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Pullback Exponential Attractors for Nonautonomous Reaction Diffusion Equations in H_0^1

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Abstract

Under the assumption that $g(t)$ is translation bounded in $L_{loc}^4(R;L^4(\Omega))$, and using the method developed in [3], we prove the existence of pullback exponential attractors in $H_0^1(\Omega)$ for nonlinear reaction diffusion equation with polynomial growth nonlinearity($p \geq 2$ is arbitrary).

1. Introduction

Attractor's theory is very important to describe the long time behavior of dissipative dynamical systems generated by evolution equations, and there are several kinds of attractors. In this article, we will study the existence of pullback exponential attractors (see [1]-[3]) for nonlinear reaction diffusion equation. This equation is written in the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + f(u) = g(t), x \in \Omega, \\ u\mid_\Omega = 0, \\ u(\tau) = u_{\tau.} \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in R^n, $g(\cdot) \in L_{loc}^2(R, L^2(\Omega))$, $f \in C^1(R,R)$ and there exist $p \geq 2$, $c_i > 0$, $i = 1,2,\cdots 5, l > 0$ such that

$$c_1 \mid u \mid^p - c_2 \leq f(u)u \leq c_3 \mid u \mid^p + c_4, \ f'(u) \geq -l, \ \mid f'(u) \mid \leq c_5(1+ \mid u \mid^{p-2}) \tag{1.2}$$

for all $u \in R$.

The Equation of (1.1) has been widely studied. For the autonomous case, *i.e.*, $g(t)$ does not depend on the time, the asymptotic behaviors of the solution have been studied extensively in the framework of global attractor, see [4]-[6]. For the nonautonomous case, the asymptotic behaviors of the solution have been studied in the framework of pullback attractor, see [7]-[9]. Recently, the theory of pullback exponential attractor have been developed, see [1]-[3], and some methods are given to prove the existence of pullback exponential attractors.

In order to obtain the existence of pullback exponential attractors of (1.1), we will need the following theorem.

Theorem 1.1. ([3]) *Let X be an uniformly convex Banach space, $B(X)$ be the set of all bounded subsets of X, $\{U(t,\tau)\,|\,t \geq \tau\}$ be a time continuous process in X. Then the process $\{U(t,\tau)\,|\,t \geq \tau\}$ exist pullback exponential attractors in X if the following conditions hold true:*

(1) There exists an uniformly bounded absorbing set $B \subset X$, that is, for any $t \geq \tau$ and $D \in B(X)$, there exists $T_0 > 0$ such that

$$U(t, \tau - s)D \subset B, \forall s \leq T_0 \tag{1.3}$$

(2) There exist $\delta, \theta, > 0, \delta + \theta < 1, T_1, l > 0$, and a finite dimension subspace $X_1 \subset X$, such that

$$\| U(t, \tau)u_1 - U(t, \tau)u_2 \| \leq l \| u_1 - u_2 \|, \forall t, \tau \in [kT_1, (k+1)T_1], \forall k \in \mathbb{Z}, \tag{1.4}$$

$$\| (I - P_m)(U(t, t - T_1)u_1 - U(t, t - T_1)u_2) \| \leq \delta \| u_1 - u_2 \|, \tag{1.5}$$

$$\| (I - P_m) \bigcup_{s \leq T_1} U(t, \tau - s)u \| \leq \theta, \forall t \geq \tau, \tag{1.6}$$

for all $u, u_1, u_2 \in B$ and $t \in R$, where δ is independent on the choice of t, and $\| \cdot \|$ is the norm in X, I is the identity operator, $P_m : X \to X_1$ is a bounded projector, m is the dimension of X_1.

2. Some Estimates of Equation (1.1)

In this section, we will derive some priori estimates for the solutions of (1.1) that will be used to construct pullback exponential attractors for the problem (1.1).

For convenience, hereafter let $|\cdot|_p$ be the norm of $L^p(\Omega)(p > 1)$, and c an arbitrary constant, which may difference from line to line and even in the same line. We define $H = L^2(\Omega)$ with scalar product $(\cdot)$ and norm $|\cdot|$; let $((\cdot))$ and $\| \cdot \|$ denote the scalar product and norm of $H_0^1(\Omega)$ and $((u,v)) = \int_\Omega \nabla u \nabla v dx$ for all $u, v \in H_0^1(\Omega)$, set λ is the first eigenvalue of $-\Delta$.

For the initial value problem (1.1), we know from [4]-[6] that for any initial datum $u_0 \in H$, there exists a unique solution $u(t) \in C([\tau, T]; H) \bigcap L^2([\tau, T]; H_0^1)$ for any $T > \tau$.

Thanks to the existence theorem, the initial value problem is equivalent to a process $\{U(t, \tau)\,|\,t \geq \tau\}$ define by

$$U(t, \tau) : H \to H_0^1.$$

In addition, we assume that the function $g(t)$ is translation bounded in $L_{loc}^3(R; L^3(\Omega))$, that is

$$\sup_{t \in R} \int_t^{t+1} | g(s) |_3^3 ds < \infty. \tag{2.1}$$

By (2.1), for $\lambda > 0$, we have

$$\sup_{t \in R} \int_t^{t+1} | g(s) |^2 ds < c, \ e^{-\lambda t} \int_\tau^t e^{\lambda s} | g(s) |^2 dx \leq \sum_{n=0}^\infty \int_{t-n-1}^{t-n} e^{-\lambda(t-s)} | g(s) |^2 \, ds \leq c. \tag{2.2}$$

Lemma 2.1. ([7]-[9]) *Assume that f and g satisfy (1.2) and (2.2), $u(t)$ be a weak solution of (1.1), then for any $t \geq \tau$, we have the following inequality:*

$$| u(t) |^2 \leq e^{-\lambda(t-s)} | u_\tau |^2 + c \tag{2.3}$$

and

$$\int_\tau^t e^{\lambda s} (\| u(s) \|^2 + 2c_1 | u(s) |_p^p) ds \leq (1 + \lambda(t - \tau))e^{\lambda \tau} | u_\tau |^2 + ce^{\lambda t} + \lambda^{-1} \int_\tau^t e^{\lambda s} | g(s) |^2 \, ds \tag{2.4}$$

Lemma 2.2. *Assume that* f *and* g *satisfy* (1.2) *and* (2.2), $u(t)$ *be a weak solution of* (1.1), *then the following inequality holds for* $t > \tau$

$$| u(t) |^2 + \| u(t) \|^2 + | u(t) |_p^p \le c(1+\frac{1}{t-\tau})(1+t-\tau)e^{-\lambda(t-\tau)} | u_\tau |^2 + c(1+\frac{1}{t-\tau}) \tag{2.5}$$

Obviously, for any bounded $D \subset H$, *there exist* $T, r > 0$, *such that*

$$| U(t,\tau)u_\tau |^2 + \| U(t,\tau)u_\tau \|^2 + | U(t,\tau)u_\tau |_p^p \le r \quad \text{for any} \quad u_\tau \in D \quad \text{and} \quad t - \tau \ge T. \tag{2.6}$$

Proof. Let $F(s) = \int_0^s f(\tau)d\tau$, then by (1.2), we get there exist $\widetilde{c}_i > 0$, $i = 1, 2, 3, 4$, such that

$$\widetilde{c}_1 | u |^p - \widetilde{c}_2 \le F(s) \le \widetilde{c}_3 | u |^p + \widetilde{c}_4. \tag{2.7}$$

Taking inner product of (1.1) with u in H and using (2.7), we get

$$\frac{d}{dt} | u |^2 + \| u \|^2 + c\int_\Omega F(u)dx \le c(1 + | g(t) |^2). \tag{2.8}$$

Multiply (1.1) by u_t, we have

$$| u_t |^2 + \frac{1}{2}\frac{d}{dt}(\| u \|^2 + 2\int_\Omega F(u)dx) = (g(t), u_t),$$

since $| (g(t), u_t) | \le \frac{1}{2}(| g(t) |^2 + | u_t |^2)$, we obtain

$$\frac{d}{dt}(\| u \|^2 + 2\int_\Omega F(u)dx) \le | g(t) |^2.$$

Combining (2.7), we get

$$\frac{d}{dt}(| u |^2 + \| u \|^2 + 2\int_\Omega F(u)dx) + \| u \|^2 + c\int_\Omega F(u)dx \le c(1 + | g(t) |^2). \tag{2.9}$$

Thanks to Poincaré inequality $\| u \| \ge \lambda | u |$, we have

$$\| u \|^2 + c\int_\Omega F(u)dx \ge \frac{\lambda}{2} | u |^2 + \frac{1}{2}\| u \|^2 + c\int_\Omega F(u)dx \ge c(| u |^2 + \| u \|^2 + c\int_\Omega F(u)dx). \tag{2.10}$$

Let $G(u) = | u |^2 + \| u \|^2 + c\int_\Omega F(u)dx$, by (2.9) and (2.10), we obtain

$$\frac{d}{dt}G(u) + cG(u) \le c(1 + | g(t) |^2),$$

which imply

$$\frac{d}{dt}((t-\tau)e^{\lambda t}G(u)) \le (1 + (\lambda - c)(t - \tau))G(u)e^{\lambda t} + c(1 + | g(t) |^2)(t - \tau)e^{\lambda t},$$

integrating, we get

$$(t-\tau)e^{\lambda t}G(u) \le (1 + c(t-\tau))\int_\tau^t G(u)e^{\lambda s}ds + c(t-\tau)e^{\lambda t} + c(t-\tau)\int_\tau^t | g(s) |^2 e^{\lambda s}ds,$$

using (2.3) and (2.4), we get the inequality (2.5).

Lemma 2.3. *Assume that* f *and* g *satisfy* (1.2) *and* (2.1), $u(t)$ *be a weak solution of* (1.1), *then the following inequality holds for* $t > \tau_0$

$$| u(t) |_{2p-2}^{2p-2} \le c(1+\frac{1}{t-\tau_0})(1+t-\tau_0)e^{-\lambda(t-\tau_0)} | u_{\tau_0} |^2 + e^{-\lambda(t-\tau_0)} | u_{\tau_0} |_p^p, \tag{2.11}$$

Here $u(t) = U(t,\tau)u_\tau, u_{\tau_0} = U(\tau_0,\tau)u_\tau$ *for any* $t > \tau_0 > \tau$.

By the assumption (2.1) and for $\lambda > 0$, we get

$$\sup_{t>\tau} \int_\tau^t e^{\lambda s} \mid g(s) \mid_{\frac{3p-4}{p-1}}^{\frac{3p-4}{p-1}} ds < \infty .$$ (2.12)

Proof. Multiply (1.1) with $\mid u \mid^{p-2} u$, we obtain

$$\frac{1}{p}\frac{d}{dt}\mid u\mid_p^p +(p-1)\int_\Omega \mid u\mid^{(p-2)}\mid \nabla u\mid^2 dx + \int_\Omega f(u)\mid u\mid^{p-2} udx = \int_\Omega g(t)\mid u\mid^{(p-2)} udx .$$ (2.13)

By (1.2) and Young's inequality, we have

$$f(u)\mid u\mid^{p-2} u \geq c_1'\mid u\mid^{2p-2}-c_2', \quad \mid \int_\Omega g(t)\mid u\mid^{p-2} udx\mid \leq \frac{c_1'}{2}\mid u\mid_{2p-2}^{2p-2}+\frac{1}{2c_1'}\mid g(t)\mid^2 .$$

By (2.13), we get

$$\frac{d}{dt}\mid u\mid_p^p +c\mid u\mid_{2p-2}^{2p-2}\leq c(1+\mid g(t)\mid^2), \quad \frac{d}{dt}(e^{\lambda t}\mid u\mid_p^p)+ce^{\lambda t}\mid u\mid_{2p-2}^{2p-2}\leq ce^{\lambda t}(1+\mid u\mid_p^p +\mid g(t)\mid^2),$$

integrating and using (2.4), we get

$$\int_{\tau_0}^t e^{\lambda s}\mid u\mid_{2p-2}^{2p-2}ds \leq c((1+t-\tau_0)e^{\lambda\tau_0}\mid u_{\tau_0}\mid^2 +e^{\lambda\tau_0}\mid u_{\tau_0}\mid_p^p +e^{\lambda t}+\int_{\tau_0}^t e^{\lambda s}\mid g(s)\mid^2 ds .$$ (2.14)

Multiply (1.1) with $\mid u\mid^{2p-4} u$, we obtain

$$\frac{1}{2p-2}\frac{d}{dt}\mid u\mid_{2p-2}^{2p-2}+(2p-3)\int_\Omega \mid u\mid^{2p-4}\mid \nabla u\mid^2 dx +\int_\Omega f(u)\mid u\mid^{2p-4} udx =\int_\Omega g(t)\mid u\mid^{2p-4} udx .$$

By (2.1), we get

$$f(u)\mid u\mid^{2p-4} u \geq c_1''\mid u\mid^{3p-4}-c_2'' .$$

Using Young's inequality

$$\mid \int_\Omega g(t)\mid u\mid^{2p-4} udx\mid \leq \frac{c_1''}{2}\mid u\mid_{3p-4}^{3p-4}+c\mid g(t)\mid_{\frac{3p-4}{p-1}}^{\frac{3p-4}{p-1}} .$$

By the above inequality, we have

$$\frac{d}{dt}\mid u\mid_{2p-2}^{2p-2}\leq c(1+\mid g(t)\mid_{\frac{3p-4}{p-1}}^{\frac{3p-4}{p-1}}), \quad \frac{d}{dt}((t-\tau_0)e^{\lambda t}\mid u\mid_{2p-2}^{2p-2}) \leq (1+\lambda(t-\tau_0)e^{\lambda t}\mid u\mid_{2p-2}^{2p-2}+c(t-\tau_0)e^{\lambda t}(1+\mid g(t)\mid_{\frac{3p-4}{p-1}}^{\frac{3p-4}{p-1}}),$$

integrating and using (2.12) and (2.14), we get (2.11) holds.

Lemma 2.1, lemma 2.2 and lemma 2.3 show that the process generated by the equation (1.1) have an uniformly pullback bounded absorbing set in $H, H_0^1, L^p(\Omega), L^{2p-2}(\Omega)$, that is

Theorem 2.4. *Assume that f and g satisfy (1.2) and (2.1), $u(t)$ be a weak solution of (1.1), then the process generated by the equation (1.1) have an uniformly pullback bounded absorbing set*
$B \subset H \cap H_0^1 \cap L^p(\Omega) \cap L^{2p-2}(\Omega)$, *that is, for any bounded set $D \subset H$, there exists $T_0 > 0$, such that $U(t,t-\tau)D \subset B$ for any $t-\tau \geq T_0$.*

In fact, using the same proof as in Lemma 2.3, we can get the following result.

Lemma 2.5. *Assume that f satisfies (1.2), $g(t)$ is translation bounded in $L_{loc}^4(R; L^4(\Omega))$, that is*

$\sup_{t\in R}\int_t^{t+1}\mid g(s)\mid_4^4 ds < \infty$, *$u(t)$ be a weak solution of (1.1), then the process generated by the equation (1.1) have an uniformly pullback bounded absorbing set $B \subset L^{4p-6}(\Omega)$, that is, for any bounded set $D \subset H$, there exists $T_0 > 0$, such that $U(t,t-\tau)D \subset B$ for any $t-\tau \geq T_0$.*

3. Pullback Exponential Attractors

In this section, we will use Theorem 1.1 to prove that the process generated by Equation (1.1) exists a pullback

exponential attractor.

First we assume that the function $g(t)$ is normal ([10]) in $L_{loc}^2(R, H)$, that is, for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\sup_{t \in R} \int_t^{t+\eta} |g(s)|^2 ds < \varepsilon. \tag{3.1}$$

Obviously, $g(t)$ is normal in $L_{loc}^2(R, H)$ implying that $g(t)$ is translation bounded in $L_{loc}^2(R, H)$.

We set $A = -\Delta$, since A^{-1} is a continuous compact operator in H, by the classical spectral theorem, there exist a sequence $\{\lambda_j\}_{j=1}^{\infty}$, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \lambda_j \to +\infty$ as $j \to +\infty$, and a family of elements $\{e_j\}_{j=1}^{\infty}$ of H_0^1 which are orthogonal in H such that $A e_j = \lambda_j e_j$, $\forall j \in \mathbb{Z}^+$. Let $H_m = span\{e_1, e_2, \cdots, e_m\}$ in H and $P : H \to H_m$ is a orthogonal projector. For any $u \in H$, we write

$$u = Pu + (I - P)u \triangleq u_1 + u_2.$$

Theorem 2.4. *Assume that* f *satisfies* (1.2), $g(t)$ *is translation bounded in* $L_{loc}^4(R; L^4(\Omega))$ *and* (3.1) *holds, then the process generated by the equation* (1.1) *have a pullback exponential attractor.*

Next, we will verify that the process generated by (1.1) satisfy all the conditions of Theorem 1.1.

Proof. By Theorem 2.4, there exists $T_0 > 0$, such that $U(t, t-\tau)B \subset B$ for any $t - \tau \geq T_0$. Let $B' = \cup_{t \in R} \cup_{\tau \leq T_0} U(t, t-\tau)B$, we obtain B' is also an uniformly pullback bounded absorbing set in $H_0^1 \cap L^{2p-2}(\Omega)$ and $U(t, t-\tau)B' \subset B'$ for any $t \geq \tau$.

We set $u_1(t) = U(t, \tau)u_{1\tau}$, $u_2(t) = U(t, \tau)u_{2\tau}$ to be solutions associated with Equation (1.1) with initial data $u_{1\tau}, u_{2\tau} \in B'$, since B' is the uniformly pullback bounded absorbing set in $H_0^1 \cap L^{2p-2}(\Omega)$, so there exists $M > 0$ such that $\| u_{i\tau} \| \leq M, \| u_i(t) \| \leq M, |u_i(t)|_{2p-2}^{2p-2} \leq M$, $i = 1, 2$. Let $w(t) = u_1(t) - u_2(t)$, by (1.1), we get

$$w_t - \Delta w + f(u_1(t) - u_1(t)) = 0. \tag{3.2}$$

Taking inner product of (3.2) with $-\Delta w$ in H, we have

$$\frac{1}{2}\frac{d}{dt} \| w \|^2 + \| \Delta w \|^2 + (f(u_1(t) - u_1(t)), -\Delta w) = 0. \tag{3.3}$$

Taking into account (1.2) and Holder inequality, it is immediate to see that

$$|(f(u_1) - f(u_2), -\Delta w)| \leq \int_{\Omega} |f(u_1) - f(u_2)| |\Delta w| dx \leq \frac{1}{2}|\Delta w|^2 + \frac{1}{2}\int_{\Omega} |f(u_1) - f(u_2)|^2 dx,$$

and

$$\int_{\Omega} |f(u_1) - f(u_2)|^2 dx = \int_{\Omega} |f'(u_1 + \theta(u_2 - u_1))^2 |u_2 - u_1|^2 dx \leq c\int_{\Omega}(1 + |u_1|^{p-2} + |u_2|^{p-2})^2 |u_2 - u_1|^2 dx$$

$$\leq c\int_{\Omega}(1 + |u_1|^{2p-3} + |u_2|^{2p-3})|u_1 - u_2| dx \leq c(1 + |u_1|_{4p-6}^{4p-6} + |u_2|_{4p-6}^{4p-6})|w|^2,$$

By Lemma 2.5, we get

$$\int_{\Omega} |f(u_1) - f(u_2)|^2 dx \leq c \| w \|^2. \tag{3.4}$$

Using (3.3), we obtain $\frac{d}{dt} \| w \|^2 \leq c \| w \|^2$, hence

$$\| w(t) \|^2 \leq \| w(\tau) \|^2 e^{c(t-\tau)}. \tag{3.5}$$

Let $w = w_1 + w_2$, w_1 be the project in PH. Taking inner product of (3.2) with $-\Delta w_2$ in H, we have

$$\frac{1}{2}\frac{d}{dt} \| w_2 \|^2 + |\Delta w_2|^2 + (f(u_1) - f(u_2), -\Delta w_2) = 0. \tag{3.6}$$

$$| (f(u_1) - f(u_2), -\Delta w_2) | \leq \int_\Omega | f(u_1) - f(u_2) | | \Delta w_2 | dx \leq \frac{1}{2} | \Delta w_2 |^2 + \frac{1}{2} \int_\Omega | f(u_1) - f(u_2) |^2 dx.$$

Taking into (3.4) account, we obtain

$$\frac{d}{dt} \| w_2 \|^2 + | \Delta w_2 |^2 \leq c \| w \|^2,$$

Using the Poincaré inequality $\lambda_n \| w_2 \|^2 \leq | \Delta w_2 |^2$, we get $\frac{d}{dt} \| w_2 \|^2 + \lambda_n \| w_2 \|^2 \leq c \| w \|^2$, by Gronwall's

Lemma, we have $\| w_2(t) \|^2 \leq e^{-\lambda_n(t-\tau)} \| w(\tau) \|^2 + ce^{-\lambda_n t} \int_\tau^t e^{-\lambda_n s} \| w(s) \|^2 ds$. Using (3.5), we get

$$\| w_2(t) \|^2 \leq (e^{-\lambda_n(t-\tau)} + c \frac{e^{c(t-\tau)}}{\lambda_n}) \| w(\tau) \|^2. \tag{3.7}$$

Let $u(t) = u_1(t) + u_2(t)$, $u_1(t)$ be the project in PH. Taking inner product of (1.1) with $-\Delta u_2$, we get

$$\frac{1}{2} \frac{d}{dt} \| u_2 \|^2 + \| \Delta u_2 \|^2 + (f(u(t)), -\Delta u_2) = (g(t), -\Delta u_2).$$

Since $| (g(t), -\Delta u_2) | \leq | g(t) |^2 + \frac{1}{4} | \Delta u_2 |^2$, $| (f(u(t)), -\Delta u_2) | \leq \int_\Omega | f(u) |^2 dx + \frac{1}{4} | \Delta u_2 |^2$, and by Poincaré in-

equality $\lambda_n \| u_2 \|^2 \leq | \Delta u_2 |^2$, we have

$$\frac{d}{dt} \| u_2 \|^2 + \lambda_n \| u_2 \|^2 \leq c(1 + | g(t) |^2).$$

By Gronwall's lemma, we get

$$| u_2(t) |^2 \leq e^{-\lambda_n(t-\tau)} | u_\tau |^2 + ce^{-\lambda_n t} \int_\tau^t e^{\lambda_n s} (1 + | g(s) |^2) ds.$$

By (3.1), we obtain that there exists $c > 0$, such that $\int_t^{t+1} | g(s) |^2 ds < c$ for any $t \in R$, and for any $\varepsilon > 0$,

there exists $\eta > 0$, such that $\int_{t-\eta}^t | g(s) |^2 ds < \frac{\varepsilon}{3}$, so we get

$$e^{-\lambda_n t} \int_\tau^t e^{-\lambda_n s} | g(s) |^2 ds = \int_\tau^t e^{-\lambda_n(t-s)} | g(s) |^2 ds$$

$$\leq \int_{t-\eta}^t e^{-\lambda_n(t-s)} | g(s) |^2 ds + \int_{t-\eta-1}^{t-\eta} e^{-\lambda_n(t-s)} | g(s) |^2 ds$$

$$+ \int_{t-\eta-2}^{t-\eta-1} e^{-\lambda_n(t-s)} | g(s) |^2 ds + \cdots + \int_{t-\eta-k-1}^{t-\eta-k} e^{-\lambda_n(t-s)} | g(s) |^2 ds + \cdots \quad \text{and} \quad e^{-\lambda_n t} \int_\tau^t e^{\lambda_n s} ds \leq \frac{1}{\lambda_n}, \text{ we have}$$

$$\leq \frac{\varepsilon}{3} + ce^{-\lambda_n \eta} (1 + e^{-\lambda_n} + e^{-2\lambda_n} + \cdots + e^{-k\lambda_n} + \cdots) \leq \frac{\varepsilon}{3} + \frac{e^{-\lambda_n \eta}}{1 - e^{-\lambda_n}}$$

$$| u_2(t) |^2 \leq e^{-\lambda_n(t-\tau)} | u_\tau |^2 + c(\frac{1}{\lambda_n} + \frac{\varepsilon}{3} + \frac{e^{-\lambda_n \eta}}{1 - e^{-\lambda_n}}). \tag{3.8}$$

Let $T_1 = t - \tau = 1$, by (3.5), we get

$$\| U(t,\tau)u_{1\tau} - U(t,\tau)u_{2\tau} \| \leq e^c \| u_{1\tau} - u_{2\tau} \|. \tag{3.9}$$

Since $\lambda_n \to +\infty$, for $0 < \varepsilon < 1$, from (3.7) and (3.8), there exist $m \in \mathbb{Z}^+$, $\delta, \theta, > 0, \delta + \theta < 1$ such that

$$\| (I - P_m)(U(t, t-T_1)u_1 - U(t, t-T_1)u_2) \| \leq \delta \| u_1 - u_2 \|, \tag{3.10}$$

$$\| (I - P_m) \bigcup_{s \leq T_1} U(t, \tau - s)u \| \leq \theta, \forall t \geq \tau. \tag{3.11}$$

By Theorem 2.4 and (3.9)-(3.11), we know that the process $\{U(t, \tau) \,|\, t \geq \tau\}$ generated by (1.1) satisfy all the conditions of Theorem 1.1.

Funds

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