$$-L\varphi_i = \lambda_i\varphi_i, \langle \varphi_i, \varphi_j \rangle = \delta_{ij}.$$

Note that $\lambda_i \geq 0$.

Assume that $b = b(t)$ is the solution to (2.3). Set

$$b = \sum \langle b, \varphi_i \rangle \varphi_i := \sum u_i \varphi_i, u_i \in R, u_i = u_i(t).$$

Then by (2.3), we obtain

$$(u_i)_t \varphi_i = L(u_i \varphi_i) = -u_i \lambda_i \varphi_i.$$

Then $(u_i)_t = -u_i \lambda_i$, and $u_i = u_i(0)e^{-\lambda_i t}$.

Hence

$$b = \sum u_i(0)e^{-\lambda_i t} \varphi_i,$$

and

$$a = \sum u_i(0)e^{-\lambda_i t + [\lambda(t)dt]} \varphi_i \tag{2.4}$$

solves (2.1) with the given $\lambda(t) \geq 0$.

Next we define a iteration relation to solve (2.1) for the unknown $\lambda(t)$ given by (2.2).

Define a_1 such that it solves the equation $a_t = La + \lambda_0 a$ with $\lambda_0 = -\frac{\langle La_0, a_0 \rangle}{\langle a_0, a_0 \rangle}$.

Let $k \geq 1$ be any integer. Define a_{k+1} such that

$$(a_{k+1})_t = La_{k+1} + \lambda_k(t)a_{k+1}, a_{k+1}(0) = a_0, \tag{2.5}$$

with

$$\lambda_k(t) = -\frac{\langle La_k, a_k \rangle}{\langle a_k, a_k \rangle}. \tag{2.6}$$

Then using the Formula (2.4), we get a sequence (a_k) .

We claim that $(a_k) \subset M_n$ is a bounded sequence and $(\lambda_k(t))$ is also a bounded sequence.

It is clear that $(a_k) \subset M_n$. If this claim is true, we may assume

$$a_k \rightarrow a, \lambda_k(t) \rightarrow \tilde{\lambda}(t).$$

Then by (2.5) and (2.6), we obtain

$$a_t = La + \tilde{\lambda}(t)a$$

and

$$\tilde{\lambda}(t) = -\frac{\langle La, a \rangle}{\langle a, a \rangle},$$

which is the same as (2.1). That is to say, $a = a(t)$ obtained above is the desired solution to (2.1).

Firstly we prove the claim in a small interval $[0, T]$. Assume $|a_k| \leq A = 1.5$ and $|\lambda_k| \leq B = \log(4/3)$ on $[0, T]$, $T = 1/2$. Then, by (2.5),

$$\frac{1}{2} |a_{k+1}|_t^2 = \langle a_{k+1}, (a_{k+1})_t \rangle = \langle a_{k+1}, La_{k+1} \rangle + \lambda_k |a_{k+1}|^2. \tag{2.7}$$

By (2.6), we obtain $\lambda_{k+1} = \frac{-\langle La_{k+1}, a_{k+1} \rangle}{|a_{k+1}|^2}$. Then

$$-\langle La_{k+1}, a_{k+1} \rangle = \lambda_{k+1} |a_{k+1}|^2.$$

By (2.7), we get

$$\frac{1}{2} |a_{k+1}|_t^2 = -\lambda_{k+1} |a_{k+1}|^2 + \lambda_k |a_{k+1}|^2 = (\lambda_k - \lambda_{k+1}) |a_{k+1}|^2.$$

Then $|a_{k+1}|^2 = e^{2\int(\lambda_k - \lambda_{k+1})dt} |a_0|^2 = e^{2Bt} \leq A$.

Note that $\exists D > 0$ such that $|\delta_\mu a|^2 \leq D |a|^2$ for any $a \in M_n$. We have

$$\begin{aligned} \lambda_{k+1} |a_{k+1}|^2 &= \langle La_{k+1}, a_{k+1} \rangle = \langle \Delta a_{k+1}, a_{k+1} \rangle + \langle \Lambda a_{k+1}, a_{k+1} \rangle \\ &\leq (D + |\Lambda|) |a_{k+1}|^2 = C |a_{k+1}|^2. \end{aligned}$$

Then $\lambda_{k+1} \leq C$. Hence the claim is true in $[0, T]$.

Therefore, (2.1) has a solution in $[0, T]$. By iteration we can get a solution in $[T, 2T]$ with $u(T)$ as the initial data. We can iterate this step on and on and we get a global solution to (2.1) with initial data a_0 .

In conclusion we have the below.

Theorem 2.1 For any given initial matrix $a_0 \in M_n$ with $|a_0|^2 = 1$, the Equation (2.1) has a global solution with a_0 as its initial data and $|a(t)|^2 = 1$ for all $t > 0$.

3. Positive Property Preserved by the Flow

In this section we show that positivity of the initial matrices is preserved along the flow. That is to say, we show that if the initial matrix is positive definite, then along the flow (2.1), the evolution matrix is also positive definite.

Theorem 3.1 Assume $a_0 > 0$, that is a_0 is a Hermitian positive definite. Then $a(t) > 0, \forall t > 0$ along the flow equation

$$a_t = La + \lambda(t)a$$

with $a(0) = a_0$, where $\lambda(t)$ is given by (2.2).

Proof. By an argument as in [4], we know $a = a(t)$ is Hermitian matrix. Then we know that $a = a(t) > 0$ for small $t > 0$ by continuity. Compute

$$\frac{d}{dt} \log \det a = \langle a^{-1}, a_t \rangle = \langle a^{-1}, La \rangle + N\lambda(t),$$

where $N = \sigma(I) = \langle 1, I \rangle$.

Since

$$\langle a^{-1}, La \rangle = \langle a^{-1}, \Delta a \rangle - \langle a^{-1}, \Lambda a \rangle = \langle a^{-1}, \Delta a \rangle - \tau(\Lambda),$$

and

$$\langle a^{-1}, \Lambda a \rangle = -\langle \delta_\mu(a^{-1}), \delta_\mu a \rangle = \langle a^{-1} \delta_\mu a \cdot a^{-1}, \delta_\mu a \rangle = |a^{-1} \delta_\mu a|^2.$$

We know that

$$\frac{d}{dt} \log e^{\tau(\Lambda)} \det a = \frac{d}{dt} [\log \det a + t\tau(\Lambda)] = |a^{-1} \delta_\mu a|^2 + \tau(\Lambda) - \tau(\Lambda) + N\lambda(t) \geq N\lambda(t) \geq 0.$$

Then we have

$$e^{\tau(\Lambda)} \det a > \det a_0 > 0.$$

Hence $\det a > 0$ and $a = a(t) > 0, \forall t > 0$. \square

Then the proof of Theorem 3.1 is complete.

Remark that by continuity, we can show that if $a_0 \geq 0$, then $a(t) \geq 0$ along the flow (2.1).

Funds

The research is partially supported by the National Natural Science Foundation of China (No. 11301158, No.11271111).

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