

Existence of Viscosity Solutions to a Parabolic Inhomogeneous Equation Associated with Infinity Laplacian

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Abstract

In this paper, we obtain the existence result of viscosity solutions to the initial and boundary value problem for a nonlinear degenerate parabolic inhomogeneous equation of the form $u_t - \Delta_\infty u = f$, where Δ_∞ denotes infinity Laplacian given by $\Delta_\infty u = \langle D^2 u Du, Du \rangle$.

Keywords

Infinity Laplacian, Viscosity Solution, Inhomogeneous Equation

1. Introduction

In this paper, we consider the nonlinear degenerate parabolic inhomogeneous equation involving infinity Laplace

$$u_t - \Delta_\infty u = f, \quad (1)$$

where

$$\Delta_\infty u = \langle D^2 u Du, Du \rangle$$

denotes the 3-homogeneous infinity Laplacian. We want to establish the existence result of viscosity solutions to the initial and Dirichlet boundary problem.

The homogeneous infinity Laplace equation $\Delta_\infty u = 0$ is the Euler-Lagrange equation associated with L^∞ -variational problem. See for details [1]-[5] and the references therein. Recently, Juutinen and Kawohl [6] con-

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sidered the degenerate and singular parabolic equation

$$u_t = \frac{1}{|Du|^2} \Delta_\infty u. \tag{2}$$

They proved the existence and uniqueness for both Dirichlet and Cauchy problems, established interior and boundary Lipschitz estimates and a Harnack inequality, and also provided numerous explicit solutions. Due to the degeneracy and the singularity of the Equation (2), they introduced the approximating equations to obtain the existence result with the aid of the uniform continuity estimates. And in [7] we considered the corresponding inhomogeneous parabolic equation. Notice that the 1-homogeneous infinity Laplacian

$$\frac{1}{|Du|^2} \Delta_\infty u \tag{3}$$

is related to game theory named tug-of-war [8]. In [9]-[12], Akagi, Suzuki, *et al.* considered the following degenerate parabolic equation

$$u_t = \Delta_\infty u.$$

They also introduced the corresponding approximating equations and got the uniform continuity estimates of approximate solutions by the barrier function arguments. By this approximate procedure, the existence of the solutions was obtained. They also proved the uniqueness and the asymptotic behavior of the viscosity solutions. In [13], Portilheiro and Vázquez considered the parabolic equation

$$u_t = \frac{1}{|Du|^h} \Delta_\infty u,$$

with $h < 2$. They proved existence and uniqueness of viscosity solutions and derived the asymptotic behavior of the solutions for the Cauchy problem and the initial and Dirichlet problem with zero boundary conditions. In [14], Portilheiro and Vázquez studied the nonlinear porous medium type equation involving the infinity Laplacian operator

$$u_t = \Delta_\infty^N (u^m). \tag{4}$$

By the density-to-pressure transformation, they transformed the Equation (4) into a new equation, then the existence, uniqueness and asymptotic behavior etc. were obtained.

In this article, we are interested in the parabolic version of the infinity Laplacian here. We think that Equation (1) is interesting, because it not only is degenerate, but also has many applications in image processing and optimal transportation etc. The parabolic equation involving infinity Laplacian operator has received a lot of attention in the last decade, notably due to its application to image processing, the main usage being in the reconstructions of damaged digital images [15]. For numerical purposes it has been necessary to consider also the evolution equation corresponding to the infinity Laplace operator. We prove the existence of viscosity solutions to the initial-Dirichlet problem by approximating procedure. The approximation process is introduced in [6] for the infinity Laplacian evolution and followed in [9] [13] [14] etc.

This paper is organized in the following order. In Section 2, we give the notations, definitions of viscosity solutions related to the Equation (1). In Section 3, we prove our main existence result by approximating procedure.

2. Preliminaries

Throughout of this paper, we use the following notation: If $\Omega \subset \mathbb{R}^n$, $Q_T = \Omega \times \{0, T\}$, $\partial_l Q_T = \partial\Omega \times [0, T)$ denotes the lateral boundary, $\partial_b Q_T = \Omega \times \{t = 0\}$ the bottom boundary, and $\partial_p Q_T = \partial_l Q_T \cup \partial_b Q_T$ (the parabolic boundary of Q_T). $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the largest and the smallest of the eigenvalues to a symmetric matrix. $C^{2,1}(Q_T)$ denotes those functions which are twice differentiable in x and once in t .

In the following paper, we adopt the definition of viscosity solutions, (see for example [16]).

Definition 2.1. Suppose that $u : Q_T \rightarrow \mathbb{R}$ is upper semi-continuous. If for every $(x_0, t_0) \in Q_T$ and $C^{2,1}(Q_T)$ test function φ such that $u - \varphi$ has a strict local maximum at point (x_0, t_0) , that is $u(x_0, t_0) = \varphi(x_0, t_0)$

and $u(x,t) < \varphi(x,t)$ in a neighborhood of (x_0, t_0) , there holds

$$\varphi_t(x_0, t_0) - \Delta_x \varphi(x_0, t_0) \leq f(x_0, t_0), \tag{5}$$

then we say that u is a viscosity sub-solution of (1).

Similarly, $u : Q_T \rightarrow R$ is lower semi-continuous. If for every $(x_0, t_0) \in Q_T$ and $C^{2,1}(Q_T)$ test function φ such that $u - \varphi$ has a strict local minimum at point (x_0, t_0) , there holds

$$\varphi_t(x_0, t_0) - \Delta_x \varphi(x_0, t_0) \geq f(x_0, t_0)$$

then we say that u is a viscosity super-solution of (1).

If $u \in C(Q_T)$ is both a viscosity sub-solution and a viscosity super-solution, then we say that u is a viscosity solution of (1).

3. Existence Theorem

In this section we will prove the existence of viscosity solutions to (1) with the initial and boundary data g . The method we adopt is the approximation procedure introduced in [6] and used in [9] [13] [14] etc. The main existence result we obtain is.

Theorem 3.1. *Let $Q_T = \Omega \times (0, T)$, where $\Omega \subset R^n$ is a bounded domain, f is continuous in \bar{Q}_T , and let $g \in C(R^{n+1})$. Then there exists a function $u \in C(\bar{Q}_T)$ such that $u = g$ on $\partial_p Q_T$ and*

$$u_t - \Delta_x u = f \tag{6}$$

in Q_T in the viscosity sense.

We use the approximate procedure, cf, [6] [9] [14]. We consider the approximating equations

$$u_t - \mathcal{L}^\varepsilon u = f, \tag{7}$$

where

$$\mathcal{L}^\varepsilon u = \left(\varepsilon + |Du|^2 \right) \Delta u + \left\langle D^2 u Du, Du \right\rangle \tag{8}$$

with $0 < \varepsilon < 1$. For this equation with smooth initial and boundary data $g(x, t)$, the existence of a smooth solution u_ε is guaranteed by classical results in [17]. Our goal is to obtain a solution of (1) as a limit of these functions as $\varepsilon \rightarrow 0$. This amounts to proving uniform estimates for u_ε that are independent of $0 < \varepsilon < 1$. The estimates we require will be obtained by using the standard barrier method.

Theorem 3.2. *(Boundary regularity at $t = 0$) Let $Q_T = \Omega \times (0, T)$, where $\Omega \subset R^n$ is a bounded domain, f is continuous in \bar{Q}_T , and let $g \in C^{2,1}(\bar{Q}_T)$. Suppose that $u = u_\varepsilon$ is a smooth solution satisfying*

$$\begin{cases} u_t - \mathcal{L}^\varepsilon u = f, & \text{in } Q_T, \\ u = g, & \text{on } \partial_p Q_T. \end{cases}$$

Then there exists a constant $C \geq 0$ depending on $\|g_t\|_\infty$, $\|D^2 g\|_\infty$ and $\|f\|_\infty$ but independent of $0 < \varepsilon < 1$ such that

$$|u(x, t) - g(x, 0)| \leq Ct \text{ in } Q_T.$$

Moreover, if g is only continuous in x (possibly discontinuous in t) and bounded in t , then the modulus of continuity of u on $\Omega \times [0, \bar{T}]$ (for small \bar{T}) can be estimated in terms of $\|g\|_\infty$, $\|f\|_\infty$ and the modulus of continuity of $g(x, 0)$ in x .

Proof. Step 1. Suppose first that $g \in C^{2,1}(\bar{Q}_T)$ and we consider the upper barrier function

$$w^+(x, t) = g(x, 0) + \lambda t,$$

where $\lambda > 0$ is to be determined. We have

$$\begin{aligned} w_t^+ - \mathcal{L}^\varepsilon w^+ &= \lambda - \left[\varepsilon \Delta g(x, 0) + \left\langle D^2 g(x, 0) Dg(x, 0), Dg(x, 0) \right\rangle \right] \\ &\geq \lambda - (1 + \varepsilon n) \|D^2 g(x, 0)\|_\infty \geq \|f\|_\infty \geq f(x, t), \end{aligned}$$

if $\lambda \geq (1+n)\|D^2 g\|_\infty + \|f\|_\infty$. Therefore w^+ is a super-solution.

Clearly, $w^+(x,0) = g(x,0)$ for all $x \in \Omega$. Moreover, for $x \in \partial\Omega$ and $t > 0$,

$$w^+(x,t) = g(x,0) + \lambda t \geq g(x,t) + (\lambda - \|g_t\|_\infty)t \geq g(x,t),$$

if $\lambda \geq \|g_t\|_\infty$. That is, $w^+ \geq u$ on $\partial_p Q_T$.

Thus, by the classical comparison principle, we obtain

$$u(x,t) \leq w^+(x,t) = g(x,0) + \lambda t$$

for every $(x,t) \in Q_T$. Similarly, by considering also the lower barrier function

$$w^-(x,t) = g(x,0) - \lambda t,$$

we obtain the symmetric inequality, and hence the Lipschitz estimate

$$|u(x,t) - g(x,0)| \leq Ct \tag{9}$$

for $0 < t < T$ and

$$C = \max\{\|g_t\|_\infty, (1+n)\|D^2 g\|_\infty + \|f\|_\infty\}.$$

Step 2. Suppose now that g is only continuous in x and let $\mu(\cdot)$ be its modulus of continuity. Let us fix a point $x_0 \in \Omega$ and $0 < \tau < \min\{\text{dist}(x_0, \partial\Omega), 2\sqrt{\|g\|_\infty}\}$. Let us consider the smooth functions

$$g^\pm(x,t) = g(x_0,0) \pm \mu(\tau) \pm \frac{2\|g\|_\infty}{\tau^2}|x-x_0|^2.$$

It is easy to check that $g^- \leq g \leq g^+$ on the parabolic boundary $\partial_p Q_T$.

Thus if u^\pm are the unique classical solutions to (7) with boundary and initial data g^\pm , respectively, we have $u^- \leq u \leq u^+$ in Q_T by the classical comparison principle again. Since g^\pm are smooth, we can use estimate (9) to conclude that

$$|u^\pm(x_0,t) - g^\pm(x_0,0)| \leq \bar{C}t,$$

where \bar{C} depends on τ , $\|f\|_\infty$ and $\|g\|_\infty$. Therefore,

$$\begin{aligned} & |u(x_0,t) - g(x_0,0)| \\ & \leq |u(x_0,t) - u^\pm(x_0,t)| + |u^\pm(x_0,t) - g^\pm(x_0,0)| + |g^\pm(x_0,0) - g(x_0,0)| \\ & \leq |u^+(x_0,t) - u^-(x_0,t)| + \bar{C}t + \mu(\tau) \\ & \leq |u^+(x_0,t) - g^+(x_0,0)| + |g^+(x_0,0) - g^-(x_0,0)| + |g^-(x_0,0) - u^-(x_0,t)| + \bar{C}t + \mu(\tau) \\ & \leq 3(\bar{C}t + \mu(\tau)) \end{aligned}$$

with this inequality it is straightforward to complete the proof. \square

The full Lipschitz estimate in time now follows easily with the aid of the comparison principle and the fact that the Equation (7) is translation invariant.

Corollary 3.3. (Lipschitz regularity in time) *If f is continuous in \bar{Q}_T , $g \in C^{2,1}(\bar{Q}_T)$ and $u = u_\varepsilon$ is as in Theorem 3.2, then there exists a constant $C \geq 0$ depending on $\|f\|_\infty$, $\|g_t\|_\infty$ and $\|D^2 g\|_\infty$ but independent of $0 < \varepsilon \leq 1$ such that*

$$|u(x,t) - u(x,s)| \leq C|t-s|$$

for all $x \in \Omega$ and $t, s \in (0, T)$. Moreover, if g is only continuous, then the modulus of continuity of u on Q_T can be estimated in terms of $\|g\|_\infty$, $\|f\|_\infty$ and the modulus of continuity of g .

Proof. Let $\tilde{u}(x,t) = u(x,t+\sigma)$, $\sigma > 0$. Then both u and \tilde{u} are smooth solutions to (7) in $Q_\sigma = \Omega \times (0, T - \sigma)$,

and hence if $g \in C^{2,1}(\bar{Q}_T)$, we have

$$\begin{aligned} \sup_{Q_\sigma} |u - \tilde{u}| &= \sup_{\partial_p Q_\sigma} |u - \tilde{u}| \\ &\leq \max \left\{ \|u(\cdot, \sigma) - u(\cdot, 0)\|_\infty, \sup_{x \in \partial\Omega} \|u(x, \cdot + \sigma) - u(x, \cdot)\|_{(0, T-\sigma)} \right\} \\ &\leq \max \{C\sigma, \|g_t\|_\infty \sigma\} = C\sigma \end{aligned}$$

by the classical comparison principle and Theorem 3.2. This implies the Lipschitz estimate asserted above, and the proof for the case when g is only continuous is analogous. \square

Theorem 3.4. (Hölder regularity at the lateral boundary) Let $Q_T = \Omega \times (0, T)$, where $\Omega \subset R^n$ is a bounded domain, f is continuous in \bar{Q}_T , and let $g \in C^{2,1}(\bar{Q}_T)$. Suppose that $u = u_\varepsilon$ is a smooth solution satisfying

$$\begin{cases} u_t - \mathcal{L}^\varepsilon u = f, & \text{in } Q_T, \\ u = g, & \text{on } \partial_p Q_T. \end{cases}$$

Then for each $0 < \alpha < 1$, there exists a constant $C_* \geq 1$ depending on $\alpha, \|g\|_\infty, \|g_t\|_\infty, \|f\|_\infty$ and $\|Dg\|_\infty$ but independent of ε and δ sufficiently small such that

$$|u(x, t_0) - g(x_0, t_0)| \leq C_* |x - x_0|^\alpha,$$

for all $(x_0, t_0) \in \partial\Omega \times (0, T)$ and $x \in \Omega \cap B_1(x_0)$.

Proof. Step 1. For every $(x_0, t_0) \in \partial\Omega \times (0, T)$ and $0 < \alpha < 1$, let

$$w^+(x, t) = g(x_0, t_0) + C_* |x - x_0|^\alpha + \lambda(t_0 - t),$$

where $C_* \geq 1, \lambda > 0$ are to be determined. Then a straightforward computation gives

$$w_t^+ - \mathcal{L}^\varepsilon w^+ = -\lambda - C_*^2 \alpha |x - x_0|^{\alpha-2} \left[C_* (\varepsilon + \alpha^2 C_*^2 |x - x_0|^{2\alpha-2}) (n + \alpha - 2) + C_*^2 \alpha^2 (\alpha - 1) |x - x_0|^{2\alpha-2} \right].$$

If $|x - x_0| \leq 1$ and $C_* \geq 1$, we have

$$-\varepsilon(n + \alpha - 2) - \alpha^2(\alpha - 1) \geq \alpha^2 \frac{1 - \alpha}{10}.$$

Therefore

$$w_t^+ - \mathcal{L}^\varepsilon w^+ \geq -\lambda + C_*^2 \alpha |x - x_0|^{\alpha-2} \cdot \frac{\alpha^2(1 - \alpha)}{10} \geq -\lambda + C_*^2 \alpha^3 \frac{1 - \alpha}{10} \geq \|f\|_\infty \geq f(x, t),$$

if $C_*^2 \geq \frac{10}{\alpha^3(1 - \alpha)} (1 + \lambda + \|f\|_\infty)$.

We have shown that w^+ is a super-solution of (7).

Step 2. Let $Q_* = (\Omega \cap B_1(x_0)) \times (t_0 - t_*, t_0)$, where $t_* = \min\{1, t_0\}$. We want to prove first $w^+ \geq u$ on $\partial_l Q^*$. Case 1. If $x \in \partial\Omega \cap B_1(x_0)$, then

$$\begin{aligned} w^+(x, t) &= g(x_0, t_0) + C_* |x - x_0|^\alpha + \lambda(t_0 - t) \\ &\geq g(x_0, t_0) + C_* |x - x_0| + \lambda(t_0 - t) \\ &\geq -\|g_t\|_\infty (t_0 - t) - \|Dg\|_\infty |x - x_0| + g(x, t) + C_* |x - x_0| + \lambda(t_0 - t) \\ &\geq g(x, t) = u(x, t), \end{aligned}$$

provided $C_* \geq \|Dg\|_\infty$ and $\lambda \geq \|g_t\|_\infty$.

Case 2. If $x \in \Omega \cap \partial B_1(x_0)$, it is easy to see that $v(x, t) = \|f\|_\infty t + \|g\|_\infty$ is a super-solution of (7) in Q_T and $v \geq g$ on $\partial_p Q_T$. Hence, we have

$$\begin{aligned} w^+(x,t) &= g(x_0,t_0) + C_* + \lambda(t_0 - t) \geq g(x_0,t_0) + C_* \\ &\geq \|g\|_\infty + \|f\|_\infty T \geq \|g\|_\infty + \|f\|_\infty t \\ &= v(x,t) \geq u(x,t), \end{aligned}$$

provided $C_* \geq 2\|g\|_\infty + \|f\|_\infty T$, and in the last inequality we have used the comparison principle.

Step 3. To prove $w^+ \geq u$ on $\partial_p Q^*$.

Case 1. If $t_* = t_0$, then $Q_* = (\Omega \cap B_1(x_0)) \times (0, t_0)$, and notice that since $u = g$ on the bottom of this cylinder,

$$\begin{aligned} w^+(x,0) &= g(x_0,t_0) + C_* |x - x_0|^\alpha + \lambda t_0 \geq g(x_0,t_0) + C_* |x - x_0| + \lambda t_0 \\ &\geq g(x_0,t_0) + \|Dg\|_\infty |x - x_0| + \|g_t\| t_0 \geq g(x,0), \end{aligned}$$

if $C_* \geq \|Dg\|_\infty$ and $\lambda \geq \|g_t\|_\infty$.

Case 2. If $t_* = t_0 - 1$, then $Q_* = (\Omega \cap B_1(x_0)) \times (t_0 - 1, t_0)$. Using the comparison principle again, we have

$$\begin{aligned} w^+(x,t_0 - 1) &= g(x_0,t_0) + C_* |x - x_0|^\alpha + \lambda \geq g(x_0,t_0) + \lambda \\ &\geq \|g\|_\infty + \|f\|_\infty (T - 1) \geq \|g\|_\infty + \|f\|_\infty (t_0 - 1) = v(x,t_0 - 1) \geq u(x,t_0 - 1), \end{aligned}$$

if $\lambda \geq 2\|g\|_\infty + \|f\|_\infty (T - 1)$.

Step 4. In conclusion, we have shown that $w^+ \geq u$ on $\partial_p Q_*$, if we choose

$$\begin{aligned} \lambda &\geq \max \{ \|g_t\|_\infty, 2\|g\|_\infty + \|f\|_\infty (T - 1) \}, \\ C_* &\geq \max \left\{ \|Dg\|_\infty, 2\|g\|_\infty + \|f\|_\infty T, \frac{10}{\alpha(1-\alpha)} (1 + \lambda + \|f\|_\infty) \right\}. \end{aligned}$$

Therefore, we have $w^+ \geq u$ in Q_* by the comparison principle. In particular,

$$u(x,t_0) \leq w^+(x,t_0) = g(x_0,t_0) + C_* |x - x_0|^\alpha$$

for $x \in \Omega \cap B_1(x_0)$. Using the lower barrier

$$w^-(x,t) = g(x_0,t_0) - C_* |x - x_0|^\alpha - \lambda(t_0 - t),$$

we get the symmetric inequality. This finishes the proof. \square

Due to the translation invariant of the equation and the comparison principle, we can extend the Hölder estimate to the interior of the domain, cf. [6] [14] etc.

Corollary 3.5. (Hölder regularity in space) Let $Q_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, f is continuous in \bar{Q}_T , and let $g \in C^{2,1}(\bar{Q}_T)$. Suppose that $u = u_\varepsilon$ a smooth solution satisfying

$$\begin{cases} u_t = \mathcal{L}^\varepsilon u, & \text{in } Q_T, \\ u = g, & \text{on } \partial_p Q_T. \end{cases}$$

Then there exists $0 < \alpha < 1$, and constants $C \geq 1$, depending on α , $\|f\|_\infty$, $\|g\|_\infty$, $\|g_t\|_\infty$ and $\|Dg\|_\infty$ but independent of ε and δ sufficiently small such that

$$|u(x,t) - u(y,t)| \leq C|x - y|^\alpha,$$

for all $(x, y) \in \Omega$.

Proof. Step 1. For fixed $0 < t < T$, take a point $z \in B_1(0)$ and let $V = \Omega \cap \Omega_z$. Define $u_z(x,t) = u(x - z, t)$. By Theorem 3.4 we have that $|u(x,t) - u_z(x,t)| \leq \bar{C}|z|^\alpha$ on $\partial_t(V \times (0, T))$ (noting that in this case $x \in \Omega$ or $x - z \in \Omega$). Hence $|u(x,t) - u_z(x,t)| \leq \bar{C}|z|^\alpha$ for every $x \in V$ by the comparison principle. This means that whenever $x, y \in \Omega \cap \Omega_{x-y}$ or $x, y \in \Omega \cap \Omega_{y-x}$ with $|x - y| \leq r$, we have $|u(x,t) - u(y,t)| \leq \bar{C}|x - y|^\alpha$.

Step 2. When $|x - y| \geq 1$, using the comparison principle we get

$$|u(x, t) - u(y, t)| \leq 2\|g\|_\infty \leq 2\|g\|_\infty |x - y|^\alpha = C|x - y|^\alpha.$$

This finishes the proof. \square

The following theorem shows that one can obtain the Lipschitz estimate when one remove the Laplacian term from the equation, cf. [6].

Theorem 3.1 follows now easily from Theorem 3.2 and 3.3 and the stability properties of viscosity solutions.

Proof. (Proof of Theorem 3.1) If $g \in C^{2,1}(\bar{Q}_T)$ and u_ε is the unique smooth solution to

$$\begin{cases} u_t - \mathcal{L}^\varepsilon u = f(x, t), & \text{in } Q_T, \\ u = g, & \text{on } \partial_p Q_T, \end{cases}$$

Corollaries 3.3 and 3.5 and the comparison principle imply that the family of functions $\{u_\varepsilon\}$ is equicontinuous and uniformly bounded. Therefore, up to a subsequence, $u_\varepsilon \rightarrow u$ as $\varepsilon \rightarrow 0$ and u is the unique viscosity solution to (7) by the stability properties of viscosity solutions.

The existence for a general continuous data g follows by approximating the data by smooth functions and using Corollaries 3.3 and 3.5 and the stability properties of viscosity solutions again. \square

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