

# Self-Consistent Sources and Conservation Laws for Super Tu Equation Hierarchy

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## Abstract

Based upon the basis of Lie super algebra  $B(0,1)$ , the super Tu equation hierarchy with self-consistent sources was presented. Furthermore, the infinite conservation laws of above hierarchy were given.

## Keywords

Super Tu Hierarchy, Self-Consistent Sources, Conservation Laws, Lie Super Algebra

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## 1. Introduction

Soliton equations with self-consistent sources have been receiving growing attention in recent years. Physically, the sources may result in solitary waves with a non-constant velocity and therefore lead to a variety of dynamics of physical models. For applications, these kinds of systems can be used to describe interactions between different solitary waves. Ma and Strampp systematically applied explicit symmetry constraint and binary nonlinearization of Lax pairs for generating soliton equation with sources [1]. Then, Ma presented the soliton solutions of the Schrödinger equation with self-consistent sources [2]. The discrete case of using variational derivatives in generating sources was discussed in [3].

With the development of soliton theory, super integrable systems associated with fermi variables have been receiving growing attention. Various methods have been developed to search for new super integrable systems, Lax pairs, soliton solutions, symmetries and conservation laws, etc. [4]-[11]. In 1997, Hu proposed the super-trace identity and applied it to establish the super Hamiltonian structures of super-integrable systems [4]. Then Professor Ma gave a systematic proof of super trace identity and presented the super Hamiltonian structures of super AKNS hierarchy and super Dirac hierarchy for application [5]. The super Tu hierarchy and its super-Hamiltonian structure was considered [6]. Recently, Yu *et al.* considered the binary nonlinearization of the super AKNS hierarchy under an implicit symmetry constraint [7] and the Bargmann symmetry constraint and binary nonlinearization of the super Dirac systems [8]. Meanwhile, various systematic methods have been developed to obtain exact solutions of the super integrable such as the inverse transformations, the Bäcklund and Darboux transformations, the bilinear transformation of Hirota and others [9]-[11].

This paper is organized as follows. In Section 2, the method for establishing super integrable soliton hierarchy with self-consistent sources by using Lie super algebra  $B(0,1)$  was presented. For application, the super Tu hierarchy with self-consistent sources was obtained in Section 3. In Section 4, conservation laws of super Tu

hierarchy were given.

## 2. A Kind of Super Integrable Soliton with Hierarchy Self-Consistent Sources

In the following. Consider a basis of Lie super algebra  $B(0,1)$  [5]

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (1)$$

We introduce the loop algebra  $\tilde{B}(0,1)$  as follows

$$\tilde{B}(0,1) = \{A \mid A \in R(\lambda) \otimes B(0,1)\} \quad (2)$$

where the loop algebra  $\tilde{B}(0,1)$  is defined by  $\{\lambda^n A \mid n \geq 0, A \in B(0,1)\}$ .

Consider the auxiliary linear problem

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = U(u, \lambda) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad U(u, \lambda) = e_0(\lambda) + \sum_{i=1}^5 u_i e_i(\lambda), \quad \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_{t_n} = V(u, \lambda) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad (3)$$

where  $u = (u_1, \dots, u_s)^T$ ,  $U(u, \lambda) = u_1 e_1 + \dots + u_p e_p$ ,  $u_i = u_i(x, t)$  ( $i = 1, 2, \dots, p$ ),  $\phi_i = \phi_i(x, t)$  are field variables defining on  $x \in R$ ,  $t \in R$ ,  $e_i(\lambda) = e_i(x, t, \lambda) \in \tilde{B}(0,1)$ .

From the spectral problem (3), the compatibility condition gives rise to the well-known zero curvature equation

$$U_{t_n} - V_x + [U, V] = 0, \quad n = 1, 2, \dots \quad (4)$$

The general scheme of searching for the consistent  $V^{(n)}$  and generating a hierarchy of nonlinear equations was proposed as follows [5]. We solve the equation

$$V_x = [U, V], \quad V = \sum_{m=0}^{\infty} V_m \lambda^{-m} = \sum_{m=0}^{\infty} \lambda^{-m} \begin{pmatrix} A_m & B_m + C_m & \rho_m \\ B_m - C_m & -A_m & \delta_m \\ \delta_m & -\rho_m & 0 \end{pmatrix}, \quad (5)$$

And search for  $\Delta_n(u, \lambda) \in \tilde{B}(0,1)$ , such that  $V^{(n)}$  can be constructed by

$$V^{(n)} = \sum_{m=0}^n V_m \lambda^{n-m} + \Delta_n(u, \lambda), \quad \Delta_n(u, \lambda) = \begin{pmatrix} \Delta_{n1} & \Delta_{n2} + \Delta_{n3} & \Delta_{n4} \\ \Delta_{n2} - \Delta_{n3} & -\Delta_{n1} & \Delta_{n5} \\ \Delta_{n5} & -\Delta_{n4} & 0 \end{pmatrix} \quad (6)$$

where  $\Delta_{ni}$  ( $1 \leq i \leq 5$ ) are linear functions of  $A_m, B_m, C_m, \rho_m, \delta_m$ .

We consider the super trace identity of super integrable systems [4] [5]

$$\frac{\delta}{\delta u} \left( \text{Str} \left( V \frac{\partial U}{\partial \lambda} \right) \right) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \text{Str} \left( \frac{\partial U}{\partial u} V \right) \quad (7)$$

where Str means the super trace. Defining a scalar  $H = H(u, \lambda)$  by the equation

$$H = \text{Str} \left( V \frac{\partial U}{\partial \lambda} \right), \quad H = \sum_{m=0}^{\infty} H_m(u, \lambda) \lambda^{-m} \quad (8)$$

The sets  $\{H_m\}$  proves the conserved densities of (4). The Hmailtonian form with  $H_{n+1}$  can be written as

$$u_{t_n} = J \frac{\delta H_{n+1}}{\delta u}, \quad \frac{\delta H_n}{\delta u} = L \frac{\delta H_{n-1}}{\delta u} = \dots = L^n \frac{\delta H_0}{\delta u}, \quad n = 1, 2, \dots \quad (9)$$

where  $L$  is a recursion operator and  $J$  is a symplectic operator, and  $\frac{\delta}{\delta u} = \left( \frac{\delta}{\delta u_1}, \dots, \frac{\delta}{\delta u_p} \right)^T$ .

According to (3) and (5), we consider the auxiliary linear problem. For  $N$  distinct  $\lambda_j, j = 1, \dots, N$ , the following systems result from (1)

$$\begin{aligned} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x &= U(u, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix} = \sum_{i=1}^5 u_i e_i(\lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \\ \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{t_n} &= V^{(n)}(u, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix} = \left[ \sum_{m=0}^n V_m(u, \lambda_j) \lambda_j^{n-m} + \Delta_n(u, \lambda_j) \right] \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}. \end{aligned} \tag{10}$$

Based on the results [11], we show that the following equations

$$\frac{\delta H_k}{\delta u} + \sum_{j=1}^N \alpha_j \frac{\delta \lambda_j}{\delta u} = 0 \tag{11}$$

where  $\alpha_j$  are constants. Equation (11) determines a finite dimensional invariant set for the flows (9).

For (10), it is known that

$$\frac{\delta \lambda_j}{\delta u} = \text{Str} \left( \psi_j \frac{\partial U(u, \lambda_j)}{\partial u} \right) = \text{Str}(\psi_j e_i(\lambda_j)), \quad i = 1, \dots, 5 \tag{12}$$

where Str denotes the super trace of a matrix and

$$\psi_j = \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 & \phi_{1j}\phi_{3j} \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} & \phi_{2j}\phi_{3j} \\ \phi_{2j}\phi_{3j} & -\phi_{1j}\phi_{3j} & 0 \end{pmatrix}, \quad j = 1, \dots, N \tag{13}$$

According to (11), for a specific  $k_0 \geq n_0$ , we demand that

$$\frac{\delta H_{k_0}}{\delta u_i} = \sum_{j=1}^N \frac{\delta \lambda_j}{\delta u_i} = \sum_{j=1}^N \text{Str}(\psi_j e_i(\lambda_j)) \tag{14}$$

From (9) and (11), a kind of super integrable hierarchy with self-consistent sources can be present as follows

$$u_{i,t_n} = J \frac{\delta H_{n+1}}{\delta u_i} + J \sum_{j=1}^N \frac{\delta \lambda_j}{\delta u_i} = JL^n \frac{\delta H_1}{\delta u_i} + J \sum_{j=1}^N \frac{\delta \lambda_j}{\delta u_i}, \quad n = 1, 2, \dots \tag{15}$$

### 3. The Super Tu Hierarchy with Self-Consistent Sources

The super Tu spectral problem associated with Lie super algebra  $B(0,1)$  is given by [6]

$$\phi_x = U\phi, \quad U = \begin{pmatrix} -\lambda + \frac{1}{2}q & r & \alpha \\ r & \lambda - \frac{1}{2}q & \beta \\ \beta & -\alpha & 0 \end{pmatrix}, \quad u = \begin{pmatrix} q \\ r \\ \alpha \\ \beta \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{16}$$

where  $\lambda$  is a spectral parameter,  $q$  and  $r$  are even variables,  $\alpha$  and  $\beta$  are odd variables [6].

Taking

$$V = \begin{pmatrix} A & B+C & \rho \\ B-C & -A & \delta \\ \delta & -\rho & 0 \end{pmatrix}$$

The co-adjoint equation associated with (16)  $V_x = [U, V]$  gives

$$\begin{cases} A_x = -2rC + \beta\rho + \alpha\delta, \\ B_x = -2\lambda C + qC - \alpha\rho + \beta\delta, \\ C_x = -2\lambda B - 2rA + qB - \alpha\rho - \beta\delta, \\ \rho_x = -\lambda\rho - \alpha A - \beta B - \beta C + \frac{1}{2}q\rho + r\delta, \\ \delta_x = \lambda\delta + \beta A - \alpha B + \alpha C + r\rho - \frac{1}{2}q\delta. \end{cases} \quad (17)$$

If we set

$$A = \sum_{i \geq 0} A_i \lambda^{-i}, \quad B = \sum_{i \geq 0} B_i \lambda^{-i}, \quad C = \sum_{i \geq 0} C_i \lambda^{-i}, \quad \rho = \sum_{i \geq 0} \rho_i \lambda^{-i}, \quad \delta = \sum_{i \geq 0} \delta_i \lambda^{-i} \quad (18)$$

Then (17) is equivalent to

$$\begin{cases} B_{i+1} = -rA_i + \frac{1}{2}qB_i - \frac{1}{2}C_{ix} - \frac{1}{2}\alpha\rho_i - \frac{1}{2}\beta\delta_i, & i \geq 0, \\ C_{i+1} = -\frac{1}{2}B_{i,x} + \frac{1}{2}qC_i - \frac{1}{2}\alpha\rho_i + \frac{1}{2}\beta\delta_i, & i \geq 0, \\ \rho_{i+1} = -\alpha A_i - \beta B_i - \beta C_i - \rho_{i,x} + \frac{1}{2}q\rho_i + r\delta_i, & i \geq 0, \\ \delta_{i+1} = -\beta A_i + \alpha B_i - \alpha C_i - r\rho_i + \frac{1}{2}q\delta_i + \delta_{i,x}, & i \geq 0, \\ A_{i+1,x} = -2rC_{i+1} + \beta\rho_{i+1} + \alpha\delta_{i+1}, & i \geq 0. \end{cases} \quad (19)$$

Which results in the recurrence relations

$$\begin{cases} (A_{i+1}, 2B_{i+1}, 2\delta_{i+1}, -2\rho_{i+1})^T = L(A_i, 2B_i, 2\delta_i, -2\rho_i)^T, \\ A_i = \delta^{-1}(-2rC_i + \beta\rho_i + \alpha\delta_i), \quad i \geq 0. \end{cases} \quad (20)$$

where

$$L = \begin{pmatrix} \frac{1}{2}\delta^{-1}q\partial & \frac{1}{2}\delta^{-1}r\partial & \frac{1}{2}\delta^{-1}\alpha\partial & \frac{1}{2}\delta^{-1}\beta\partial \\ \frac{1}{2}\delta^{-1}\frac{1}{r}\partial - 2r & \frac{1}{2}q & -\frac{1}{2}\beta - \frac{1}{4}\partial\frac{\alpha}{r} & \frac{1}{2}\alpha + \frac{1}{4}\partial\frac{\beta}{r} \\ \frac{\alpha}{r}\partial - 2\beta & \alpha & \partial + \frac{1}{2}q & r + \frac{\alpha\beta}{2r} \\ 2\alpha - \frac{\beta}{r}\partial & \beta & -r - \frac{\alpha\beta}{2r} & -\partial + \frac{1}{2}q \end{pmatrix} \quad (21)$$

Upon choosing the initial conditions

$$B_0 = C_0 = \rho_0 = \delta_0 = 0, \quad A_0 = 1$$

All other  $A_i, B_i, C_i, \rho_i, \delta_i$  ( $i \geq 1$ ) can be worked out by the recurrence relations (20). The first few sets are as follows:

$$A_1 = 0, \quad B_1 = -r, \quad C_1 = 0, \quad \rho_1 = -\alpha, \quad \delta_1 = -\beta,$$

$$A_2 = -\frac{1}{2}r^2 - \alpha\beta, \quad B_2 = -\frac{1}{2}qr, \quad C_2 = \frac{1}{2}r_x, \quad \rho_2 = \alpha_x - \frac{1}{2}q\alpha, \quad \delta_2 = -\beta_x - \frac{1}{2}q\beta,$$

$$A_3 = -\frac{1}{2}qr^2 + \alpha\beta_x - \alpha_x\beta - q\alpha\beta, \quad B_3 = -\frac{1}{4}r_{xx} - \frac{1}{4}q^2r + \frac{1}{2}r^3 + r\alpha\beta - \frac{1}{2}\alpha\alpha_x + \frac{1}{2}\beta\beta_x,$$

$$C_3 = \frac{1}{4}q_xr + \frac{1}{2}qr_x - \frac{1}{2}\alpha\alpha_x - \frac{1}{2}\beta\beta_x,$$

$$\rho_3 = -\alpha_{xx} + \frac{1}{2}q_x\alpha + q\alpha_x - \frac{1}{4}q^2\alpha + \frac{1}{2}r^2\alpha - \frac{1}{2}r_x\beta - r\beta_x, \quad \delta_3 = -\beta_{xx} - \frac{1}{2}r_x\alpha - r\alpha_x - q\beta_x - \frac{1}{4}q^2\beta + \frac{1}{2}r^2\beta - \frac{1}{2}q_x\beta.$$

Let us associate the problem (16) with the following auxiliary problem

$$\phi_{t_n} = V^{(n)}\phi, \quad V^{(n)} = \sum_{i=0}^n \begin{pmatrix} A_i & B_i + C_i & \rho_i \\ B_i - C_i & -A_i & \delta_i \\ \delta_i & -\rho_i & 0 \end{pmatrix} \lambda^{n-i} + \begin{pmatrix} \frac{B_{n+1}}{r} & 0 & 0 \\ 0 & -\frac{B_{n+1}}{r} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (22)$$

The compatible conditions of the spectral problem (16) and the auxiliary problem (22) are

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0 \quad (23)$$

Which refer the super Tu equation hierarchy

$$u_{t_n} = K_n = \left( 2 \left( \frac{B_{n+1}}{r} \right)_x, -2C_{n+1}, -\rho_{n+1} + \frac{\alpha}{r} B_{n+1}, \delta_{n+1} - \frac{\beta}{r} B_{n+1} \right)^T. \quad (24)$$

Here  $u_{t_n} = K_n$  in (24) is called the  $n$ -th Tu flow of this hierarchy.

Using the super trace identity (7), we have

$$(A_{i+1} \quad 2B_{i+1} \quad 2\delta_{i+1} \quad -2\rho_{i+1})^T = \frac{\delta}{\delta u} H_i, \quad H_i = \int \frac{2A_{i+2}}{i+1} dx, \quad i \geq 0 \quad (25)$$

Therefore, the super Tu soliton hierarchy Equation (24) can be written as the following super Hamiltonian form:

$$u_{t_n} = J \frac{\delta H_n}{\delta u} \quad (26)$$

where

$$J = \begin{pmatrix} 0 & \frac{\partial}{\partial r} & 0 & 0 \\ \frac{\partial}{\partial r} & 0 & -\frac{\alpha}{2r} & -\frac{\beta}{2r} \\ 0 & \frac{\alpha}{2r} & 0 & \frac{1}{2} \\ 0 & \frac{\beta}{2r} & \frac{1}{2} & 0 \end{pmatrix}$$

Is a super symplectic operator, and  $H_n$  is given by (25).

The first non-trivial nonlinear of super Tu hierarchy is given by its second flow

$$\begin{cases} q_{t_2} = -\frac{1}{2} \left( \frac{r_{xx}}{r} \right)_x - qq_x + 2rr_x + 2(\alpha\beta)_x - \left( \frac{\alpha\alpha_x}{r} \right)_x + \left( \frac{\beta\beta_x}{r} \right)_x, \\ r_{t_2} = -\frac{1}{2} q_x r - qr_x + \frac{1}{2} \alpha\alpha_x + \beta\beta_x, \\ \alpha_{t_2} = \alpha_{xx} - \frac{1}{2} q_x \alpha - q\alpha_x - r^2 \alpha + \frac{1}{2} r_x \beta + r\beta_x - \frac{r_{xx}}{4r} \alpha + \frac{1}{2r} \alpha\beta\beta_x, \\ \beta_{t_2} = -\beta_{xx} - \frac{1}{2} r_x \alpha - r\alpha_x - q\beta_x + r^2 \beta - \frac{1}{2} q_x \beta + \frac{r_{xx}}{4r} \beta + \frac{1}{2r} \alpha\alpha_x \beta. \end{cases} \quad (27)$$

Which possesses a Lax pair of  $U$  defined in (16) and  $V^{(2)}$  defined by

$$V^{(2)} = \begin{pmatrix} \lambda^2 - \frac{r_{xx}}{4r} - \frac{1}{4} q^2 - \frac{\alpha\alpha_x}{2r} + \frac{\beta\beta_x}{2r} & -r\lambda - \frac{1}{2} qr + \frac{1}{2} r_x & -\alpha\lambda + \alpha_x - \frac{1}{2} q\alpha \\ -r\lambda - \frac{1}{2} qr - \frac{1}{2} r_x & -\lambda^2 + \frac{r_{xx}}{4r} + \frac{1}{4} q^2 + \frac{\alpha\alpha_x}{2r} - \frac{\beta\beta_x}{2r} & -\beta\lambda - \beta_x - \frac{1}{2} q\beta \\ -\beta\lambda - \beta_x - \frac{1}{2} q\beta & \alpha\lambda - \alpha_x + \frac{1}{2} q\alpha & 0 \end{pmatrix}$$

Next we will establish the super Tu hierarchy with self-consistent sources. Consider the linear system

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x = U \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix} = \begin{pmatrix} -\lambda + \frac{1}{2}q & r & \alpha \\ r & \lambda - \frac{1}{2}q & \beta \\ \beta & -\alpha & 0 \end{pmatrix} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \quad \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{t_n} = V \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix} = \begin{pmatrix} A & B+C & \rho \\ B-C & -A & \delta \\ \delta & -\rho & 0 \end{pmatrix} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix} \quad (28)$$

For the system (28), we consider the  $\frac{\delta H}{\delta u} = \sum_{j=1}^N \frac{\delta \lambda_j}{\delta u}$  in the Lie super algebra  $B(0,1)$  and obtain

$$\frac{\delta \lambda_j}{\delta u} = \begin{pmatrix} \text{Str} \left( \Psi_j \frac{\partial U}{\partial q} \right) \\ \text{Str} \left( \Psi_j \frac{\partial U}{\partial r} \right) \\ \text{Str} \left( \Psi_j \frac{\partial U}{\partial \alpha} \right) \\ \text{Str} \left( \Psi_j \frac{\partial U}{\partial \beta} \right) \end{pmatrix} = \begin{pmatrix} \langle \Phi_1, \Phi_2 \rangle \\ \langle \Phi_2, \Phi_2 \rangle - \langle \Phi_1, \Phi_1 \rangle \\ -2 \langle \Phi_2, \Phi_3 \rangle \\ 2 \langle \Phi_1, \Phi_3 \rangle \end{pmatrix} \quad (29)$$

where  $\Phi_i = (\phi_{i1}, \dots, \phi_{iN})^T$  ( $i = 1, 2, 3$ ).

According to the results in (15), the super Tu hierarchy with self-consistent sources is presented as

$$u_{t_n} = \begin{pmatrix} q \\ r \\ \alpha \\ \beta \end{pmatrix}_{t_n} = JL^n \begin{pmatrix} 0 \\ -2r \\ -2\beta \\ 2\alpha \end{pmatrix} + J \begin{pmatrix} \langle \Phi_1, \Phi_2 \rangle \\ \langle \Phi_2, \Phi_2 \rangle - \langle \Phi_1, \Phi_1 \rangle \\ -2 \langle \Phi_2, \Phi_3 \rangle \\ 2 \langle \Phi_1, \Phi_3 \rangle \end{pmatrix} \quad (30)$$

The first nontrivial integrable super Tu hierarchy with self-consistent sources is its second flow

$$\begin{cases} q_{t_2} = -\frac{1}{2} \left( \frac{r_{xx}}{r} \right)_x - qq_x + 2rr_x + 2(\alpha\beta)_x - \left( \frac{\alpha\alpha_x}{r} \right)_x + \left( \frac{\beta\beta_x}{r} \right)_x + \left( \frac{\langle \Phi_2, \Phi_2 \rangle - \langle \Phi_1, \Phi_1 \rangle}{r} \right)_x, \\ r_{t_2} = -\frac{1}{2} q_x r - qr_x + \frac{1}{2} \alpha\alpha_x + \beta\beta_x + \left( \frac{\langle \Phi_1, \Phi_2 \rangle}{r} \right)_x + \frac{\alpha}{r} \langle \Phi_2, \Phi_3 \rangle + \frac{\beta}{r} \langle \Phi_1, \Phi_3 \rangle, \\ \alpha_{t_2} = \alpha_{xx} - \frac{1}{2} q_x \alpha - q\alpha_x - r^2 \alpha + \frac{1}{2} r_x \beta + r\beta_x - \frac{r_{xx}}{4r} \alpha + \frac{1}{2r} \alpha\beta\beta_x + \frac{\alpha}{2r} (\langle \Phi_2, \Phi_2 \rangle - \langle \Phi_1, \Phi_1 \rangle) + \langle \Phi_1, \Phi_3 \rangle, \\ \beta_{t_2} = -\beta_{xx} - \frac{1}{2} r_x \alpha - r\alpha_x - q\beta_x + r^2 \beta - \frac{1}{2} q_x \beta + \frac{r_{xx}}{4r} \beta + \frac{1}{2r} \alpha\alpha_x \beta - \frac{\beta}{2r} (\langle \Phi_2, \Phi_2 \rangle - \langle \Phi_1, \Phi_1 \rangle) - \langle \Phi_2, \Phi_3 \rangle. \end{cases} \quad (31)$$

When  $\alpha = \beta = 0$ , it is the well known nonlinear Tu equation with self-consistent sources. So system (30) is a novel super integrable equation hierarchy.

### 4. Conservation Laws for the Super Tu Hierarchy

In what follows, we will construct conservation laws of the super coupled Burgers equation. Introduce the variables:

$$K = \frac{\phi_2}{\phi_1}, \quad G = \frac{\phi_3}{\phi_1}, \quad (32)$$

where  $p(K) = 0, p(G) = 1$ . From (10), we have

$$K_x = r + (2\lambda - q)K + \beta G - rK^2 - \alpha KG, \quad G_x = \beta - \alpha K + \left( \lambda - \frac{1}{2}q \right)G - rKG - \alpha G^2 \quad (33)$$

We expand  $K, G$  in powers of  $\lambda^{-1}$  as follows

$$K = \sum_{j=0}^{\infty} k_j \lambda^{-j}, \quad G = \sum_{j=0}^{\infty} g_j \lambda^{-j}, \quad (34)$$

where  $p(k_j) = 0$ ,  $p(g_j) = 1$ . Substituting (34) into (33) and comparing the coefficients of the same powers of  $\lambda$ , we obtain

$$\begin{aligned} k_1 &= -\frac{1}{2}r, \quad g_1 = -\beta, \quad k_2 = -\frac{1}{4}r_x - \frac{1}{4}qr, \quad g_2 = -\beta_x - \frac{1}{2}r\alpha - \frac{1}{4}qr, \\ k_3 &= -\frac{1}{8}r_x - \frac{1}{8}q_x r - \frac{1}{4}qr_x - \frac{1}{8}q^2 r + \frac{1}{2}\beta\beta_x + \frac{1}{8}qr\beta + \frac{1}{8}r^3, \\ g_3 &= -\beta_{xx} - \frac{3}{4}r_x\alpha - \frac{1}{2}qr\alpha - \frac{1}{2}r\alpha_x - \frac{1}{4}q_x r - \frac{1}{4}qr_x - \frac{1}{2}q\beta_x - \frac{1}{8}q^2 r + \frac{1}{2}r^2\beta. \end{aligned} \quad (35)$$

And a recursion formula for  $k_n$  and  $g_n$ ,

$$\begin{aligned} k_{n+1} &= \frac{1}{2}k_{nx} + \frac{1}{2}qk_n - \frac{1}{2}\beta g_n + \frac{1}{2}r \left( \sum_{l=0}^n k_l k_{n-l} \right) + \frac{1}{2}\alpha \left( \sum_{l=0}^n k_l g_{n-l} \right), \\ g_{n+1} &= g_{nx} + \alpha k_n + \frac{1}{2}qg_n + r \left( \sum_{l=0}^n k_l g_{n-l} \right) + \alpha \left( \sum_{l=0}^n g_l g_{n-l} \right). \end{aligned} \quad (36)$$

Because of

$$\frac{\partial \phi_{1,x}}{\partial t} = \frac{\partial \phi_{1,t}}{\partial x} \quad (37)$$

we derive the conservation laws of (27)

$$\frac{\partial}{\partial t} \left( -\lambda + \frac{1}{2}q + rK + \alpha G \right) = \frac{\partial}{\partial x} \left( A + (B + C)K + G \right) \quad (38)$$

where

$$\begin{aligned} A &= c_0 \lambda^2 + c_1 \lambda - \frac{1}{2}c_0 r^2 - c_0 \alpha \beta, \\ B &= -c_0 r \lambda - c_1 r - \frac{1}{2}c_0 q r, \\ C &= \frac{1}{2}c_0 r_x, \\ \rho &= -c_0 \alpha \lambda - c_1 \alpha + c_0 \alpha_x - \frac{1}{2}c_0 q \alpha. \end{aligned}$$

Assume that  $\sigma = -\lambda + \frac{1}{2}q + rK + \alpha G$ ,  $\theta = A + (B + C)K + \rho G$ , then (38) can be written as  $\sigma_t = \theta_x$ , which is the right form of conservation laws. We expand  $\sigma$  and  $\theta$  as series in powers of  $\lambda$  according with the coefficients, which are called conserved densities and currents respectively

$$\sigma = -\lambda + \sum_{j=0}^{\infty} \sigma_j \lambda^{-j}, \quad \theta = c_0 \lambda^2 + c_1 \lambda + \sum_{j=0}^{\infty} \theta_j \lambda^{-j}, \quad (39)$$

where  $c_0, c_1$  are constants of integration. Then the first two conserved densities and currents are

$$\begin{aligned} \sigma_1 &= -\frac{1}{2}r^2 - \alpha\beta, \\ \sigma_2 &= -\frac{1}{4}rr_x - \frac{1}{4}qr^2 - \alpha\beta_x - \frac{1}{4}qr\alpha, \\ \theta_1 &= c_0 \left( \frac{1}{2}qr^2 + \alpha\beta_x + \frac{1}{4}qr\alpha - \alpha_x\beta + \frac{1}{2}q\alpha\beta \right) + c_1 \left( \frac{1}{2}r^2 + \alpha\beta \right), \\ \theta_2 &= c_0 \left( \frac{1}{8}rr_x + \frac{1}{8}q_x r^2 + \frac{1}{4}qrr_x + \frac{1}{4}q^2 r^2 - \frac{1}{2}r\beta\beta_x - \frac{1}{8}qr^2\beta - \frac{1}{8}r^4 + \alpha\beta_{xx} + r\alpha\alpha_x \right. \\ &\quad \left. + \frac{1}{4}q_x r\alpha + \frac{1}{4}qr_x\alpha + q\alpha\beta_x + \frac{1}{8}q^2 r\alpha - \frac{1}{2}r^2\alpha\beta - \alpha_x\beta_x + \frac{1}{8}qr\alpha \right) + c_1 \left( \frac{1}{4}rr_x + \frac{1}{4}qr^2 + \alpha\beta_x + \frac{1}{4}qr\alpha \right). \end{aligned}$$

The recursion relations for  $\sigma_n$  and  $\theta_n$  are

$$\sigma_n = rk_n + \alpha g_n, \quad \theta_n = c_0 \left( -rk_{n+1} - \frac{1}{2}qrk_n + \frac{1}{2}r_x k_n - \alpha g_{n+1} - \alpha_x g_n - \frac{1}{2}q\alpha g_n \right) + c_1 (-rk_n - \alpha g_n) \quad (40)$$

where  $k_n$  and  $g_n$  can be calculated from (36). The infinitely conservations laws of (36) can be easily obtained in (32)-(40) respectively.

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