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The Existence and Multiplicity of Solutions for Singular Boundary Value Systems with *p*-Laplacian

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Abstract

This paper presents sufficient conditions for the existence of positive solutions for the fourth-order boundary value problem system with *p*-Laplacian operator. The existence of single or multiple positive solutions for the system is showed through the fixed point index theory in cones under some assumptions.

Keywords

Coupled Singular Boundary Value Problem, Positive Solution, Fixed Point Index Theorem

1. Introduction

In this paper, we are concerned with the existence and multiplicity of positive solutions for the system (BVP):

$$\begin{cases}\left(\varphi_p\left(u''(t)\right)\right)'' - a_1(t) f_1\left(u(t), v(t)\right) = 0, & 0 < t < 1,\\ \left(\varphi_p\left(v''(t)\right)\right)'' - a_2(t) f_2\left(u(t), v(t)\right) = 0, & 0 < t < 1,\\ u(0) = u(1) = u''(0) = u''(1) = 0,\\ v(0) = v(1) = v''(0) = v''(1) = 0,\end{cases}$$

where $\varphi_p(s) = |s|^{p-2}$, $s, p > 1$, $a_i(t) \in C\left((0,1),(0,+\infty)\right)$ and $f_i(u,v) \in C\left([0,+\infty),[0,+\infty)\right)$, $a_i(t)$ is allowed to have singularity at $t = 0,1$, $i = 1,2$.

Several papers ([1]-[4]) have studied the solution of fourth-order boundary value problems. But results about fourth-order differential eguations with *p*-Laplacian have rarely seen. Recently, several papers ([6]-[8]) have been devoted to the study of the coupled boundary value problem.

Motivated by the results mentioned above, here we establish some sufficient conditions for the existence of to (BVP) (1.1) under certain suitable weak conditions. The main results in this paper improve and generalize the results by others.

The following fixed-point index theorem in cones is fundamental.

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Theorem A [9] Assume that X is a Banach space, $K \subseteq X$ is a cone in X, and $0 < r < +\infty$, $\Omega_r = \{x \in k : \|x\| \le r\}$, if $T : \Omega_r \mapsto X$ is a completely operator and $Tx \ne x$, $\forall x \in \partial\Omega_r$.

1) If for $\forall u \in \partial\Omega_r$, $\|x\| \le \|Tx\|$, then i $(T, \Omega_r, K) = 0$;
2) If for $\forall u \in \partial\Omega_r$, $\|x\| \ge \|Tx\|$ then i $(T, \Omega_r, K) = 1$.

2. Preliminaries and Lemmas

In this paper, let $E = C[0,1]$ and $E^+ = \{u \in E; u(t) \ge 0 \text{ is a concave function}\}$ then $E^+ \times E^+$ is a Banach space with the norm $\|(u,v)\|_0 = \|u\| + \|v\|$, $\forall (u,v) \in E^+ \times E^+$, where $\|u\| = \max_{0 \le t \le 1}\{u(t)\}$, $\|v\| = \max_{0 \le t \le 1}\{v(t)\}$, then $X := E^+ \times E^+$ is a cone of $(E \times E, \|\cdot\|_0)$. In tnhis paper, $(u_1, v_1) \ge (u_2, v_2)$ *i.e.* $u_1 \ge u_2$, $v_1 \ge v_2$

Suppose $G(t,s)$ is the Green function of the following boundary problem: z = 0, $0 < t < 1$, $z(0) = z(1) = 0$, then

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1, \end{cases}$$

Obviously, $t(1-t)s(1-s) \le G(t,s) = G(s,t) \le t(1-t)$, $0 \le t$, $s \le 1$

Define a cone $K \subset X$ as follows $K = \{(u,v) \in X \,|\, (u,v) \ge (0,0), u(t) + v(t) \ge \|(u,v)\|_0 t(1-t), t \in [0,1]\}$ and define an integral operator $A : K \to K$ by $A(u(t), v(t)) = (A_1(u,v)(t), A_2(u,v)(t))$, where

$$A_i(u,v)(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^1 G(s,\tau) a_i(\tau) f_i(u(\tau), v(\tau)) \mathrm{d}\tau\right)\mathrm{d}s, \quad i = 1, 2$$

Let us list the following assumptions for convenience.

(H) $a_i \in C((0,1), [0,+\infty))$, $a_i(i = 1,2)$ is singular at $t = 0$ or 1, and

$$0 < \int_0^1 G(\tau,\tau) a_i(\tau)\mathrm{d}\tau < +\infty, \quad 0 < \int_0^1 G(s,\tau) a_i(\tau)\mathrm{d}\tau < +\infty, \quad i = 1, 2$$

Lemma 2.1 (u,v) is a solution of BVP (1.1) if and only if $(u,v) \in K$, $A(u,v) = (u,v)$ has fixed points.
It is easy to see that $(u,v) \in K$, $A(u,v) = (u,v)$ if (u,v) is a solution of BVP (1.1).
Lemma 2.2 Suppose that (H) hold, then $AK \subset K$.
Lemma 2.3 Suppose that H hold. Then $A : K \to K$ is completely continuous.
Proof Firstly, assume $D \subset K$ is a bounded set, we have

$$A_i(u,v)(t) \le \int_0^1 G(t,s)\phi_q\left(\int_0^1 G(s,\tau) a_i(\tau)\right)\mathrm{d}s\,\phi_q\left(\sup f_i(u,v) : (u,v) \in D\right) < +\infty$$

Then $A_i(D)(i = 1,2)$ is bounded, therefore $A(D)$ is bounded.
Secondly, suppose (u_n, v_n), $(u_0, v_0) \in D$, $(u_n, v_n) \to (u_0, v_0)(n \to \infty)$ then (u_n, v_n) is bounded, we get

$$\begin{aligned}
&\left|A(u_n, v_n)(t) - A(u_0, v_0)(t)\right| \\
&\le \int_0^1 G(s,s)\left|\phi_q\left(\int_0^1 G(s,\tau) a_1(\tau) f_1(u_n, v_n)(\tau)\mathrm{d}\tau\right) - \phi_q\left(\int_0^1 G(s,\tau) a_1(\tau) f_1(u_0, v_0)(\tau)\mathrm{d}\tau\right)\right|\mathrm{d}s \\
&\quad + \int_0^1 G(s,s)\left|\phi_q\left(\int_0^1 G(s,\tau) a_2(\tau) f_2(u_n, v_n)(\tau)\mathrm{d}\tau\right) - \phi_q\left(\int_0^1 G(s,\tau) a_2(\tau) f_2(u_0, v_0)(\tau)\mathrm{d}\tau\right)\right|\mathrm{d}s \\
&\le \frac{1}{4}\max_{0 \le t \le 1}\left|f_1^{q-1}(u_n(t), v_n(t)) - f_1^{q-1}(u_0(t), v_0(t))\right|\phi_q\int_0^1 \tau(1-\tau) a_1(\tau)\mathrm{d}\tau \\
&\quad + \frac{1}{4}\max_{0 \le t \le 1}\left|f_2^{q-1}(u_n(t), v_n(t)) - f_2^{q-1}(u_0(t), v_0(t))\right|\phi_q\int_0^1 \tau(1-\tau) a_2(\tau)\mathrm{d}\tau.
\end{aligned}$$

Due to the continuity of f_1, f_2, by H and above fomula together with Lebesgue Dominated Convergence Theorem, then $\left|A(u_n, v_n)(t) - A(u_0, v_0)(t)\right| \to 0$ when $n \to \infty$. Therefore A is continuous.

Lastly, since $G(t,s)$ is continuous in $[0,1] \times [0,1]$, so it is uniformly continous. For all $\varepsilon > 0$, $\exists \delta > 0$ for all $s \in [0,1]$, when $|t_1 - t_2| < \delta$, we get

$$\left|G(t_1,s)-G(t_2,s)\right|<\frac{\varepsilon}{2}\cdot\left\{\phi_q\left(\int_0^1 G(\tau,\tau)a_i(\tau)\mathrm{d}\tau\right)\phi_q\left(\sup f_i(u,v):(u,v)\in D\right)\right\}^{-1},\quad i=1,2$$

Then for all $(u,v)\in D$, we have

$$\begin{aligned}\left|A(u,v)(t_1)-A(u,v)(t_2)\right| &\le \int_0^1\left|G(t_1,s)-G(t_2,s)\right|\cdot\phi_q\left(\int_0^1 G(\tau,\tau)a_1(\tau)\mathrm{d}\tau\right)\cdot\phi_q\left(\sup f_1(u,v):(u,v)\in D\right)\mathrm{d}s\\ &\quad+\int_0^1\left|G(t_1,s)-G(t_2,s)\right|\cdot\phi_q\left(\int_0^1 G(\tau,\tau)a_2(\tau)\mathrm{d}\tau\right)\cdot\phi_q\left(\sup f_2(u,v):(u,v)\in D\right)\mathrm{d}s\\ &<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned}$$

So A is equicontinuous, by Arzela-Ascoli theorem we know AD is relatively compact.

Therefore, $A:\overline{K}_{r,R}\to K$ is completely continuous.

For convenience we denote

$$f_{i0}(u,v):=\underline{\lim}_{u+v\to 0}\frac{f_i(u,v)}{(u+v)^{p-1}},\quad f_{i\infty}(u,v):=\underline{\lim}_{u+v\to 0}\frac{f_i(u,v)}{(u+v)^{p-1}},\quad i=1,2;$$

$$f_i^0(u,v):=\overline{\lim}_{u+v\to 0}\frac{f_i(u,v)}{(u+v)^{p-1}},\quad f_i^\infty(u,v):=\overline{\lim}_{u+v\to 0}\frac{f_i(u,v)}{(u+v)^{p-1}},\quad i=1,2;$$

$$\mu_i:=\phi_q\left(\int_0^1 G(\tau,\tau)a_i(\tau)\mathrm{d}\tau\right),\quad i=1,2;$$

$$\nu_i:=\min_{\frac14\le t\le\frac12}\int_{\frac14}^{\frac12}G(t,s)\phi_q\left(\int_0^1 G(s,\tau)a_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s,\quad i=1,2.$$

3. Main Results

Theorem 3.1 Suppose that H holds. If the following conditions are satisfied:

$$(H_1)\ \ f_i^0(u,v)=0,\quad i=1,2;\ \ (H_2)\ \ f_{1\infty}(u,v)=\infty\ \text{ or }\ f_{2\infty}(u,v)=\infty$$

Then the system (1.1) has at least one positive solution $(u(t),v(t))$, $t\in(0,1)$

Proof By Lemma 2.3, we know A is completely continuous. By (H_1), there exists $r_1>0$, when $0\le u(t)+v(t)\le r_1$, $t\in[0,1]$, we have $f_i(u,v)\le(\alpha_i(u+v))^{p-1}$, where $\alpha_i>0\,(i=1,2)$ satisfies $\max\{\alpha_1\mu_1,\alpha_2\mu_2\}\le 3$. Let $\Omega_{r_1}=\left\{(u,v)\in K;\|(u,v)\|_0<r_1\right\}$, when $(u,v)\in\partial\Omega_{r_1}\cap K$, we get

$$A_1(u(t),v(t))\le\frac{\alpha_1}{6}\|u+v\|\phi_q\left(\int_0^1 G(\tau,\tau)a_1(\tau)\mathrm{d}\tau\right)\le\frac{\alpha_1}{6}\|(u,v)\|_0\,\mu_1\le\frac{\|(u,v)\|_0}{2}$$

Hence, $\|A_1(u,v)\|\le\frac{\|(u,v)\|_0}{2}$. Similarly, we have $A_2(u(t),v(t))\le\frac{\|(u,v)\|_0}{2}$ then $\|A_2(u,v)\|\le\frac{\|(u,v)\|_0}{2}$, therefore $\|A(u,v)\|_0=\left\{\|A_1(u,v)\|+\|A_2(u,v)\|\right\}\le\|(u,v)\|_0$, $\forall(u,v)\in\partial\Omega_{r_1}\cap K$. By Theorem A, i $\left(A,\Omega_{r_1}\cap K,K\right)=1$.

On the other hand, from (H_2), if $f_{1\infty}(u,v)=\infty$, there exists $R_0>r_1>0$, for $\beta_1>0$ satisfing $\beta_1\nu_1\ge 8$, we get $f_1(u,v)\ge(\beta_1(u+v))^{p-1}$ when $u(t)+v(t)\ge R_0$. Set $R_1>R_0$ such that $R_0\le\|u\|+\|v\|\le R_1$, let $\Omega_{R_1}=\left\{(u,v)\in K;\|(u,v)\|_0<R_1\right\}$, when $(u,v)\in\partial\Omega_{R_1}\cap K$, $t\in\left[\frac14,\frac12\right]$, we get $u+v\ge t(1-t)\|(u,v)\|_0\ge\frac18\|(u,v)\|_0$, so

$$A_1(u(t),v(t))\ge\frac{\beta_1}{8}\|(u,v)\|_0\int_{\frac14}^{\frac12}G(t,s)\phi_q\left(\int_0^1 G(s,\tau)a_1(\tau)\mathrm{d}\tau\right)\mathrm{d}s\ge\frac{\beta_1}{8}\|(u,v)\|_0\,\nu_1\ge\|(u,v)\|_0$$

Hence, $\|A_1(u,v)\| \ge \|(u,v)\|_0$. then $\|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \ge \|(u,v)\|_0$, $\forall (u,v) \in \partial\Omega_{R_1} \cap K$

If $f_{2\infty}(u,v) = \infty$, with the similar proofs of the condition $f_{1\infty}(u,v) = \infty$, we get $\|A_2(u,v)\| \ge \|(u,v)\|_0$. Then $\|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \ge \|(u,v)\|_0$, $\forall (u,v) \in \partial\Omega_{R_1} \cap K$. In either case, we always may set $\|A(u,v)\|_0 \ge \|(u,v)\|_0$, $\forall (u,v) \in \partial\Omega_{R_1} \cap K$ By Theorem A, $i(A, \Omega_{R_1} \cap K, K) = 0$ Through the additivity of the fixed point index we know that

$$i\left(A, \left(\Omega_{R_1} \cap K\right) \setminus \left(\overline{\Omega_{r_1} \cap K}, K\right)\right) = i\left(A, \Omega_{R_1} \cap K, K\right) - i\left(A, \Omega_{r_1} \cap K, K\right) = 0 - 1 = -1$$

Therefore it follows from the fixed-point theorem that A has a fixed point $(u,v) \in \left(\Omega_{R_1} \cap K\right) \setminus \left(\overline{\Omega_{r_1} \cap K}\right)$, and thus $(u(t), v(t))$, $t \in (0,1)$ is a positive solution of BVP (1.1).

Theorem 3.2 Suppose that H holds. If the following conditions are satisfied:

$$(H_3)\ \ f_i^{\infty}(u,v) = 0,\quad i = 1,2;\ \ (H_4)\ \ f_{10}(u,v) = \infty \ \text{ or } \ f_{20}(u,v) = \infty,$$

Then the system (1.1) has at least one positive solution $(u(t), v(t))$, $t \in (0,1)$

Proof By lemma 2.3, we know A is completely continuous. From (H_4), if $f_{10}(u,v) = \infty$, for $\xi_1 > 0$ satisfying $\xi_1 \nu_1 \ge 8$, there exists $r_1' > 0$, when $0 \le u(t) + v((t) \le r_1'$, $t \in [0,1]$, we have $f_1(u,v) \ge \left(\xi_1 (u+v)\right)^{p-1}$. Let $\Omega_{r'} = \left\{(u,v) \in K; \|(u,v)\|_0 < r_1'\right\}$, when $(u,v) \in \partial\Omega_{r_1'} \cap K$, $t \in \left[\frac{1}{4}, \frac{1}{2}\right]$, we get $u + v \ge t(1-t)\|(u,v)\|_0 \ge \frac{1}{8}\|(u,v)\|_0$, then

$$A_1(u(t), v(t)) \ge \frac{\xi_1}{8}\|(u,v)\|_0 \int_{\frac{1}{4}}^{\frac{1}{2}} G(t,s)\phi_q\left(\int_0^1 G(s,\tau) a_1(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \ge \frac{\xi_1}{8}\|(u,v)\|_0 \nu_1 \ge \|(u,v)\|_0$$

Hence, $\|A_1(u,v)\| \ge \|(u,v)\|_0$. then $\|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \ge \|(u,v)\|_0$, $\forall (u,v) \in \partial\Omega_{r_1'} \cap K$

If $f_{20}(u,v) = \infty$, take $\xi_2 > 0$ satisfying $\xi_2 \nu_2 \ge 8$, such taht $f_2(u,v) \ge \left(\xi_2 (u+v)\right)^{p-1}$. Similarly, we get $\|A_2(u,v)\| \ge \|(u,v)\|_0$, then $\|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \ge \|(u,v)\|_0$, $\forall (u,v) \in \partial\Omega_{r_1'} \cap K$ In either case, we always may set $\|A(u,v)\|_0 \ge \|(u,v)\|_0$, $\forall (u,v) \in \partial\Omega_{r_1'} \cap K$. By Theorem A, $i(A, \Omega_{r_1'} \cap K, K) = 0$.

On the other hand, from (H_3), there exists $R_0' > r_1'$ such that $f_i(u,v) \le \left(\theta_i (u+v)\right)^{p-1}$, when $u + v \ge R_0$, where $\theta_i > 0\,(i = 1,2)$ satisfies $\max\{\theta_1\mu_1, \theta_2\mu_2\} \le 3$. There are two cases to consider.

Case (i). Suppose that $\max_{1 \le t \le 1} f_i(u,v)\,(i = 1,2)$ is bounded, then there exists $M_i > 0$ satisfying $f_i(u,v) \le M_i^{p-1}$, $i = 1,2$. Taking $R_1' > \max\left\{R_0', \frac{M_1}{3}\mu_1, \frac{M_2}{3}\mu_2\right\}$, let $\Omega_{R_1'} = \left\{(u,v) \in K; \|(u,v)\|_0 < R_1'\right\}$, when $(u,v) \in \partial\Omega_{R_1'} \cap K$, we get

$$A_1(u(t), v(t)) \le \frac{M_1}{6}\phi_q\left(\int_0^1 G(\tau,\tau) a_1(\tau)\,\mathrm{d}\tau\right) \le \frac{\|(u,v)\|_0}{2}$$

Hence, $\|A_1(u,v)\| \le \frac{\|(u,v)\|_0}{2}$. Similarly, we have $A_2(u(t), v(t)) \le \frac{\|(u,v)\|_0}{2}$, hence $\|A_2(u,v)\| \le \frac{\|(u,v)\|_0}{2}$, then $\|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \le \|(u,v)\|_0$, $\forall (u,v) \in \partial\Omega_{R_1'} \cap K$.

Case (ii). Suppose that $\max_{1 \le t \le 1} f_i(u,v)\,(i = 1,2)$ is unbounded, since $f_i\,(i = 1,2)$ is continuous in $[0,+\infty) \times [0,+\infty)$, so there exists constant $R_1' \ge R_0'$ and two points $(u_i, v_i) \in [0,+\infty) \times [0,+\infty)$ such that $R_0' \le u_i + v_i \le R_1'$, and $f_i(u,v) \le f_i(u_i, v_i)$. Then we get $f_i(u,v) \le f_i(u_i, v_i) \le \left(\theta_i (u_i + v_i)\right)^{p-1} \le \left(\theta_i R_1'\right)^{p-1}$, i = 1, 2. Let $\Omega_{R_1'} = \left\{(u,v) \in K; \|(u,v)\|_0 < R_1'\right\}$, when $(u,v) \in \partial\Omega_{R_1'} \cap K$, we get

$$A_1(u(t), v(t)) \le \frac{\theta_1 R_1'}{6}\phi_q\left(\int_0^1 G(\tau,\tau) a_1(\tau)\,\mathrm{d}\tau\right) \le \frac{R_1'}{2} = \frac{\|(u,v)\|_0}{2}$$

Hence, $\|A_1(u,v)\| \le \frac{\|(u,v)\|_0}{2}$. Similarly, we have $A_2(u(t),v(t)) \le \frac{R_2'}{2} = \frac{\|(u,v)\|_0}{2}$, then $\|A_2(u,v)\| \le \frac{\|(u,v)\|_0}{2}$, so $\|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \le \|(u,v)\|_0$, $\forall (u,v) \in \partial\Omega_{R_1'} \cap K$. In either case, we always may set $\|A(u,v)\|_0 \le \|(u,v)\|_0$, $\forall (u,v) \in \partial\Omega_{R_1'} \cap K$. By Theorem A, $i(A,\Omega_{R_1'} \cap K, K) = 1$. Through the additivity of the fixed point index we know that

$$i\left(A,\left(\Omega_{R_1'} \cap K\right) \backslash \left(\overline{\Omega_{r_1'} \cap K}, K\right)\right) = i\left(A,\Omega_{R_1'} \cap K, K\right) - i\left(A,\Omega_{r_1'} \cap K, K\right) = 0 - 1 = -1$$

Therefore it follows from the fixed-point theorem that A has a fixed point $(u,v) \in \left(\Omega_{R_1'} \cap K\right) \backslash \left(\overline{\Omega_{r_1'} \cap K}\right)$, and thus $(u(t),v(t))$, $t \in (0,1)$ is a positive solution of BVP (1.1). This completes the proof.

Remark 3.1 Note that if f is superlinear or sublinear, our conclusions hold. Limit conditions of f in this paper are more weak and general.

Remak 3.2 When $u = v$, $f_1 = f_2$ and $w_1 = w_2$, our results generalize and improve the results of [1]-[4].

Theorem 3.3 Suppose that H holds. If the following conditions are satisfied:

(H_5) $f_i^0(u,v) = \rho_i \in [0,+\infty)$, $i = 1,2$ where $\rho_i\,(i=1,2)$ satisfies $\max\left\{\rho_1^{q-1}\mu_1, \rho_2^{q-1}\mu_1\right\} \le 3$;

(H_6) $f_{1\infty}(u,v) = \lambda_1 \in (0,+\infty)$ or $f_{2\infty}(u,v) = \lambda_2 \in (0,+\infty)$, where λ_i satisfies $\lambda_i^{q-1}\nu_i \ge 8$ $(i = 1 \text{ or } i = 2)$

then the system (1.1) has at least one positive solution $(u(t),v(t))$, $t \in (0,1)$

Proof. Choosing $\varepsilon_i > 0\,(i=1,2)$ such that $\max\left\{(\rho_1+\varepsilon_1)^{q-1}\mu_1, (\rho_2+\varepsilon_2)^{q-1}\mu_2\right\} \le 3$ and $(\lambda_i - \varepsilon)^{q-1}\nu_i \ge 8$, $i = 1$ or $i = 2$ From (H_5), there exists $r_1^* > 0$ such that $f_i(u,v) \le (\rho_i + \varepsilon_i)(u,v)^{p-1}$ $(i=1,2)$ when $0 \le u + v \le r_1^*$. Let $\Omega_{r_1^*} = \left\{(u,v) \in K; \|(u,v)\|_0 < r_1^*\right\}$ when $(u,v) \in \partial\Omega_{r_1^*} \cap K$, we get

$$A_1(u(t),v(t)) \le \frac{(\rho_1+\varepsilon_1)^{q-1}}{6}\|u+v\|\phi_q\left(\int_0^1 G(\tau,\tau)a_1(\tau)\,\mathrm{d}\tau\right) \le \frac{(\rho_1+\varepsilon_1)^{q-1}}{6}\|(u,v)\|_0\,\mu_1 \le \frac{\|(u,v)\|_0}{2}$$

Hence, $\|A_1(u,v)\| \le \frac{\|(u,v)\|_0}{2}$. Similarly, we have $A_2(u(t),v(t)) \le \frac{\|(u,v)\|_0}{2}$, so $\|A_2(u,v)\| \le \frac{\|(u,v)\|_0}{2}$, then $\|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \le \|(u,v)\|_0$, $\forall (u,v) \in \partial\Omega_{r_1^*} \cap K$. By Theorem A, $i\left(A,\Omega_{r_1^*} \cap K, K\right) = 1$.

On the other hand, From (H_6), if $f_{1\infty}(u,v) = \lambda_1$, there exists $R_0^* > r_1^*$, such that $f_1(u,v) \ge (\lambda_1 - \varepsilon_1)(u,v)^{p-1}$ when $u(t) + v(t) \ge R_0^*$. Set $R_1^* > R_0^*$ such that $R_0 \le \|u\| + \|v\| \le R_1^*$, let $\Omega_{R_1} = \left\{(u,v) \in K; \|(u,v)\|_0 < R_1^*\right\}$, when $(u,v) \in \partial\Omega_{R_1^*} \cap K$, $t \in \left[\frac{1}{4},\frac{1}{2}\right]$, we get $u + v \ge t(1-t)\|(u,v)\|_0 \ge \frac{1}{8}\|(u,v)\|_0$, then

$$\begin{aligned} A_1(u(t),v(t)) &\ge \frac{(\lambda_1-\varepsilon_1)^{q-1}}{8}\|(u,v)\|_0 \int_{\frac{1}{4}}^{\frac{1}{2}} G(t,s)\phi_q\left(\int_0^1 G(s,\tau)a_1(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\ge \frac{(\lambda_1-\varepsilon_1)^{q-1}}{8}\|(u,v)\|_0\,\nu_1 \ge \|(u,v)\|_0, \end{aligned}$$

Hence, $\|A_1(u,v)\| \ge \|(u,v)\|_0$. then $\|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \ge \|(u,v)\|_0$, $\forall (u,v) \in \partial\Omega_{R_1^*} \cap K$.

If $f_{2\infty}(u,v) = \lambda_2$, by $(\lambda_2 - \varepsilon)^{q-1}\nu_2 \ge 8$, with the similar proofs of the condition $f_{1\infty}(u,v) = \lambda_1$, we get $\|A_2(u,v)\| \ge \|(u,v)\|_0$. Then $\|A(u,v)\|_0 = \|A_1(u,v)\| + \|A_2(u,v)\| \ge \|(u,v)\|_0$, $\forall (u,v) \in \partial\Omega_{R_1^*} \cap K$ In either case, we always may set $\|A(u,v)\|_0 \ge \|(u,v)\|_0$, $\forall (u,v) \in \partial\Omega_{R_1^*} \cap K$. By Theorem A, $i\left(A,\Omega_{R_1^*} \cap K, K\right) = 0$. Through the additivity of the fixed point index we know that

$$i\left(A,\left(\Omega_{R_1^*} \cap K\right) \backslash \left(\overline{\Omega_{r_1^*} \cap K}, K\right)\right) = i\left(A,\Omega_{R_1^*} \cap K, K\right) - i\left(A,\Omega_{r_1^*} \cap K, K\right) = 0 - 1 = -1$$

Therefore it follows from the fixed-point theorem that A has a fixed point $(u,v) \in \left(\Omega_{R_1^*} \cap K\right) \backslash \left(\overline{\Omega_{r_1^*} \cap K}\right)$, and thus $(u(t),v(t))$, $t \in (0,1)$ is a positive solution of BVP (1.1). This completes the proof.

Theorem 3.4 Suppose that H holds. If the following conditions are satisfied:

(H_7) $f_{i0}(u,v) = \rho_i^* \in (0,+\infty)$, $i = 1$ or $i = 2$ where $\rho_i^*(i = 1,2)$ satisfies $\rho_i^{q-1} v_i \geq 8$, $(i = 1$ or $i = 2)$;

(H_8) $f_i^\infty(u,v) = \lambda_i^* \in [0,+\infty)$ where λ_i^* satisfies $\max\{\lambda_1^{q-1}\mu_1, \lambda_2^{q-1}\mu_2\} \leq 3$, $(i = 1,2)$, then the system (1.1) has at least one positive solution $(u(t), v(t))$, $t \in (0,1)$

The proofs are similar to that of Theorem 3.2 and are omitted.

Theorem 3.5 Assume that H, H_2 holds. If the following conditions are satisfied:

$$(H_9)\ f_{i0}(u,v) = 0,\quad i = 1,2;\ (H_{10})\ f_1^0(u,v) = \infty \ \text{ or } \ f_2^0(u,v) = \infty,$$

Then the system (1.1) has at least two positive solutions $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ satisfying $0 < \|(u_1, v_1)\|_0 \leq r_1 \leq \|(u_2, v_2)\|_0$.

Theorem 3.6 Assume that H, H_3, H_9, H_{10} hold. then the system (1.1) has at least two positive solutions $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ satisfying $0 < \|(u_1, v_1)\|_0 \leq r_1 \leq \|(u_2, v_2)\|_0$.

Remark 3.3 Under suitable weak conditions, the multiplicity results for fourth-order singular boundary value problem with p-Laplacian are established. Our results extend and improve the results of [5]-[8].

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