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Some New Delay Integral Inequalities Based on Modified Riemann-Liouville Fractional Derivative and Their Applications

Zhimin Zhao, Run Xu

Department of Mathematics, Qufu Normal University, Qufu, China
Email: 782493982@qq.com, xurun2005@163.com

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Abstract

By using the properties of modified Riemann-Liouville fractional derivative, some new delay integral inequalities have been studied. First, we offered explicit bounds for the unknown functions, then we applied the results to the research concerning the boundness, uniqueness and continuous dependence on the initial for solutions to certain fractional differential equations.

Keywords

Modified, Riemann-Liouville, Fractional Derivative, Integral Inequalities, Delay Fractional Differential Equation

1. Introduction

The common differential and integral inequalities are playing an important role in the qualitative analysis of differential equations. At the same time, delay integral and differential inequality have been studied due to their wide applications [1]-[3]. In recent years, the fractional differential and fractional integrals are adopted in various fields of science and engineering. In addition, the fractional differential inequalities have also been studied [4]-[10]. We also need to study the delay differential equation and delay differential inequalities when dealing with certain problems. However, to the best of our knowledge, very little is known regarding this problem [11]. In this paper, we will investigate some delay integral inequalities.

In 2008, Zhiling Yuan, *et al.* [3] studied the following form delay integral inequality

$$x^p(t) \le a(t) + b(t)\int_0^t \left[f(s)x^q(\sigma(s)) + g(s)x^r(s) + \int_0^s h(\tau)x^m(\tau)\,\mathrm{d}\tau \right]\mathrm{d}s, \quad t \in R_+ \tag{1}$$

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then they offered an explicit estimate for $x(t)$, and applied this result to research the properties of solution to certain differential equations.

In 2013, Bin Zheng and Qinghua Feng [6] put forward the following form of fractional integral inequality

$$u^p(t) \le a(t) + \frac{b(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[g(s) u^q(s) + \int_0^s h(\xi) u^r(\xi) \mathrm{d}\xi \right] \mathrm{d}s, \quad t \ge 0, \tag{2}$$

and they applied the obtained results to study the properties of solution $u(t)$.

In this paper, combining (1) and (2), we will explore the following form of delay integral inequality

$$u^p(t) \le k(t) + \frac{b(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s) u^q(\sigma(s)) + g(s) u^r(s) + \int_0^s h(\tau) u^m(\tau) \mathrm{d}\tau \right] \mathrm{d}s, \quad t \ge 0. \tag{3}$$

Now we list some Definitions and Lemmas which can be used in this paper.

Definition 1. [6] The modified Riemann-Liouville derivative of order α is defined by

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_0^t (t-\xi)^{-\alpha} \left(f(\xi) - f(0) \right) \mathrm{d}\xi, & 0 < \alpha < 1, \\ \left(f^{(n)}(t) \right)^{(\alpha-n)}, & n \le \alpha < n+1, \quad n \ge 1. \end{cases}$$

Definition 2. [6] The Riemann-Liouville fractional integral of order α on the interval $[0,t]$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s.$$

Some important properties for the modified Riemann-Liouville derivative and fractional integral are listed as follows [6] (the interval concerned below is always defined by $[0,t]$).

$$(1) \quad D_t^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha},$$

$$(2) \quad D_t^\alpha \left(f(t) g(t) \right) = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t),$$

$$(3) \quad D_t^\alpha f\left[g(t) \right] = f_g'\left[g(t) \right] D_t^\alpha g(t) = D_g^\alpha f\left[g(t) \right] \left(g'(t) \right)^\alpha,$$

$$(4) \quad I^\alpha \left(D_t^\alpha f(t) \right) = f(t) - f(0), \quad D_t^\alpha \left(I^\alpha f(t) \right) = f(t),$$

$$(5) \quad I^\alpha \left(g(t) D_t^\alpha f(t) \right) = f(t) g(t) - f(0) g(0) - I^\alpha \left(f(t) D_t^\alpha g(t) \right).$$

Lemma 1. [3] Assume that $a \ge 0$, $p \ge q \ge 0$, and $p \ne 0$, then

$$a^{\frac{q}{p}} \le \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}} \quad \text{for any } K > 0.$$

Lemma 2. [6] Let $\alpha > 0$, $a(t)$, $b(t)$, $u(t)$ be continuous functions defined on $t \ge 0$.

Then for $t \ge 0$,

$$D_t^\alpha u(t) \le a(t) + b(t) u(t)$$

Implies

$$u(t) \le u(0) \exp\left\{ \int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} b\left(\left(s\Gamma(1+\alpha) \right)^{\frac{1}{\alpha}} \right) \mathrm{d}s \right\} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} a(\tau) \exp\left\{ -\int_{\frac{t^\alpha}{\Gamma(1+\alpha)}}^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} b\left(\left(s\Gamma(1+\alpha) \right)^{\frac{1}{\alpha}} \right) \mathrm{d}s \right\} \mathrm{d}\tau.$$

2. Main Results

Theorem 1 *Assume that* $\alpha > 0$, $u(t)$, $k(t)$, $a(t)$, $f(t)$, $g(t)$, $h(t) \in C(R_+, R_+)$, *and* $k(t)$, $a(t)$ *are* *nondecreasing functions in* $t \geq 0$. *If* $u(t)$ *satisfies the following form of delay integral inequality*

$$u^p(t) \leq k(t) + \frac{a(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s) u^q(\sigma(s)) + g(s) u^r(s) + \int_0^s h(\tau) u^m(\tau) \mathrm{d}\tau \right] \mathrm{d}s, \quad t \geq 0, \tag{4}$$

with the initial condition

$$u(t) = \phi(t), \quad t \in [\beta, 0],$$

$$\phi(\sigma(t)) \leq k^{\frac{1}{p}}(t), \quad \text{for } t \in R_+ \quad \text{with} \quad \sigma(t) \leq 0 \tag{5}$$

where $p \neq 0$, $p \geq q \geq 0$, $p \geq r \geq 0$, $p \geq m \geq 0$, p, q, r, m are constants, $\sigma(t) \in C(R_+, R)$, $\sigma(t) \leq t$, $-\infty < \beta = \inf\{\sigma(t), t \in R_+\} \leq 0$ and $\phi(t) \in C([\beta, 0], R_+)$, then we have

$$u(t) \leq \left\{ k(t) \exp\left[a(t) \int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} G_1\left(\left(s\Gamma(1+\alpha) \right)^{\frac{1}{\alpha}} \right) \mathrm{d}s \right] \right.$$
$$\left. + \frac{a(t)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} H_1(\tau) \exp\left[-a(t) \int_{\frac{\tau^\alpha}{\Gamma(1+\alpha)}}^{\frac{t^\alpha}{\Gamma(1+\alpha)}} G_1\left(\left(s\Gamma(1+\alpha) \right)^{\frac{1}{\alpha}} \right) \mathrm{d}s \right] \mathrm{d}\tau \right\}^{\frac{1}{p}}, \quad t \geq 0. \tag{6}$$

for any $K > 0$, where

$$H_1(t) = \frac{p-q}{p} K^{\frac{q}{p}} f(t) + \frac{p-r}{p} K^{\frac{r}{p}} g(t) + \int_0^t \frac{p-m}{p} K^{\frac{m}{p}} h(\tau) \mathrm{d}\tau,$$

$$G_1(t) = \frac{q}{p} K^{\frac{q-p}{p}} f(t) + \frac{r}{p} K^{\frac{r-p}{p}} g(t) + \int_0^t \frac{m}{p} K^{\frac{m-p}{p}} h(\tau) \mathrm{d}\tau.$$

Proof. Fix $T \geq 0$, let

$$z(t) = k(T) + \frac{a(T)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s) u^q(\sigma(s)) + g(s) u^r(s) + \int_0^s h(\tau) u^m(\tau) \mathrm{d}\tau \right] \mathrm{d}s, \quad t \in [0, T], \tag{7}$$

$$z'(t) = \frac{a(T)}{\Gamma(\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{\alpha-1} \left[f(s) u^q(\sigma(s)) + g(s) u^r(s) + \int_0^s h(\tau) u^m(\tau) \mathrm{d}\tau \right] \mathrm{d}s,$$

Since $\alpha > 0$, $u(t)$, $k(t)$, $a(t)$, $f(t)$, $g(t)$, $h(t) \in C(R^+, R^+)$, there's exist a constant $C \geq 0$, such that

$$C = \min\left\{ f(s) u^q(\sigma(s)) + g(s) u^r(s) + \int_0^s h(\tau) u^m(\tau) \mathrm{d}\tau \right\},$$

and $\int_0^t (t-s)^{\alpha-1} \left[f(s) u^q(\sigma(s)) + g(s) u^r(s) + \int_0^s h(\tau) u^m(\tau) \mathrm{d}\tau \right] \mathrm{d}s$ is convergence integral,

so we have

$$\begin{aligned} z'(t) &= \frac{a(T)}{\Gamma(\alpha)} \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} (t-s)^{\alpha-1} \left[f(s) u^q(\sigma(s)) + g(s) u^r(s) + \int_0^s h(\tau) u^m(\tau) \mathrm{d}\tau \right] \mathrm{d}s \\ &\geq \frac{a(T) C}{\Gamma(\alpha)} \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} (t-s)^{\alpha-1} \mathrm{d}s \\ &= \frac{a(T) C}{\Gamma(\alpha)} t^{\alpha-1} \\ &\geq 0. \end{aligned}$$

we have $z(t)$ is a nonnegative and nondecreasing. From (4) and (7) we get

$$u(t) \le z^{\frac{1}{p}}(t), \quad t \in [0,T], \tag{8}$$

and

$$D_t^{\alpha} z(t) = a(T)\left[f(t)u^q(\sigma(t)) + g(t)u^r(t) + \int_0^t h(\tau)u^m(\tau)\mathrm{d}\tau \right], \quad t \in [0,T]. \tag{9}$$

So for $t \ge 0$ with $\sigma(t) \ge 0$, we have

$$u(\sigma(t)) \le z^{\frac{1}{p}}(\sigma(t)) \le z^{\frac{1}{p}}(t), \quad t \in [0,T], \tag{10}$$

for $t \ge 0$ with $\sigma(t) \le 0$, we have

$$u(\sigma(t)) = \phi(\sigma(t)) \le k^{\frac{1}{p}}(t) \le k^{\frac{1}{p}}(T) \le z^{\frac{1}{p}}(t), \quad t \in [0,T]. \tag{11}$$

Combining (10) and (11), we obtain

$$u(\sigma(t)) \le z^{\frac{1}{p}}(t), \quad t \in [0,T], \tag{12}$$

From (8), (9) and (12) we get

$$D_t^{\alpha} z(t) \le a(T)\left[f(t)z^{\frac{q}{p}}(\sigma(t)) + g(t)z^{\frac{r}{p}}(t) + \int_0^t h(\tau)z^{\frac{m}{p}}(\tau)\mathrm{d}\tau \right], \quad t \in [0,T], \tag{13}$$

By Lemma 1 we have

$$\begin{aligned} D_t^{\alpha} z(t) &\le a(T)\Bigg\{ f(t)\left[\frac{q}{p}K^{\frac{q-p}{p}} z(t) + \frac{p-q}{p}K^{\frac{q}{p}} \right] + g(t)\left[\frac{r}{p}K^{\frac{r-p}{p}} z(t) + \frac{p-r}{p}K^{\frac{r}{p}} \right] \\ &\quad + \int_0^t h(\tau)\left[\frac{m}{p}K^{\frac{m-p}{p}} z(\tau) + \frac{p-m}{p}K^{\frac{m}{p}} \right]\mathrm{d}\tau \Bigg\} \\ &\le a(T)\left[\frac{p-q}{p}K^{\frac{q}{p}} f(t) + \frac{p-r}{p}K^{\frac{r}{p}} g(t) + \int_0^t \frac{p-m}{p}K^{\frac{m}{p}} h(\tau)\mathrm{d}\tau \right] \\ &\quad + a(T)\left[\frac{q}{p}K^{\frac{q-p}{p}} f(t) + \frac{r}{p}K^{\frac{r-p}{p}} g(t) + \int_0^t \frac{m}{p}K^{\frac{m-p}{p}} h(\tau)\mathrm{d}\tau \right] z(t) \\ &= a(T)H_1(t) + a(T)G_1(t)z(t), \quad t \in [0,T]. \end{aligned} \tag{14}$$

Since $u(t)$, $g(t)$, $h(t)$ are continuous and there exists a constant M satisfies

$$\left| f(t)u^q(\sigma(t)) + g(t)u^r(t) + \int_0^t h(s)u^m(s)\mathrm{d}s \right| \le M$$

for $t \in [0,\varepsilon]$, where $\varepsilon > 0$. Then we get

$$\int_0^t (t-s)^{\alpha-1} \left| f(s)u^q(\sigma(s)) + g(s)u^r(s) + \int_0^s h(\tau)u^m(\tau)\mathrm{d}\tau \right| \mathrm{d}s \le \frac{M}{\alpha}t^{\alpha}, \quad t \in [0,\varepsilon],$$

so we have

$$z(0) = k(T), \tag{15}$$

Using Lemma 2 to (14) we get

$$z(t) \le k(T)\exp\left\{a(T)\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} G_1\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right)\mathrm{d}s\right\} + \frac{a(T)}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} H_1(\tau)\exp\left\{-a(T)\int_{\frac{\tau^\alpha}{\Gamma(1+\alpha)}}^{\frac{t^\alpha}{\Gamma(1+\alpha)}} G_1\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right)\mathrm{d}s\right\}\mathrm{d}\tau, \quad t\in[0,T]. \tag{16}$$

Letting $t=T$ in (16) and considering $T>0$ is arbitary, after substituting T with t, we get

$$z(t) \le k(t)\exp\left\{a(t)\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} G_1\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right)\mathrm{d}s\right\} + \frac{a(t)}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} H_1(\tau)\exp\left\{-a(t)\int_{\frac{\tau^\alpha}{\Gamma(1+\alpha)}}^{\frac{t^\alpha}{\Gamma(1+\alpha)}} G_1 G\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right)\mathrm{d}s\right\}\mathrm{d}\tau, \quad t\ge 0. \tag{17}$$

Combining (8) and (17), we get (6).

Remark 1. Assume $k(t)=0$, then the inequalities in Theorem 1 reduce to Lemma 5 in [6].

Theorem 2. Assume that $\alpha>0$, $u(t)$, $k(t)$, $a(t)$, $f(t)$, $g(t)$, $h(t)$, $\sigma(t)$ are defined as in Theorem 1. If $u(t)$ satisfies the following form of integral inequality,

$$u^p(t) \le k(t) + \frac{a(t)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(s)u^q(s) + L\left(s,u(\sigma(s))\right) + \int_0^s h(\tau)u^m(\tau)\mathrm{d}\tau\right]\mathrm{d}s, \quad t\ge 0, \tag{18}$$

where $p\ge q>0$, $p\ge m>0$, $p\ge 1$, $p,\ q,\ m$ are constants and $L,M\in C\left(R_+^2,R\right)$ satisfy

$$0\le L(t,x)-L(t,y)\le M(t,y)(x-y), \quad x\ge y\ge 0, \tag{19}$$

with the condition (5) in Theorem 1, then we have

$$u(t) \le \left\{k(t) + \frac{a(t)}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} H_2(\tau)\exp\left[-\int_{\frac{\tau^\alpha}{\Gamma(1+\alpha)}}^{\frac{t^\alpha}{\Gamma(1+\alpha)}} G_2\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right)\mathrm{d}s\right]\mathrm{d}\tau\right\}^{\frac{1}{p}}, \quad t\ge 0, \tag{20}$$

where

$$H_2(t) = f(t)\left(\frac{p-q}{p}K^{\frac{q}{p}} + \frac{q}{p}K^{\frac{q-p}{p}}k(t)\right) + \int_0^t\left(\frac{p-m}{p}K^{\frac{m}{p}}h(s) + \frac{m}{p}K^{\frac{m-p}{p}}h(s)k(s)\right)\mathrm{d}s + L\left(t,\frac{p-1}{p}K^{\frac{1}{p}} + \frac{1}{p}K^{\frac{1-p}{p}}k(t)\right), \quad t\ge 0,$$

$$G_2(t) = \frac{q}{p}K^{\frac{q-p}{p}}f(t)a(t) + \int_0^t \frac{m}{p}K^{\frac{m-p}{p}}h(s)a(s)\mathrm{d}s + M\left(t,\frac{p-1}{p}K^{\frac{1}{p}} + \frac{1}{p}K^{\frac{1-p}{p}}k(t)\right)\frac{1}{p}K^{\frac{1-p}{p}}a(t), \quad t\ge 0.$$

Proof. Let

$$z(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(s)u^q(s) + L\left(s,u(\sigma(s))\right) + \int_0^s h(\tau)u^m(\tau)\mathrm{d}\tau\right]\mathrm{d}s, \quad t\ge 0, \tag{21}$$

Since $u(t)$, $f(t)$, $h(t)$, $L(t,s)$ are nonegative functions, $z(t)$ is also nonegative and nondecreasing function, in addition, there exists a constant N satisfying

$$\left|f(t)u^q(t) + L\left(t,u(\sigma(t))\right) + \int_0^t h(s)u^m(s)\mathrm{d}s\right| \le N$$

for $t\in[0,\varepsilon]$, where $\varepsilon>0$. Then we get

$$\int_0^t (t-s)^{\alpha-1}\left| f(t)u^q(t)+L\big(t,u(\sigma(t))\big)+\int_0^t h(s)u^m(s)\mathrm{d}s\right| \le \frac{N}{\alpha}t^\alpha,\quad t\in[0,\varepsilon],$$

so we can get $z(0)=0$. From (18) we have

$$u(t)\le \big(k(t)+a(t)z(t)\big)^{\frac{1}{p}},\quad t\ge 0, \tag{22}$$

and

$$D_t^\alpha z(t)= f(t)u^q(t)+L\big(t,u(\sigma(t))\big)+\int_0^t h(s)u^m(s)\mathrm{d}s,\quad t\ge 0. \tag{23}$$

By Lemma 1 we get for any $K>0$,

$$u(t)\le \frac{p-1}{p}K^{\frac{1}{p}}+\big(k(t)+a(t)z(t)\big)\frac{1}{p}K^{\frac{1-p}{p}},\quad t\ge 0. \tag{24}$$

Proceeding the similar proof of Theorem 3 in [3], we can get

$$u(\sigma(t))\le \frac{p-1}{p}K^{\frac{1}{p}}+\big(k(t)+a(t)z(t)\big)\frac{1}{p}K^{\frac{1-p}{p}},\quad t\ge 0. \tag{25}$$

From (23), (24), (25) and condition (19) we have

$$\begin{aligned}
D_t^\alpha z(t)&\le f(t)\big[k(t)+a(t)z(t)\big]^{\frac{q}{p}}+L\left(t,\frac{p-1}{p}K^{\frac{1}{p}}+\big[k(t)+a(t)z(t)\big]\frac{1}{p}K^{\frac{1-p}{p}}\right)+\int_0^t h(s)\big[k(s)+a(s)z(s)\big]^{\frac{m}{p}}\mathrm{d}s\\
&\le f(t)\left(\frac{p-q}{p}K^{\frac{q}{p}}+\big[k(t)+a(t)z(t)\big]\frac{q}{p}K^{\frac{q-p}{p}}\right)+\int_0^t h(s)\left(\frac{p-m}{p}K^{\frac{m}{p}}+\big[k(s)+a(s)z(s)\big]\frac{m}{p}K^{\frac{m-p}{p}}\right)\mathrm{d}s\\
&\quad+L\left(t,\frac{p-1}{p}K^{\frac{1}{p}}+\big[k(t)+a(t)z(t)\big]\frac{1}{p}K^{\frac{1-p}{p}}\right)+L\left(t,\frac{p-1}{p}K^{\frac{1}{p}}+k(t)\frac{1}{p}K^{\frac{1-p}{p}}\right)\\
&\quad-L\left(t,\frac{p-1}{p}K^{\frac{1}{p}}+k(t)\frac{1}{p}K^{\frac{1-p}{p}}\right)\\
&\le f(t)\left(\frac{p-q}{p}K^{\frac{q}{p}}+\frac{q}{p}K^{\frac{q-p}{p}}k(t)+\frac{q}{p}K\frac{q-p}{p}a(t)z(t)\right)\\
&\quad+\int_0^t\left(\frac{p-m}{p}K^{\frac{m}{p}}h(s)+\frac{m}{p}K^{\frac{m-p}{p}}h(s)k(s)+\frac{m}{p}K^{\frac{m-p}{p}}h(s)a(s)z(s)\right)\mathrm{d}s\\
&\quad+M\left(t,\frac{p-1}{p}K^{\frac{1}{p}}+\frac{1}{p}K^{\frac{1-p}{p}}k(t)\right)\frac{1}{p}K^{\frac{1-p}{p}}a(t)z(t)+L\left(t,\frac{p-1}{p}K^{\frac{1}{p}}+\frac{1}{p}K^{\frac{1-p}{p}}k(t)\right)\\
&\le f(t)\left(\frac{p-q}{p}K^{\frac{q}{p}}+\frac{q}{p}K^{\frac{q-p}{p}}k(t)\right)+\int_0^t\left(\frac{p-m}{p}K^{\frac{m}{p}}h(s)+\frac{m}{p}K^{\frac{m-p}{p}}h(s)k(s)\right)\mathrm{d}s\\
&\quad+L\left(t,\frac{p-1}{p}K^{\frac{1}{p}}+\frac{1}{p}K^{\frac{1-p}{p}}k(t)\right)+\left[\frac{q}{p}K^{\frac{q-p}{p}}f(t)a(t)+\int_0^t\frac{m}{p}K^{\frac{m-p}{p}}h(s)a(s)\mathrm{d}s\right.\\
&\quad\left.+M\left(t,\frac{p-1}{p}K^{\frac{1}{p}}+\frac{1}{p}K^{\frac{1-p}{p}}k(t)\right)\frac{1}{p}K^{\frac{1-p}{p}}a(t)\right]z(t)\\
&=H_2(t)+G_2(t)z(t).
\end{aligned} \tag{26}$$

By Lemma 2 we have

$$z(t) \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} H_2(\tau) \exp\left\{ -\int_{\frac{\tau^\alpha}{\Gamma(1+\alpha)}}^{\frac{t^\alpha}{\Gamma(1+\alpha)}} G_2\left(\left(s\Gamma(1+\alpha) \right)^{\frac{1}{\alpha}} \right) \mathrm{d}s \right\} \mathrm{d}\tau, \quad t \ge 0. \tag{27}$$

Combining (22) and (27), (20) can be obtained subsequently.

Theorem 3. Assume that $a(t)$, $b(t)$, $k(t)$, $u(t) \in C(R_+, R_+)$, $\alpha > 0$, $\beta(t) \in C^1(R_+, R_+)$ and $\beta(t)$ is nondecreasing with $\beta(t) \le t$ for $t \ge 0$. If $u(t)$ satisfies

$$u(t) \le k(t) + \frac{a(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^{\beta(s)} b(\xi) u(\xi) \mathrm{d}\xi \mathrm{d}s, \quad t \ge 0, \tag{28}$$

then

$$u(t) \le k(t) + \frac{a(t)}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha} H_3(\tau) \exp\left\{ -\int_{\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}}^{\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}} G_3\left(\left(s\Gamma(\alpha+2) \right)^{\frac{1}{\alpha+1}} \right) \mathrm{d}s \right\} \mathrm{d}\tau, \quad t \ge 0, \tag{29}$$

where

$$H_3(t) = b(\beta(t)) k(\beta(t)) \beta'(t), \quad G_3(t) = a(\beta(t)) b(\beta(t)) \beta'(t).$$

Proof. Let

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^{\beta(s)} b(\xi) u(\xi) \mathrm{d}\xi \mathrm{d}s,$$

then we get

$$u(t) \le k(t) + a(t) z(t), \quad t \ge 0. \tag{30}$$

Since $b(t)$, $u(t)$ are both continuous functions, $\int_0^{\beta(s)} b(\xi) u(\xi) \mathrm{d}\xi$ is continuous and there exists a constant M satisfying $\left| \int_0^{\beta(s)} b(\xi) u(\xi) \mathrm{d}\xi \right| \le M$ for $t \in [0, \varepsilon]$, where $\varepsilon > 0$. Then we get

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^{\beta(s)} b(\xi) u(\xi) \mathrm{d}\xi \mathrm{d}s \le \frac{M}{\Gamma(\alpha+1)} t^\alpha,$$

so we get $z(0) = 0$ and

$$\begin{aligned} D_t^{\alpha+1} z(t) &= b(\beta(t)) u(\beta(t)) \beta'(t) \\ &\le b(\beta(t)) \left[k(\beta(t)) + a(\beta(t)) z(\beta(t)) \right] \beta'(t) \\ &\le b(\beta(t)) \left[k(\beta(t)) + a(\beta(t)) z(t) \right] \beta'(t) \\ &= b(\beta(t)) k(\beta(t)) \beta'(t) + a(\beta(t)) b(\beta(t)) \beta'(t) z(t) \\ &= H_3(t) + G_3(t) z(t). \end{aligned}$$

By Lemma 2 we have

$$z(t) \le \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha} H_3(\tau) \exp\left\{ -\int_{\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}}^{\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}} G_3\left(\left(s\Gamma(\alpha+2) \right)^{\frac{1}{\alpha+1}} \right) \mathrm{d}s \right\} \mathrm{d}\tau, \quad t \ge 0. \tag{31}$$

Combining (30) and (31), we get (29).

Remark 2. Considering $\beta(t) = t$ in Theorem 3, proceeding the similar proof of Theorem 3, we can get

$$u(t) \le k(t) + \frac{a(t)}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha} b(\tau) k(\tau) \exp\left\{ -\int_{\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}}^{\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}} \left[a\left(\left(s\Gamma(\alpha+2) \right)^{\frac{1}{\alpha}} \right) b\left(\left(s\Gamma(\alpha+2) \right)^{\frac{1}{\alpha}} \right) \right] \mathrm{d}s \right\} \mathrm{d}\tau, \quad t \ge 0.$$

Theorem 4. Assume that $a(t)$, $k(t)$, $u(t) \in C(R_+, R_+)$, $\alpha > 0$, $\beta(t) \in C^1(R_+, R_+)$ and $\beta(t)$ is nondecreasing with $\beta(t)\le t$ for $t\ge 0$, $b(t,s)\in C(R_+\times R_+,R_+)$ with $(t,s)\to\partial_t b(t,s)\in C(R_+\times R_+,R_+)$. If $u(t)$ satisfies the following form of delay integral inequality

$$u(t)\le k(t)+\frac{a(t)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^{\beta(s)} b(s,\xi)u(\xi)\mathrm{d}\xi\mathrm{d}s,\quad t\ge 0, \tag{32}$$

then

$$u(t)\le k(t)+\frac{a(t)}{\Gamma(\alpha+1)}\int_0^t (t-\tau)^{\alpha} H_4(\tau)\exp\left\{-\int_{\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}}^{\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}} G_4\left(\left(s\Gamma(\alpha+2)\right)^{\frac{1}{\alpha+1}}\right)\mathrm{d}s\right\}\mathrm{d}\tau,\quad t\ge 0, \tag{33}$$

where

$$H_4(t)=b(t,\beta(t))k(\beta(t))\beta'(t)+\int_0^{\beta(t)} k(s)\partial_t b(t,s)\mathrm{d}s,$$

$$G_4(t)=a(\beta(t))b(t,\beta(t))\beta'(t)+\int_0^{\beta(t)} a(s)\partial_t b(t,s)\mathrm{d}s.$$

Proof. Let

$$z(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^{\beta(s)} b(s,\xi)u(\xi)\mathrm{d}\xi\mathrm{d}s,\quad t\ge 0,$$

then we get

$$u(t)\le k(t)+a(t)z(t),\quad t\ge 0, \tag{34}$$

The assumptions on $b(t)$, $u(t)$ and $\beta(t)$ imply that $z(t)$ is nondecreasing and there exists a constant M satisfying $\left|\int_0^{\beta(s)} b(s,\xi)u(\xi)\mathrm{d}\xi\right|\le M$ for $t\in[0,\varepsilon]$, where $\varepsilon>0$.

Then we get

$$z(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^{\beta(s)} b(s,\xi)u(\xi)\mathrm{d}\xi\mathrm{d}s\le\frac{M}{\Gamma(\alpha+1)}t^{\alpha},\quad t\ge 0,$$

so we have $z(0)=0$. For $t\ge 0$, we have

$$D_t^{\alpha}z(t)=\int_0^{\beta(t)} b(t,s)u(s)\mathrm{d}s,$$

and

$$\begin{aligned}
&D_t^{\alpha+1}z(t)\\
&=b(t,\beta(t))u(\beta(t))\beta'(t)+\int_0^{\beta(t)}\partial_t b(t,s)u(s)\mathrm{d}s\\
&\le b(t,\beta(t))\left[k(\beta(t))+a(\beta(t))z(\beta(t))\right]\beta'(t)+\int_0^{\beta(t)}\partial_t b(t,s)\left[k(s)+a(s)z(s)\right]\mathrm{d}s\\
&\le b(t,\beta(t))\left[k(\beta(t))+a(\beta(t))z(t)\right]\beta'(t)+\int_0^{\beta(t)}\partial_t b(t,s)\left[k(s)+a(s)z(t)\right]\mathrm{d}s\\
&=b(t,\beta(t))k(\beta(t))\beta'(t)+\int_0^{\beta(t)} k(s)\partial_t b(t,s)\mathrm{d}s+\left[a(\beta(t))b(t,\beta(t))\beta'(t)+\int_0^{\beta(t)} a(s)\partial_t b(t,s)\mathrm{d}s\right]z(t)\\
&=H_4(t)+G_4(t)z(t).
\end{aligned} \tag{35}$$

Using Lemma 4 to (35), we can get

$$z(t)\le\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-\tau)^{\alpha} H_4(\tau)\exp\left\{-\int_{\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}}^{\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}} G_4\left(\left(s\Gamma(\alpha+2)\right)^{\frac{1}{\alpha+1}}\right)\mathrm{d}s\right\}\mathrm{d}\tau,\quad t\ge 0. \tag{36}$$

Combining (34) and (36), we get (33).

Remark 3. Considering $\partial_t b(t,s)\equiv 0$ and $\beta(t)=t$ in Theorem 4, we can get Remark 2.

3. Applications

In this section, we will show that the inequalities established above are useful in the research concerning the boundness, uniqueness and continuous dependence on the initial value for solutions to fractional differential equations.

3.1. Consider the Following Fractional Differential Equation

$$u^p(t) = u(0) + I^\alpha H\left(t, u(t), u(\sigma(t)), \int_0^t u^p(s)\,\mathrm{d}s\right), \quad t \ge 0,$$

$$u(0) = C. \tag{37}$$

with the condition

$$u(t) = \phi(t), \quad t \in [\beta, 0],$$

$$|\phi(\sigma(t))| \le |u(0)|^{\frac{1}{p}}, \quad t \ge 0, \quad \sigma(t) \le 0, \tag{38}$$

where $H(t,x,y,z) \in C(R_+^4, R)$, $p > 1$ is a constant, $\sigma(t) \in C(R_+, R)$, $\sigma(t) \le t$, $\alpha > 0$,
And

$$-\infty < \beta = \inf\{\sigma(t), t \in R_+\} \le 0, \quad \phi(t) \in C([\beta, 0], R_+),$$

Example 1. Assume that $H(t,x,y,z)$ satisfies

$$|H(t,x,y,z)| \le g(t)|x| + f(t)|y| + |z|, \tag{39}$$

where $f(t)$, $g(t)$ are nonnegative continuous functions on $t \ge 0$, then we have the following estimate for $u(t)$,

$$\begin{aligned}|u(t)| \le \Bigg\{ &|C| \exp\left[\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} G\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)\mathrm{d}s\right] \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} M(\tau) \exp\left[-\int_{\frac{\tau^\alpha}{\Gamma(1+\alpha)}}^{\frac{t^\alpha}{\Gamma(1+\alpha)}} G\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)\mathrm{d}s\right]\mathrm{d}\tau \Bigg\}^{\frac{1}{p}}, \quad t \ge 0,\end{aligned} \tag{40}$$

where

$$M(t) = \frac{p-1}{p} K^{\frac{1}{p}} \left(f(t) + g(t)\right), \quad G(t) = \frac{1}{p} K^{\frac{1-p}{p}} \left(f(t) + g(t)\right) + t.$$

Proof. By Equation (37), we have

$$u^p(t) = C + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} H\left(s, u(s), u(\sigma(s)), \int_0^s u^p(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s, \quad t \ge 0. \tag{41}$$

By (39) and (41) we can get

$$\begin{aligned}|u^p(t)| &\le |C| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|H\left(s, u(s), u(\sigma(s)), \int_0^s u^p(\xi)\,\mathrm{d}\xi\right)\right|\mathrm{d}s \\ &\le |C| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s)|u(\sigma(s))| + g(s)|u(s)| + \int_0^s u^p(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s.\end{aligned} \tag{42}$$

With a suitable application of Theorems 1 to (42) (with $m = p$, $q = r = 1$, $k(t) = |C|$, $a(t) = 1$, $h(t) = 1$), we can obtain the desired result. This complete the proof of Example 1.

Example 2. Assume that

$$\left|H(t,x_1,y_1,z_1)-H(t,x_2,y_2,z_2)\right|\le g(t)\left|x_1^p-x_2^p\right|+f(t)\left|y_1^p-y_2^p\right|+\left|z_1-z_2\right|, \tag{43}$$

where $f(t)$, $g(t)$ are nonnegative continuous functions defined $t\ge 0$, p is the quotient of two odd numbers. Equation (37) has a unique solution.

Proof. Suppose $u_1(t)$, $u_2(t)$ are two solutions of Equation (37), then we have

$$u_1^p(t)=C+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}H\left(s,u_1(s),u_1(\sigma(s)),\int_0^s u_1^p(\xi)\mathrm{d}\xi\right)\mathrm{d}s,\quad t\ge 0,$$

$$u_2^p(t)=C+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}H\left(s,u_2(s),u_2(\sigma(s)),\int_0^s u_2^p(\xi)\mathrm{d}\xi\right)\mathrm{d}s,\quad t\ge 0.$$

Furthermore,

$$u_1^p(t)-u_2^p(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[H\left(s,u_1(s),u_1(\sigma(s)),\int_0^s u_1^p(\xi)\mathrm{d}\xi\right)-H\left(s,u_2(s),u_2(\sigma(s)),\int_0^s u_2^p(\xi)\mathrm{d}\xi\right)\right]\mathrm{d}s$$

which implies

$$\begin{aligned}&\left|u_1^p(t)-u_2^p(t)\right|\\&\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[g(s)\left|u_1^p(s)-u_2^p(s)\right|+f(s)\left|u_1^p(\sigma(s))-u_2^p(\sigma(s))\right|+\int_0^s\left|u_1^p(\xi)-u_2^p(\xi)\right|\mathrm{d}\xi\right]\mathrm{d}s,\end{aligned} \tag{44}$$

Through a suitable application of Theorem 1 to (44) (with $p=q=r=m=1$, $k(t)=0$, $a(t)=h(t)=1$), we can obtain

$$\left|u_1^p(t)-u_2^p(t)\right|\le 0,$$

which implies $u_1(t)\equiv u_2(t)$. So Equation (37) has a unique solution.

Example 3. Suppose that $u(t)$ is the solution of (37) and $\tilde{u}(t)$ be the solution of the following fractional integral equation,

$$\tilde{u}^p(t)=\tilde{u}(0)+I^\alpha H\left(f\left(t,\tilde{u}(t),\tilde{u}(\sigma(t))\right),\int_0^t\tilde{u}^p(s)\mathrm{d}s\right),\quad t\ge 0,$$

$$\tilde{u}(0)=\tilde{C}. \tag{45}$$

If $H(t,x,y,z)$ satisfies the condition (43), then the solution of Equation (37) depends on the initial value C continuously.

Proof. By Equation (45), we have

$$\tilde{u}^p(t)=\tilde{C}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}H\left(s,\tilde{u}(s),\tilde{u}(\sigma(s)),\int_0^s\tilde{u}^p(\xi)\mathrm{d}\xi\right)\mathrm{d}s,\quad t\ge 0, \tag{46}$$

so we get

$$\begin{aligned}&u^p(t)-\tilde{u}^p(t)\\&=C-\tilde{C}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[H\left(s,u(s),u(\sigma(s)),\int_0^s u^p(\xi)\mathrm{d}\xi\right)-H\left(s,\tilde{u}(s),\tilde{u}(\sigma(s)),\int_0^s\tilde{u}^p(\xi)\mathrm{d}\xi\right)\right]\mathrm{d}s.\end{aligned}$$

Furthermore

$$\begin{aligned}&\left|u^p(t)-\tilde{u}^p(t)\right|\\&\le\left|C-\tilde{C}\right|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[g(s)\left|u^p(s)-\tilde{u}^p(s)\right|+f(s)\left|u^p(\sigma(s))-\tilde{u}^p(\sigma(s))\right|+\int_0^s\left|u^p(\xi)-\tilde{u}^p(\xi)\right|\mathrm{d}\xi\right]\mathrm{d}s.\end{aligned} \tag{47}$$

Apply Theorem 1 to (47) (with $p=q=r=m=1$, $h(t)=1$, $k(t)=\left|C-\tilde{C}\right|$, $a(t)=1$), we get

$$\left|u^p(t)-\tilde{u}^p(t)\right| \le \left|C-\tilde{C}\right| \exp\left[\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} G\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right)\mathrm{d}s\right],$$

where $G(t)=f(t)+g(t)+t$. This gives that the solutions of Equation (37) depends on the initial value C continuously.

3.2. Consider the Following Fractional Differential Equation

$$u(t)=u(0)+I^{0.5}\int_0^{t^{0.5}} su(s)\mathrm{d}s, \quad t\ge 0,$$
$$u(0)=C. \tag{48}$$

Example 4. Assume that $u(t)$ is a solution of Equation (48), then $u(t)$ is bounded.
Proof. By Equation (48) we can get

$$u(t)=C+\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\int_0^{s^{0.5}} \xi u(\xi)\mathrm{d}\xi\mathrm{d}s, \quad t\ge 0. \tag{49}$$

with a suitable application of Theorem 3 to (49) (with $k(t)=C$, $a(t)=1$, $\alpha=0.5$, $b(t)=t$, $\beta(t)=t^{0.5}\le t$ for $t\ge 0$, $H_3(t)=\frac{C}{2}$, $G_3(t)=\frac{1}{2}$), we have

$$\begin{aligned} u(t) &\le C+\frac{C}{2\Gamma(1.5)}\int_0^t (t-\tau)^{0.5}\exp\left\{-\int_{\frac{\tau^{1.5}}{2\Gamma(2.5)}}^{\frac{t^{1.5}}{2\Gamma(2.5)}}\frac{1}{2}\mathrm{d}s\right\}\mathrm{d}\tau \\ &\le C+\frac{C}{2\Gamma(1.5)}\int_0^t (t-\tau)^{0.5}\mathrm{d}\tau \\ &= C+\frac{C}{2\Gamma(1.5)}B(1,1.5)t^{1.5} \\ &= C\left[1+\frac{t^{1.5}}{2\Gamma(2.5)}\right], \end{aligned}$$

where we used $\exp\left\{-\int_{\frac{\tau^{1.5}}{2\Gamma(2.5)}}^{\frac{t^{1.5}}{2\Gamma(2.5)}}\frac{1}{2}\mathrm{d}s\right\}\le 1$. This complete the proof of Example 4.

Example 5. If $u(t)$ is a solution of (48), then it has a unique solution.
Proof. Suppose $u_1(t)$, $u_2(t)$ are two solutions of Equation (48), then we have

$$u_1(t)=C+\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\int_0^{s^{0.5}} \xi u_1(\xi)\mathrm{d}\xi\mathrm{d}s, \quad t\ge 0,$$
$$u_2(t)=C+\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\int_0^{s^{0.5}} \xi u_2(\xi)\mathrm{d}\xi\mathrm{d}s, \quad t\ge 0.$$

Furthermore,

$$u_1(t)-u_2(t)=\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\int_0^{s^{0.5}} \xi\left[u_1(\xi)-u_2(\xi)\right]\mathrm{d}\xi\mathrm{d}s,$$

which implies

$$\left|u_1(t)-u_2(t)\right|\le\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\int_0^{s^{0.5}} \xi\left|u_1(\xi)-u_2(\xi)\right|\mathrm{d}\xi\mathrm{d}s. \tag{50}$$

With a suitable application of 3 to (50) (with $u(t)=\left|u_1(t)-u_2(t)\right|$, $k(t)=0$, $a(t)=1$, $\alpha=0.5$, $b(t)=t$,

$\beta(t)=t^{0.5}\le t$ for $t\ge 0$), we can obtain

$$\left|u_1(t)-u_2(t)\right|\le 0,$$

which implies $u_1(t)\equiv u_2(t)$. So Equation (48) has a unique solution.

Example 6. Suppose that $u(t)$ is the solution of (48) and $\tilde{u}(t)$ is the solution of the following fractional integral equation,

$$\tilde{u}(t)=\tilde{u}(0)+I^{0.5}\int_0^{t^{0.5}} s\tilde{u}(s)\mathrm{d}s,\quad t\ge 0,$$
$$\tilde{u}(0)=\tilde{C}. \tag{51}$$

Then all the solutions of Equation (48) depend on the initial value C continuously.

Proof. By Equation (51), we have

$$\tilde{u}(t)=\tilde{C}+\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\int_0^{s^{0.5}}\xi\tilde{u}(\xi)\mathrm{d}\xi\mathrm{d}s,\quad t\ge 0,$$

so we get

$$u(t)-\tilde{u}(t)=C-\tilde{C}+\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\int_0^{s^{0.5}}\xi\left[u(\xi)-\tilde{u}(\xi)\right]\mathrm{d}\xi\mathrm{d}s,\quad t\ge 0$$

Furthermore

$$\left|u(t)-\tilde{u}(t)\right|=\left|C-\tilde{C}\right|+\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\int_0^{s^{0.5}}\xi\left|u(\xi)-\tilde{u}(\xi)\right|\mathrm{d}\xi\mathrm{d}s,\quad t\ge 0. \tag{52}$$

Apply Theorem 3 to (52) (with $k(t)=\left|C-\tilde{C}\right|$, $a(t)=1$, $b(t)=t$, $\alpha=0.5$, $\beta(t)=t^{0.5}\le t$ for $t\ge 0$, $H_4(t)=\frac{\left|C-\tilde{C}\right|}{2}$, $G_4(t)=\frac{1}{2}$), we get

$$\begin{aligned}\left|u(t)-\tilde{u}(t)\right| &\le \left|C-\tilde{C}\right|+\frac{\left|C-\tilde{C}\right|}{2\Gamma(1.5)}\int_0^t (t-\tau)^{-0.5}\exp\left\{-\int_{\frac{\tau^{1.5}}{\Gamma(2.5)}}^{\frac{t^{1.5}}{\Gamma(2.5)}}\frac{1}{2}\mathrm{d}s\right\}\mathrm{d}\tau\\ &\le \left|C-\tilde{C}\right|\left[1+\frac{1}{2\Gamma(1.5)}\int_0^t (t-\tau)^{0.5}\,d\tau\right]\\ &=\left|C-\tilde{C}\right|\left[1+\frac{1}{2\Gamma(1.5)}B(1,1.5)t^{1.5}\right]\\ &=\left|C-\tilde{C}\right|\left[1+\frac{t^{1.5}}{2\Gamma(2.5)}\right],\end{aligned} \tag{53}$$

where we use the fact that $\exp\left\{-\int_{\frac{\tau^{1.5}}{\Gamma(2.5)}}^{\frac{t^{1.5}}{\Gamma(2.5)}}\frac{1}{2}\mathrm{d}s\right\}\le 1$

This gives that $u(t)$ depends on the initial value C continuously.

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