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Generalized Darboux Transformation and Rational Solutions for the Nonlocal Nonlinear Schrödinger Equation with the Self-Induced Parity-Time Symmetric Potential

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Abstract

In this paper, I construct a generalized Darboux transformation for the nonlocal nonlinear Schrödinger equation with the self-induced parity-time symmetric potential. The *N*-order rational solution is derived by the iterative rule and it can be expressed by the determinant form. In particular, I calculate first-order and second-order rational solutions and obtain their figures according to different parameters.

Keywords

Generalized Darboux Transformation, Rational Solutions, Nonlocal Nonlinear Schrödinger Equation

1. Introduction

Parity-time (PT) symmetry was firstly proposed by Bender and Boettcher in quantum mechanics [1]. And it has been widespread concerned in optical solitons [2], non-reciprocal light propagation [3], unidirectional invisibility [4], perfect absorber [5] and so on. Since then, there has been attracted more and more attentions in the non-Hermitian systems with PT symmetry [6]-[13]. Generally, the non-Hermitian Hamiltonian $H=\hat{p}^2/2+V(x)$ is deemed to be PT symmetric if $V(x)=V^*(-x)$, where $\hat{p}$ denotes the momentum operator; $V(x)$ is the complex potential [1] [6]; the asterisk denotes the complex conjugate. According to the PT symmetry condition,

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the real part of a PT symmetry complex potential must be an even function whereas the imaginary part should be odd. In optical system, the PT symmetric potential can be realized by controlling the complex refractive index distribution $n(x)=n_R(x)+in_I(x)$, where the refractive index profile $n_R(x)$ is an even function in the transverse direction, the gain or loss component $n_I(x)$ is an odd one [10]-[12].

In the nonlinear optics, the PT symmetric and Kerr nonlinearity linear potentials have been intensively researched in the nonlinear Schrödinger (NLS) equations. For example, [14] has studied the soliton in PT symmetric potential with competing nonlinearity; [15] has studied the dynamical behaviors of 2D nonautonomous solitons in PT symmetric potentials; and [16] has studied stable dark solitons in PT symmetric dual-core waveguides.

In this paper, I will consider a nonlocal NLS equation [17]

$$iu_z(x,z)=u_{xx}(x,z)+2\sigma u^*(-x,z)u(x,z),\ (\sigma=\pm1), \tag{1}$$

which is non-Hermitian but PT symmetric, where $u(x,z)$ is a complex valued function of real variables *x* and *z*, and $\sigma=\pm1$. The signs $(+)$ and $(-)$ denote the focusing and defocusing respectively. In Equation (1), the nonlinear term brings a self-induced potential of the form $V(x,z)=u(x,z)u^*(-x,z)$, which satisfies the PT symmetric condition $V(x,z)=V^*(-x,z)$. The exact moving one-soliton solution of Equation (1) has been obtained in [17] via the inverse scattering transform. The dark and antidark soliton interactions have been given in [18] via the classical Darboux transformation (DT) method. However, there are no papers on high-order rational solutions of Equation (1) by generalized Darboux transformation (gDT).

The organization of this paper is as follows: In Section 2, a determinant expression of *N*-order gDT will be constructed based on the Lax pair. In Section 3, I will obtain a general determinant expression of *N*-order rational solution of Equation (1). In addition, I calculate first-order and second-order rational solutions and obtain their figures according to different parameters. The conclusions will be given in Section 4.

2. Lax Pair and Generalized Darboux Transformation

The Lax pair of Equation (1) can be expressed as follows [17]:

$$\psi_x=U\psi,\ \psi_z=V\psi, \tag{2}$$

where $\psi=(\varphi,\phi)^{\mathrm{T}}$ is the vector eigenfunction of Lax pair (2), and *T* signifies the vector transpose. Matrices *U* and *V* have the following forms:

$$U=\begin{pmatrix}\lambda & u(x,z)\\ -\sigma u^*(-x,z) & -\lambda\end{pmatrix},$$

$$V=\begin{pmatrix}-2i\lambda^2-i\sigma u(x,z)u^*(-x,z) & -2i\lambda u(x,z)-iu_x(x,z)\\ 2i\sigma\lambda u^*(-x,z)+i\sigma u_x^*(-x,z) & 2i\lambda^2+i\sigma u(x,z)u^*(-x,z)\end{pmatrix},$$

where λ is a spectral parameter, the asterisk denotes the complex conjugate. The compatibility condition $U_z-V_x+UV-VU=0$ is equivalent to Equation (1) by a direct computation.

The classical DT for Equation (1) has been constructed in [18]:

$$\psi[1]=T[1]\psi,$$

$$u_1(x,z)=u(x,z)+2(\lambda_1-\lambda_1^*)\frac{\sigma\varphi_1(x,z)\phi_1^*(-x,z)}{\sigma\phi_1(x,z)\phi_1^*(-x,z)-\varphi_1(x,z)\varphi_1^*(-x,z)}, \tag{3}$$

where

$$T[1]=\lambda I-H_1\Lambda_1H_1^{-1},\ H_1=\begin{pmatrix}\varphi_1(x,z) & \sigma\phi_1^*(-x,z)\\ \phi_1(x,z) & \varphi_1^*(-x,z)\end{pmatrix},\ \Lambda_1=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_1^*\end{pmatrix}.$$

Here $\psi_1=(\varphi_1(x,z),\phi_1(x,z))^{\mathrm{T}}$ is an eigenfunction of Lax pair (2) with a seeding solution $u(x,z)$ and $\lambda=\lambda_1$, then $\bar{\psi}_1=(\sigma\phi_1^*(-x,z),\varphi_1^*(-x,z))^{\mathrm{T}}$ is also a solution of the Lax pair (2) with $\lambda=\lambda_1^*$ [17]. Thus I choose different eigenfunctions $\psi_i=(\varphi_i,\phi_i)^{\mathrm{T}}$ separately at $\lambda=\lambda_i$, the above DT procedure can be easily iterated. Based on Crum theorem [19], I can obtain a general case for Equation (1) in the form of determinant.

Next, I suppose $\psi_i\,(1\le i\le N)$ are N different eigenfunctions of Lax pair (2) with $\lambda=\lambda_i\,(1\le i\le N)$ then iterate the above DT N times, I obtain the N-fold DT for Equation (1) in the form of a determinant as

$$u_N(x,z)=u(x,z)+2\frac{|\omega_2|}{|\omega_1|}, \tag{4}$$

where

$$\omega_1=\begin{pmatrix} \lambda_1^{N-1}\phi_1 & \cdots & \lambda_N^{N-1}\phi_N & \lambda_1^{*N-1}\varphi_1^* & \cdots & \lambda_N^{*N-1}\varphi_N^* \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_1 & \cdots & \phi_N & \varphi_1^* & \cdots & \varphi_N^* \\ \lambda_1^{N-1}\varphi_1 & \cdots & \lambda_N^{N-1}\varphi_N & \sigma\lambda_1^{*N-1}\phi_1^* & \cdots & \sigma\lambda_N^{*N-1}\phi_N^* \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \varphi_1 & \cdots & \varphi_N & \sigma\phi_1^* & \cdots & \sigma\phi_N^* \end{pmatrix},$$

$$\omega_2=\begin{pmatrix} \lambda_1^{N}\varphi_1 & \cdots & \lambda_N^{N}\varphi_N & \sigma\lambda_1^{*N}\phi_1^* & \cdots & \sigma\lambda_N^{*N}\phi_N^* \\ \lambda_1^{N-2}\phi_1 & \cdots & \lambda_N^{N-2}\phi_N & \lambda_1^{*N-2}\varphi_1^* & \cdots & \lambda_N^{*N-2}\varphi_N^* \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_1 & \cdots & \phi_N & \varphi_1^* & \cdots & \varphi_N^* \\ \lambda_1^{N-1}\varphi_1 & \cdots & \lambda_N^{N-1}\varphi_N & \sigma\lambda_1^{*N-1}\phi_1^* & \cdots & \sigma\lambda_N^{*N-1}\phi_N^* \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \varphi_1 & \cdots & \varphi_N & \sigma\phi_1^* & \cdots & \sigma\phi_N^* \end{pmatrix},$$

$$\left(\varphi_i=\varphi_i(x,z),\phi_i=\phi_i(x,z),\varphi_i^*=\varphi_i^*(-x,z),\phi_i^*=\phi_i^*(-x,z),1\le i\le N\right).$$

In the following, I derive the determinant form of the gDT for Equation (1). Considering N different eigenfunctions $\psi_i=\left(\varphi_i(x,z),\phi_i(x,z)\right)^{\mathrm{T}}\,(i=1,2,\cdots,n)$ for the Lax pair (2) with $\lambda=\lambda_i\,(1\le i\le n)$ and Taylor expansion

$$(\lambda_i+\delta_i)^j\varphi(\lambda_i+\delta_i)=\varphi_i[j,0]+\varphi_i[j,1]\delta_i+\cdots+\varphi_i[j,m_i]\delta_i^{m_i}+\cdots,$$

$$(\lambda_i+\delta_i)^j\phi(\lambda_i+\delta_i)=\phi_i[j,0]+\phi_i[j,1]\delta_i+\cdots+\phi_i[j,m_i]\delta_i^{m_i}+\cdots,$$

where

$$\varphi_i[j,k]=\frac{1}{k!}\frac{\partial^k}{\partial\delta_i^k}\left[(\lambda_i+\delta_i)^j\varphi_i(\lambda_i+\delta_i)\right]\Big|_{\delta_i=0},$$

$$\phi_i[j,k]=\frac{1}{k!}\frac{\partial^k}{\partial\delta_i^k}\left[(\lambda_i+\delta_i)^j\phi_i(\lambda_i+\delta_i)\right]\Big|_{\delta_i=0}$$

$$\left(j=0,1,\cdots,N,k=1,2,\cdots,m_i,N=n+\sum_{i=1}^{n}m_i\right).$$

Thus, on the basis of the work in [20] [21], I can perform the limit on Formula (4), then obtain the following result:

$$u_N(x,z)=u(x,z)+2\frac{|\Delta_2|}{|\Delta_1|},\ \Delta_1=\det\left(\left[\Delta_{1,1},\cdots,\Delta_{1,n}\right]\right),\ \Delta_2=\det\left(\left[\Delta_{2,1},\cdots,\Delta_{2,n}\right]\right), \tag{5}$$

where

$$\Delta_{1,i}=\begin{pmatrix} \lambda_i^{N-1}\phi_i & \cdots & \phi_i[N-1,m_i] & \lambda_i^{*N-1}\varphi_i^* & \cdots & \varphi_i^*[N-1,m_i] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_i & \cdots & \phi_i[0,m_i] & \varphi_i^* & \cdots & \varphi_i^*[0,m_i] \\ \lambda_i^{N-1}\varphi_i & \cdots & \varphi_i[N-1,m_i] & \sigma\lambda_i^{*N-1}\phi_i^* & \cdots & \sigma\phi_i^*[N-1,m_i] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \varphi_i & \cdots & \varphi_i[0,m_i] & \sigma\phi_i^* & \cdots & \sigma\phi_i^*[0,m_i] \end{pmatrix},$$

$$\Delta_{2,i}=\begin{pmatrix} \lambda_i^{N}\varphi_i & \cdots & \varphi_i[N,m_i] & \sigma\lambda_i^{*N}\phi_i^* & \cdots & \sigma\phi_i^*[N,m_i] \\ \lambda_i^{N-2}\phi_i & \cdots & \phi_i[N-2,m_i] & \lambda_i^{*N-2}\varphi_i^* & \cdots & \varphi_i^*[N-2,m_i] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_i & \cdots & \phi_i[0,m_i] & \varphi_i^* & \cdots & \varphi_i^*[0,m_i] \\ \lambda_i^{N-1}\varphi_i & \cdots & \varphi_i[N-1,m_i] & \sigma\lambda_i^{*N-1}\phi_i^* & \cdots & \sigma\phi_i^*[N-1,m_i] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \varphi_i & \cdots & \varphi_i[0,m_i] & \sigma\phi_i^* & \cdots & \sigma\phi_i^*[0,m_i] \end{pmatrix},$$

$$\left(\varphi_i=\varphi_i(x,z),\phi_i=\phi_i(x,z),\varphi_i^*=\varphi_i^*(-x,z),\phi_i^*=\phi_i^*(-x,z)\right).$$

3. Rational Solutions

To construct the rational solutions of Equation (1), I take a plane wave solution

$$u_0=a\mathrm{e}^{i\omega z},\tag{6}$$

where a is real constant, and the frequency ω satisfies the nonlinear dispersion relation

$$\omega=-2a^2\sigma.\tag{7}$$

Then inserting Equation (7) into the Lax pair (2) and taking $\lambda=\lambda_1$, I obtain

$$\psi=\begin{pmatrix}\varphi\\ \phi\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\left[\left(c_1+\frac{\lambda_1c_1}{s}+\frac{ac_2}{s}\right)\mathrm{e}^{s\chi}+\left(c_1-\frac{\lambda_1c_1}{s}-\frac{ac_2}{s}\right)\mathrm{e}^{-s\chi}\right]\mathrm{e}^{\frac{1}{2}i\omega z}\\ \frac{1}{2}\left[\left(c_2-\frac{\lambda_1c_2}{s}-\frac{a\sigma c_1}{s}\right)\mathrm{e}^{s\chi}+\left(c_2+\frac{\lambda_1c_2}{s}+\frac{a\sigma c_1}{s}\right)\mathrm{e}^{-s\chi}\right]\mathrm{e}^{-\frac{1}{2}i\omega z}\end{pmatrix},\tag{8}$$

with

$$s=\sqrt{\lambda_1^2-a^2\sigma},\ \ \chi=x-2i\lambda_1z,$$

where c_1 and c_2 are both complex constants. In order to obtain the rational solitonic structure, I must impose s to be real numbers, which is satisfied only when $\sigma=-1$, $\mathrm{Re}(\lambda_1)=0$. I point out that, a special seed solution and suitable eigenvalue enable us to get higher rational solutions in determinant forms according to Formula (5). In the following discussions, I may set $a=1$ to simply our calculation process. Then set $\lambda_1=i(1+f^2)$ in Formula (8), I obtain

$$\varphi_1(x,z)=l_1\mathrm{e}^{A}+l_2\mathrm{e}^{-A},\ \ \phi_1(x,z)=l_3\mathrm{e}^{A}+l_4\mathrm{e}^{-A},\tag{9}$$

where

$$l_1=\frac{c_1}{2}+\frac{(1+f^2)c_1}{2f\sqrt{f^2+2}}+\frac{c_2}{2if\sqrt{f^2+2}},\ \ l_2=\frac{c_1}{2}-\frac{(1+f^2)c_1}{2f\sqrt{f^2+2}}-\frac{c_2}{2if\sqrt{f^2+2}},$$

$$l_3=\frac{c_2}{2}-\frac{(1+f^2)c_2}{2f\sqrt{f^2+2}}+\frac{c_1}{2if\sqrt{f^2+2}},\ \ l_4=\frac{c_2}{2}+\frac{(1+f^2)c_2}{2f\sqrt{f^2+2}}-\frac{c_1}{2if\sqrt{f^2+2}},$$

$$A = if\sqrt{f^2+2}\left[x+2\left(1+f^2\right)z+\Phi(f)\right],\ \ \Phi(f)=\sum_{i=0}^{N} s_i f^{2i},\ \ s_i \in C.$$

The relevant Taylor expansions are

$$(i)^j\left(1+f^2\right)^j \varphi_1(f) = \varphi_1[j,0]+\varphi_1[j,1]f^2+\cdots+\varphi_1[j,N]f^{2N}+\cdots,$$

$$(i)^j\left(1+f^2\right)^j \phi_1(f) = \phi_1[j,0]+\phi_1[j,1]f^2+\cdots+\phi_1[j,N]f^{2N}+\cdots,$$

where

$$\varphi_1[j,k]=\frac{1}{(2k)!}\frac{\partial^{2k}}{\partial f^{2k}}\left[(i)^j\left(1+f^2\right)^j\varphi_1(f)\right]\Bigg|_{f=0},$$

$$\phi_1[j,k]=\frac{1}{(2k)!}\frac{\partial^{2k}}{\partial f^{2k}}\left[(i)^j\left(1+f^2\right)^j\phi_1(f)\right]\Bigg|_{f=0},$$

$$\left(j=0,1,\cdots,N, k=1,2,\cdots,N-1\right).$$

It follows that the *N*-order rational solution for Equation (1), reads

$$u_N(x,z)=\left(1+2\frac{|\Omega_2|}{|\Omega_1|}\right)\mathrm{e}^{i\omega z}, \tag{10}$$

where

$$\Omega_1=\begin{pmatrix} i^{N-1}\phi_1 & \cdots & \phi_1[N-1,N-1] & (-i)^{N-1}\varphi_1^* & \cdots & \varphi_1^*[N-1,N-1] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_1 & \cdots & \phi_1[0,N-1] & \varphi_1^* & \cdots & \varphi_1^*[0,N-1] \\ i^{N-1}\varphi_1 & \cdots & \varphi_1[N-1,N-1] & -(-i)^{N-1}\phi_1^* & \cdots & -\phi_1^*[N-1,N-1] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \varphi_1 & \cdots & \varphi_1[0,N-1] & -\phi_1^* & \cdots & -\phi_1^*[0,N-1] \end{pmatrix},$$

$$\Omega_2=\begin{pmatrix} i^{N}\varphi_1 & \cdots & \varphi_1[N,N-1] & -(-i)^{N}\phi_1^* & \cdots & -\phi_1^*[N,N-1] \\ i^{N-2}\phi_1 & \cdots & \phi_1[N-2,N-1] & (-i)^{N-2}\varphi_1^* & \cdots & \varphi_1^*[N-2,N-1] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_1 & \cdots & \phi_1[0,N-1] & \varphi_1^* & \cdots & \varphi_1^*[0,N-1] \\ i^{N-1}\varphi_1 & \cdots & \varphi_1[N-1,N-1] & -(-i)^{N-1}\phi_1^* & \cdots & -\phi_1^*[N-1,N-1] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \varphi_1 & \cdots & \varphi_1[0,N-1] & -\phi_1^* & \cdots & -\phi_1^*[0,N-1] \end{pmatrix},$$

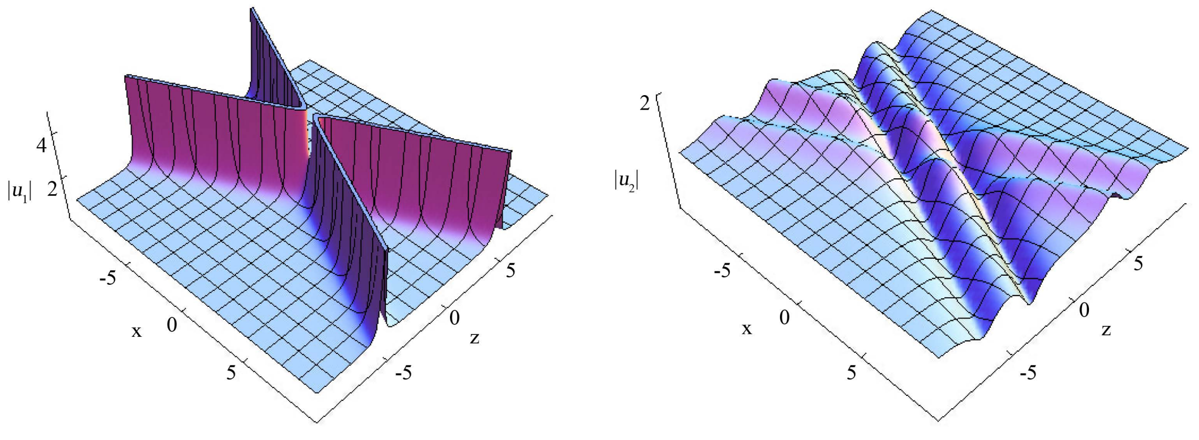
$$\left(\varphi_i=\varphi_i(x,z),\phi_i=\phi_i(x,z),\varphi_i^*=\varphi_i^*(-x,z),\phi_i^*=\phi_i^*(-x,z)\right).$$

Setting $N = 1$ in Formula (10), then I obtain the first-order rational solution (see **Figure 1(a)**) with the parameters $c_1=c_2=i$, $\Phi(f)=0$ as follows

$$u_1(x,z)=\mathrm{e}^{2iz}\left[1+\frac{2i+4x^2-(8-8i)z-16z^2}{1-2x^2+4z+8z^2}\right]. \tag{11}$$

Then with $N = 2$, the second-order rational solution (see **Figure 1(b)**) with the parameters $c_1=c_2=i$, $\Phi(f)=50if^2$ is obtained, namely,

$$u_2(x,z)=\left(1+\frac{G_1+izG_2}{H}\right)\mathrm{e}^{iz}, \tag{12}$$



(a) (b)

Figure 1. (a) Plot of the first-order rational solution; (b) Plot of the second-order rational solution.

where

$$G_1 = -\frac{64}{3}\Big[-3i + (600-600i) - (12+6i)x^2 + (4+4i)x^4 + 36z + 2400xz - 48x^2z + 192z^2 - 96x^2z^2 + 320z^3 + 320z^4\Big],$$

$$G_2 = -\frac{512}{3}\left(-3 + 2x^4 + 3z - 12x^2z + 24z^2 - 16x^2z^2 + 40z^3 + 32z^4\right),$$

$$H = -\frac{16}{9}\Big[-90009 + 2400ix^3 + 16x^6 - 72z - 864z^2 - 1728z^3 - 2688z^4 - 1536z^5 - 1024z^6 + \left(3600ix - 24x^4\right)\left(1 + 4z + 8z^2\right) + 24x^2\left(3 + 6z + 32z^3 + 32z^4\right)\Big].$$

4. Conclusion

In this paper, I have studied the nonlocal nonlinear Schrödinger equation with the self-induced parity-time-symmetric potential. Then I have constructed a gDT for Equation (1) and derived the *N*-fold rational solutions in determinant forms. In particular, I have calculated first-order and second-order rational solutions from a plane-wave solution and obtained their figures according to different parameters.

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