Journal of Applied Mathematics and Physics, 2014, 2, 1224-1232
Published Online December 2014 in SciRes. http://www.scirp.org/journal/jamp
http://dx.doi.org/10.4236/jamp.2014.213143

Scientific Research

On the Cauchy Problem for Von Neumann-Landau Wave Equation

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Received 21 October 2014; revised 16 November 2014; accepted 11 December 2014

Abstract

In present paper we prove the local well-posedness for Von Neumann-Landau wave equation by the T. Kato's method.

Keywords

Von Neumann-Landau Wave Equation, Strichartz Estimate, Cauchy Problem

1. Introduction

For the stationary Von Neumann-Landau wave equation, Chen investigated the Dirichlet problems [1], where the generalized solution is studied by Function-analytic method. The present paper is related to the Cauchy problem: the Von Neumann-Landau wave equation

$$\begin{cases} i\partial_t u = \left(-\Delta_x + \Delta_y\right)u + f(u) \\ u(0,x,y) = u_0(x,y) \end{cases}, \tag{1}$$

where $\Delta_x = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ for $x = (x_1,\cdots,x_n) \in \mathbb{R}^n$, $u(t,x,y)$ is an unknown complex valued function on $\mathbb{R}^{1+2n}$ and f is a nonlinear complex valued function.

If the plus "+" is replaced by the minus "−" on right hand in Equation (1), then the resulted equation is the Schrödinger equation. For the Schrödinger equation, the well-posedness problem is investigated for various nonlinear terms f. In terms of the nonlinear terms f, the problem (1) can be divided into the subcritical case and the critical case for H^1 solutions. We are concerned with the subcritical case and obtain a local well-

How to cite this paper: Liu, C.Y. and Liu, M.M. (2014) On the Cauchy Problem for Von Neumann-Landau Wave Equation. *Journal of Applied Mathematics and Physics*, **2**, 1224-1232. http://dx.doi.org/10.4236/jamp.2014.213143

posedness result by the T. Kato's method.

The paper is organized as follows. Section 2 contains the list of assumptions on the interaction term f and the main result is presented. Section 3 is concerned with the Strichartz estimates. Finally, in Section 4, the main result is proved.

2. Statement of the Main Result

In this section we list the assumptions on the interaction term f and state the main result. Firstly, we recall that the definition of admissible pair [2].

Definition 2.1. *Fix* $d = 2n$, $n \geq 1$. *We say that a pair* (q, r) *of exponents is admissible if*

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right), \tag{2}$$

and

$$2 \leq r \leq \frac{2d}{d-2} \ (2 \leq r < \infty, \text{if } d = 2). \tag{3}$$

Remark 2.1. *The pairs* $(\infty, 2)$ *is always admissible, so is the* $\left(2, \frac{2d}{d-2}\right)$ *if* $d > 2$. *The two pairs are called the endpoint cases.*

Secondly, let $f \in C(\mathbb{C}, \mathbb{C})$ satisfy

$$f(0) = 0, \tag{4}$$

and

$$|f(u) - f(v)| \leq K(M)|u - v|, \tag{5}$$

for all $u, v \in \mathbb{C}$ such that $|u|, |v| \leq M$, with

$$K(t) \leq C_1(1 + t^{\alpha}), \ 0 < \alpha < \frac{4}{d-2}, \tag{6}$$

where C_1 is a constant independent of t. Set

$$f(u)(x) = f(u(x)), \tag{7}$$

for all measurable function u and a.e. $x \in \mathbb{R}^{1+2n}$.

Finally, let us make the notion of solution more precise.

Definition 2.2. *Let* I *be an interval such that* $0 \in I$. *We say that* u *is a strong* H^1*-solution of* (1) *on* I *if* $u \in C\left(I, H^1\left(\mathbb{R}^d\right)\right)$ *satisfies the integral equation*

$$u(t) = \mathrm{e}^{-itL}u_0 - i\int_0^t \mathrm{e}^{-i(t-s)L} f(u(s))\,\mathrm{d}s, \tag{8}$$

for all $t \in I$, *where* $L =: -\Delta_x + \Delta_y$.

The main result is the following theorem:

Theorem 1. *Suppose* $n \geq 1$. *Let* $f \in C(\mathbb{C}, \mathbb{C})$ *satisfy* (4)-(6). *If* f *(considered as a function* $\mathbb{R}^2 \to \mathbb{R}^2$*) is of class* C^1, *then the Cauchy problem* (1) *is locally well posed in* $H^1(\mathbb{R}^d)$. *More specially, the following properties hold*:

(i) *For any* $R > 0$ *there exists a time* $T = T(d, \alpha, R) > 0$ *and constant* $c = c(d, \alpha)$ *such that for each* u_0 *in the ball* $B_R := \left\{\varphi \in H^1\left(\mathbb{R}^d\right) : \|\varphi\|_{H^1\left(\mathbb{R}^d\right)} \leq R\right\}$ *there exists a unique strong* H^1*-solution* u *to the Equation* (1) *in* $C\left([-T, T], H^1\left(\mathbb{R}^d\right)\right)$ *such that*

$$\|u\|_{L^\infty\left((-T,T),H^1(\mathbb{R}^d)\right)}+\|u\|_{L^q\left((-T,T),W^{1,r}(\mathbb{R}^d)\right)}\le c\|u_0\|_{H^1(\mathbb{R}^d)}, \tag{9}$$

where $r=\alpha+2$, *and* (q,r) *is an admissible pair.*

(ii) *The map* $u_0\mapsto u$ *is continuous from* B_R *to* $C\left([-T,T],H^1(\mathbb{R}^d)\right)$;

(iii) *For every* $u_0\in H^1(\mathbb{R}^d)$, *the unique solution* u *is defined on a maximal interval* $(-T_{\min},T_{\max})$, *with* $T_{\max}=T_{\max}(u_0)\in(0,\infty]$ *and* $T_{\min}=T_{\min}(u_0)\in(0,\infty]$;

(iv) *There is the blowup alternative: If* $T_{\max}<\infty$, *then* $\|u(t)\|_{H^1(\mathbb{R}^d)}\to+\infty$ *as* $t\nearrow T_{\max}$ (*respectively, if* $T_{\min}<\infty$, *then* $\|u(t)\|_{H^1(\mathbb{R}^d)}\to+\infty$ *as* $t\searrow -T_{\min}$).

Remark 2.2. *It follows from Strichartz estimates that*

$$u\in L^\gamma_{loc}\left((-T_{\min},T_{\max}),W^{1,\rho}(\mathbb{R}^d)\right),$$

for any admissible pair (γ,ρ).

Remark 2.3. *For the Schrödinger equations, the similar results hold* [2]. *It implies a fact that the ellipticity of the operator* $-\Delta_x-\Delta_y$ *is not the key point in the local well-posedness problem.*

3. Strichartz Estimates

In this subsection, we recall that the Strichartz estimates. Let (ξ,η) denote a general Fourier variable in $\mathbb{R}^{2n}$, $\xi=(\xi_1,\cdots,\xi_n)$, $\eta=(\eta_1,\cdots,\eta_n)$. Let $L=:-\Delta_x+\Delta_y$, then by Fourier transform(denoting by $\mathcal{F}$ or $\wedge$) we have

$$Lu=\mathcal{F}^{-1}\left(\left(|\xi|^2-|\eta|^2\right)\hat{u}\right), \tag{10}$$

for any $u\in H^2(\mathbb{R}^{2n})$. It is easy to verify that the L is a self-adjoint unbounded operator on $L^2(\mathbb{R}^{2n})$ with the domain $H^2(\mathbb{R}^{2n})$. Then, by Stone theorem we see that e^{itL} is an unitary group on $L^2(\mathbb{R}^{2n})$. Moreover, e^{itL} can be expressed explicitly by Fourier transform.

$$e^{itL}\varphi=\mathcal{F}^{-1}\left(e^{it\left(|\xi|^2-|\eta|^2\right)}\hat{\varphi}\right), \tag{11}$$

for any $\varphi\in L^2(\mathbb{R}^{2n})$. By the direct compute, we conclude

$$\left(e^{itL}\varphi\right)(x,y)=\frac{1}{\left(4\pi i|t|\right)^n}\int_{\mathbb{R}^{2n}}e^{\frac{-i|x-x'|^2}{4t}}e^{\frac{i|y-y'|^2}{4t}}\varphi(x',y')\,dx'dy'. \tag{12}$$

The following result is the fundamental estimate for e^{itL}.

Lemma 1. *If* $p\in[1,2]$ *and* $t\neq0$, *then* e^{itL} *maps* $L^p(\mathbb{R}^d)$ *continuously to* $L^{p'}(\mathbb{R}^d)$ *and*

$$\left\|e^{itL}\varphi\right\|_{L^{p'}(\mathbb{R}^d)}\le\left(4\pi|t|\right)^{-d\left(\frac{1}{p}-\frac{1}{2}\right)}\|\varphi\|_{L^p(\mathbb{R}^d)} \tag{13}$$

where p' *is the dual exponent of* p, *defined by the formula* $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. For the proof please see [3] or [4]. □

The following estimates, known as Strichartz estimates, are key points in the method introduced by T. Kato [5].

Lemma 2. *Let* (q,r) *and* $(\tilde{q},\tilde{r})$ *be any admissible exponents. Then, we have the homogeneous Strichartz* *estimate*

$$\left\| e^{itL}\varphi \right\|_{L^q\left(\mathbb{R}, L^r\left(\mathbb{R}^d\right)\right)} \lesssim_{d,q,r} \left\|\varphi\right\|_{L^2\left(\mathbb{R}^d\right)}, \tag{14}$$

the dual homogeneous Strichartz estimate

$$\left\| \int_{\mathbb{R}} e^{-itL}\phi(t)\,dt \right\|_{L^2\left(\mathbb{R}^d\right)} \lesssim_{d,q,r} \left\|\phi\right\|_{L^{q'}\left(\mathbb{R}, L^{r'}\left(\mathbb{R}^d\right)\right)}, \tag{15}$$

and the inhomogeneous Strichartz estimate

$$\left\| \int_{t_0}^{t} e^{i(t-s)L}\phi(s)\,ds \right\|_{L^q\left(J, L^r\left(\mathbb{R}^d\right)\right)} \lesssim_{d,q,r,\tilde{q},\tilde{r}} \left\|\phi\right\|_{L^{\tilde{q}'}\left(J, L^{\tilde{r}'}\left(\mathbb{R}^d\right)\right)}, \tag{16}$$

for any interval J *and real number* t_0.

Proof. For the proof please see [3] or [4] in the non-endpoint case. On the other hand, the proof in the endpoint case follows from the theorem 1.2 in [6] and the lemma 1 in the present paper. □

4. The Proof of Theorem

Proof. Let $\chi \in C_0^\infty\left(\mathbb{R}^2\right)$ be such that $\chi(z)=1$ for $|z|\le 1$ and $\chi(z)=0$ for $|z|\ge 2$. Setting

$$f_1(z) = \chi(z) f(z),$$
$$f_2(z) = \left(1-\chi(z)\right) f(z),$$

one easily verifies that for any $z, w \in \mathbb{C}$

$$\left| f_1(z) - f_1(w)\right| \lesssim_\alpha |z-w|,$$
$$\left| f_2(z) - f_2(w)\right| \lesssim_\alpha \left(|z|^\alpha + |w|^\alpha\right)|z-w|. \tag{17}$$

Set $f_l(u)(x) = f_l(u(x))$ for $l=1,2$. Using (17), we deduce from Hölder's inequality that

$$\left\| f_1(u) - f_1(v)\right\|_{L^2\left(\mathbb{R}^d\right)} \lesssim_\alpha \left\|u-v\right\|_{L^2\left(\mathbb{R}^d\right)},$$
$$\left\| f_2(u) - f_2(v)\right\|_{L^{r'}\left(\mathbb{R}^d\right)} \lesssim_\alpha \left(\left\|u\right\|_{L^r\left(\mathbb{R}^d\right)}^\alpha + \left\|v\right\|_{L^r\left(\mathbb{R}^d\right)}^\alpha\right)\left\|u-v\right\|_{L^r\left(\mathbb{R}^d\right)}. \tag{18}$$

And it follows from Remark 1.3.1 (vii) in [2] that

$$\left\|\nabla f_1(u)\right\|_{L^2\left(\mathbb{R}^d\right)} \lesssim_\alpha \left\|\nabla u\right\|_{L^2\left(\mathbb{R}^d\right)},$$
$$\left\|\nabla f_2(u)\right\|_{L^{r'}\left(\mathbb{R}^d\right)} \lesssim_\alpha \left\|u\right\|_{L^r\left(\mathbb{R}^d\right)}^\alpha \left\|\nabla u\right\|_{L^r\left(\mathbb{R}^d\right)}. \tag{19}$$

We now proceed in four steps.

Step 1. Proof of (i). Fix $A, T > 0$, to be chosen later, and let $r = \alpha + 2$, q be such that (q, r) is an admissible pair, and set $I = (-T, T)$. Consider the set

$$E = \left\{ u \in L^\infty\left(I, H^1\left(\mathbb{R}^d\right)\right) \cap L^q\left(I, W^{1,r}\left(\mathbb{R}^d\right)\right) : \left\|u\right\|_{L^\infty\left(I, H^1\left(\mathbb{R}^d\right)\right)} \le A, \left\|u\right\|_{L^q\left(I, W^{1,r}\left(\mathbb{R}^d\right)\right)} \le A \right\}, \tag{20}$$

equipped with the distance

$$d(u,v) = \left\|u-v\right\|_{L^\infty\left(I, L^2\left(\mathbb{R}^d\right)\right)} + \left\|u-v\right\|_{L^q\left(I, L^r\left(\mathbb{R}^d\right)\right)}. \tag{21}$$

We claim that (E, d) is a complete metric space. Indeed, let $\{u_k\}_{k\ge1} \subset E$ be a Cauchy sequence. Clearly, $\{u_k\}_{k\ge1}$ is also a Cauchy sequence in $L^\infty\left(I, L^2\left(\mathbb{R}^d\right)\right)$ and $L^q\left(I, L^r\left(\mathbb{R}^d\right)\right)$. In particular, there exists a function $u \in L^{\infty}\left(I, L^{2}\left(\mathbb{R}^{d}\right)\right) \cap L^{q}\left(I, L^{r}\left(\mathbb{R}^{d}\right)\right)$ such that $u_{k} \to u$ in $L^{\infty}\left(I, L^{2}\left(\mathbb{R}^{d}\right)\right)$ and $L^{q}\left(I, L^{r}\left(\mathbb{R}^{d}\right)\right)$ as $k \to \infty$. Applying theorem 1.2.5 in [2] twice, we conclude that

$$u \in L^{\infty}\left(I, H^{1}\left(\mathbb{R}^{d}\right)\right) \cap L^{q}\left(I, W^{1,r}\left(\mathbb{R}^{d}\right)\right),$$

and that

$$\|u\|_{L^{\infty}\left(I, H^{1}\left(\mathbb{R}^{d}\right)\right)} \le \liminf_{k\to\infty} \|u_{k}\|_{L^{\infty}\left(I, H^{1}\left(\mathbb{R}^{d}\right)\right)} \le A,$$

$$\|u\|_{L^{q}\left(I, W^{1,r}\left(\mathbb{R}^{d}\right)\right)} \le \liminf_{k\to\infty} \|u_{k}\|_{L^{q}\left(I, W^{1,r}\left(\mathbb{R}^{d}\right)\right)} \le A;$$

thus, $u_{k} \to u$ in E as $k \to \infty$.

Taking up any $u, v \in E$. Since f_1 is continuous $L^{2}\left(\mathbb{R}^{d}\right) \to L^{2}\left(\mathbb{R}^{d}\right)$, it follows that $f_1(u): I \to L^{2}\left(\mathbb{R}^{d}\right)$ is measurable, and we deduce easily that $f_1(u) \in L^{\infty}\left(I, L^{2}\left(\mathbb{R}^{d}\right)\right)$. Similarly, since f_2 is continuous $L^{r}\left(\mathbb{R}^{d}\right) \to L^{r'}\left(\mathbb{R}^{d}\right)$, we see that $f_2(u) \in L^{q}\left(I, L^{r'}\left(\mathbb{R}^{d}\right)\right)$. Using inequalities (18) and (19) and Remark 1.2.2 (iii) in [2], We deduce the following:

$$f_1(u) \in L^{\infty}\left(I, H^{1}\left(\mathbb{R}^{d}\right)\right),\ f_2(u) \in L^{q}\left(I, w^{1,r'}\left(\mathbb{R}^{d}\right)\right),$$

$$\|f_1(u)\|_{L^{\infty}\left(I, H^{1}\left(\mathbb{R}^{d}\right)\right)} \lesssim_{\alpha} \|u\|_{L^{\infty}\left(I, H^{1}\left(\mathbb{R}^{d}\right)\right)},$$

$$\|f_2(u)\|_{L^{q}\left(I, w^{1,r'}\left(\mathbb{R}^{d}\right)\right)} \lesssim_{\alpha} \|u\|^{\alpha}_{L^{\infty}\left(I, L^{r}\left(\mathbb{R}^{d}\right)\right)} \|u\|_{L^{q}\left(I, W^{1,r}\left(\mathbb{R}^{d}\right)\right)},$$

and

$$\|f_1(u) - f_1(v)\|_{L^{\infty}\left(I, L^{2}\left(\mathbb{R}^{d}\right)\right)} \lesssim_{\alpha} \|u - v\|_{L^{\infty}\left(I, L^{2}\left(\mathbb{R}^{d}\right)\right)},$$

$$\|f_2(u) - f_2(v)\|_{L^{q}\left(I, L^{r'}\left(\mathbb{R}^{d}\right)\right)} \lesssim_{\alpha} \left(\|u\|^{\alpha}_{L^{\infty}\left(I, L^{r}\left(\mathbb{R}^{d}\right)\right)} + \|v\|^{\alpha}_{L^{\infty}\left(I, L^{r}\left(\mathbb{R}^{d}\right)\right)}\right) \|u - v\|_{L^{q}\left(I, L^{r}\left(\mathbb{R}^{d}\right)\right)}.$$

Using the embedding $H^{1}\left(\mathbb{R}^{d}\right) \hookrightarrow L^{r}\left(\mathbb{R}^{d}\right)$ and Hölder's inequality in time, we deduce from the above estimates that

$$\|f_1(u)\|_{L^{1}\left(I, H^{1}\left(\mathbb{R}^{d}\right)\right)} + \|f_2(u)\|_{L^{q'}\left(I, W^{1,r'}\left(\mathbb{R}^{d}\right)\right)} \lesssim_{d,\alpha} \left(T + T^{\frac{q-q'}{qq'}}\right)\left(1 + A^{\alpha}\right) A, \tag{22}$$

and

$$\|f_1(u) - f_1(v)\|_{L^{1}\left(I, L^{2}\left(\mathbb{R}^{d}\right)\right)} + \|f_2(u) - f_2(v)\|_{L^{q'}\left(I, L^{r'}\left(\mathbb{R}^{d}\right)\right)} \lesssim_{d,\alpha} \left(T + T^{\frac{q-q'}{qq'}}\right)\left(1 + A^{\alpha}\right) d(u, v). \tag{23}$$

Given $u_0 \in H^{1}\left(\mathbb{R}^{d}\right)$. For any $u \in E$, let $\mathcal{H}(u)$ be defined by

$$\mathcal{H}(u)(t) = \mathrm{e}^{-itL} u_0 - i\int_0^t \mathrm{e}^{-i(t-s)L} f\left(u(s)\right) \mathrm{d}s. \tag{24}$$

It follows from (22) and Strichartz estimates (lemma 2) that

$$\mathcal{H}(u) \in C\left(\left[-T, T\right], H^{1}\left(\mathbb{R}^{d}\right)\right) \cap L^{q}\left(\left(-T, T\right), W^{1,r}\left(\mathbb{R}^{d}\right)\right), \tag{25}$$

and

$$\|\mathcal{H}(u)\|_{L^{\infty}\left(I, H^{1}\left(\mathbb{R}^{d}\right)\right)} + \|\mathcal{H}(u)\|_{L^{q}\left(I, W^{1,r}\left(\mathbb{R}^{d}\right)\right)} \le C_1(d, \alpha)\left[\|u_0\|_{H^{1}\left(\mathbb{R}^{d}\right)} + \left(T + T^{\frac{q-q'}{qq'}}\right)\left(1 + A^{\alpha}\right) A\right]. \tag{26}$$

Also, we deduce from (23) that

$$\|\mathcal{H}(u)-\mathcal{H}(v)\|_{L^\infty(I,L^2(\mathbb{R}^d))}+\|\mathcal{H}(u)-\mathcal{H}(v)\|_{L^q(I,L^r(\mathbb{R}^d))}\le C_1(d,\alpha)\left(T+T^{\frac{q-q'}{qq'}}\right)(1+A^\alpha)d(u,v). \tag{27}$$

Finally, note that $q>q'$. We now proceed as follows. For any $R\ge\|u_0\|_{H^1(\mathbb{R}^d)}$, we set $A=2C_1(d,\alpha)R$, and we let $T=T(d,\alpha,R)$ be the unique positive number so that

$$C_1(d,\alpha)\left(T+T^{\frac{q-q'}{qq'}}\right)(1+A^\alpha)=\frac{1}{2}. \tag{28}$$

It then follows from (26) and (28) that for any $u_0\in B_R$

$$\|\mathcal{H}(u)\|_{L^\infty(I,H^1(\mathbb{R}^d))}+\|\mathcal{H}(u)\|_{L^q(I,W^{1,r}(\mathbb{R}^d))}\le C_1(d,\alpha)\|u_0\|_{H^1(\mathbb{R}^d)}+\frac{1}{2}A\le C_1(d,\alpha)R+\frac{1}{2}A=A. \tag{29}$$

Thus, $\mathcal{H}(u)\in E$ and by (27) we obtain

$$d(\mathcal{H}(u),\mathcal{H}(v))\le\frac{1}{2}d(u,v). \tag{30}$$

In particular, $\mathcal{H}$ is a strict contraction on E. By Banach's fixed-point theorem, $\mathcal{H}$ has a unique fixed point $u\in E$; that is u satisfies (8). By (25), $u=\mathcal{H}(u)\in C([-T,T],H^1(\mathbb{R}^d))$. By the definition 2.2, we con- clude that u is a strong H^1-solution of (1) on $[-T,T]$. Note that $T(d,\alpha,R)$ is decreasing on R, then the estimate (9) holds for $c=2C_1(d,\alpha)$ by letting $R=\|u_0\|_{H^1(\mathbb{R}^d)}$ in (29).

For uniqueness, assume that u,v are two strong H^1-solution of (1) on $[-T,T]$ with the same initial value u_0. Then, we have

$$u(t)-v(t)=-i\int_0^t \mathrm{e}^{-i(t-s)L}\left[f(u(s))-f(v(s))\right]\mathrm{d}s. \tag{31}$$

For simplicity, we set

$$w_l(t)=-i\int_0^t \mathrm{e}^{-i(t-s)L}\left[f_l(u(s))-f_l(v(s))\right]\mathrm{d}s,$$

for $l=1,2$, and $w=u-v$. For any interval $J\subset(-T,T)$, by (18) and Strichartz estimates (16), then we obtain

$$\|w_1\|_{L^\infty(J,L^2(\mathbb{R}^d))}+\|w_1\|_{L^q(J,L^r(\mathbb{R}^d))}\lesssim_{d,\alpha}\|f_1(u)-f_1(v)\|_{L^1(J,L^2(\mathbb{R}^d))}\lesssim_{d,\alpha}\|w\|_{L^1(J,L^2(\mathbb{R}^d))}. \tag{32}$$

Similarly, for w_2 we have

$$\begin{aligned}&\|w_2\|_{L^\infty(J,L^2(\mathbb{R}^d))}+\|w_2\|_{L^q(J,L^r(\mathbb{R}^d))}\lesssim_{d,\alpha}\|f_2(u)-f_2(v)\|_{L^{q'}(J,L^{r'}(\mathbb{R}^d))}\\&\lesssim_{d,\alpha}\left(\|u\|^\alpha_{L^\infty(I,H^1(\mathbb{R}^d))}+\|v\|^\alpha_{L^\infty(I,H^1(\mathbb{R}^d))}\right)\|w\|_{L^{q'}(J,L^r(\mathbb{R}^d))}.\end{aligned} \tag{33}$$

Note that $w=w_1+w_2$. Then, it follows from that

$$\|w\|_{L^\infty(J,L^2(\mathbb{R}^d))}+\|w\|_{L^q(J,L^r(\mathbb{R}^d))}\le C_2(1+B)\left(\|w\|_{L^1(J,L^2(\mathbb{R}^d))}+\|w\|_{L^{q'}(J,L^r(\mathbb{R}^d))}\right), \tag{34}$$

where the constant $B=\|u\|^\alpha_{L^\infty(I,H^1(\mathbb{R}^d))}+\|v\|^\alpha_{L^\infty(I,H^1(\mathbb{R}^d))}$ and the constant C_2 is independent of J by above inequalities. Note that $q'<q$, we conclude that $w=0$ by the lemma 4.2.2 in [2]. So $u=v$.

Step 2. Proof of (ii). Suppose that $u_0^{(k)}\to u_0$ in B_R as $k\to\infty$. By the part (i), we denote u_k and u by the unique solution of (1) corresponding to the initial value $u_0^{(k)}$ and u, respectively. We will show that $u_k \to u$ in $C\left(\left[-T,T\right],H^1\left(\mathbb{R}^d\right)\right)$ as $k\to\infty$. Note that

$$u_k(t)-u(t)=\mathrm{e}^{-itL}\left(u_0^{(k)}-u_0\right)+\mathcal{H}(u_k)-\mathcal{H}(u), \tag{35}$$

and the estimate (29) which implies that (27) holds for $v=u_k$. Note that the choosing of the time T in (28), it follows from (27) with (30) that

$$d(u_k,u)\lesssim_{d,\alpha}\left\|u_0^{(k)}-u_0\right\|_{L^2\left(\mathbb{R}^d\right)}+\frac{1}{2}d(u_k,u). \tag{36}$$

Hence, we have

$$\left\|u_k-u\right\|_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}+\left\|u_k-u\right\|_{L^q\left((-T,T),L^r\left(\mathbb{R}^d\right)\right)}\lesssim_{d,\alpha}\left\|u_0^{(k)}-u_0\right\|_{L^2\left(\mathbb{R}^d\right)}. \tag{37}$$

Next, we need to estimate $\left\|\nabla(u_k-u)\right\|_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}$. Note that ∇ commutes with e^{-itL}, and so

$$\nabla u(t)=\mathrm{e}^{-itL}\nabla u_0-i\int_0^t \mathrm{e}^{-i(t-s)L}\nabla f\left(u(s)\right)\mathrm{d}s. \tag{38}$$

A similar identity holds for u_k. We use the assumption $f\in C^1\left(\mathbb{R}^2,\mathbb{R}^2\right)$, which implies that $\nabla f(u)=f'(u)\nabla u$, where $f'(u)$ is a 2×2 real matrix. Therefore, we may write

$$\begin{aligned}\nabla(u_k-u)(t)=\mathrm{e}^{-itL}\nabla\left(u_0^{(k)}-u_0\right)&-i\int_0^t \mathrm{e}^{-i(t-s)L}f'(u_k)\nabla(u_k-u)\mathrm{d}s\\&-i\int_0^t \mathrm{e}^{-i(t-s)L}\left(f'(u_k)-f'(u)\right)\nabla u\mathrm{d}s.\end{aligned} \tag{39}$$

Note that f_1 and f_2 are also C^1, so that $f'=f_1'+f_2'$, and from (17) we deduce that $\left|f_1'(z)\right|\le C_3$ and $\left|f_2'(z)\right|\lesssim_\alpha |z|^\alpha$ for any $z\in\mathbb{C}$ and some constant C_3. Therefore, arguing as in Step 1, we obtain the estimate

$$\begin{aligned}&\left\|\nabla(u_k-u)\right\|_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}+\left\|\nabla(u_k-u)\right\|_{L^q\left((-T,T),L^r\left(\mathbb{R}^d\right)\right)}\\&\lesssim_{d,\alpha}\Bigg[\left\|\nabla\left(u_0^{(k)}-u_0\right)\right\|_{L^2\left(\mathbb{R}^d\right)}+T\left\|\nabla(u_k-u)\right\|_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}\\&\quad+T^{\frac{q-q'}{qq'}}\left\|u_k\right\|^\alpha_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}\left\|\nabla(u_k-u)\right\|_{L^q\left((-T,T),L^r\left(\mathbb{R}^d\right)\right)}\\&\quad+\left\|\left(f_1'(u_k)-f_1'(u)\right)\nabla u\right\|_{L^1\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}\\&\quad+\left\|\left(f_2'(u_k)-f_2'(u)\right)\nabla u\right\|_{L^{q'}\left((-T,T),L^{r'}\left(\mathbb{R}^d\right)\right)}\Bigg].\end{aligned} \tag{40}$$

By choosing $T=T(d,\alpha,R)$ as (28) and noting that $u_k\in B_R$, from (40) we obtain that

$$\begin{aligned}&\left\|\nabla(u_k-u)\right\|_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}+\left\|\nabla(u_k-u)\right\|_{L^q\left((-T,T),L^r\left(\mathbb{R}^d\right)\right)}\\&\lesssim_{d,\alpha}\Bigg[\left\|\nabla\left(u_0^{(k)}-u_0\right)\right\|_{L^2\left(\mathbb{R}^d\right)}+\left\|\left(f_1'(u_k)-f_1'(u)\right)\nabla u\right\|_{L^1\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}\\&\quad+\left\|\left(f_2'(u_k)-f_2'(u)\right)\nabla u\right\|_{L^{q'}\left((-T,T),L^{r'}\left(\mathbb{R}^d\right)\right)}\Bigg].\end{aligned} \tag{41}$$

There, if we prove that

$$\left\|\left(f_1'(u_k)-f_1'(u)\right)\nabla u\right\|_{L^1\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}+\left\|\left(f_2'(u_k)-f_2'(u)\right)\nabla u\right\|_{L^{q'}\left((-T,T),L^{r'}\left(\mathbb{R}^d\right)\right)}\to 0, \tag{42}$$

as $k\to\infty$, then we have

$$\left\|\nabla\left(u_k-u\right)\right\|_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}+\left\|\nabla\left(u_k-u\right)\right\|_{L^q\left((-T,T),L^r\left(\mathbb{R}^d\right)\right)}\to 0, \tag{43}$$

as $k\to\infty$, which, combined with (37), yields the desired convergence. we prove (42) by contradiction, and we assume that there exists $\varepsilon_0>0$, and a subsequence, which we still denote by $\{u_k\}_{k\ge 1}$ such that

$$\left\|\left(f_1'(u_k)-f_1'(u)\right)\nabla u\right\|_{L^1\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}+\left\|\left(f_2'(u_k)-f_2'(u)\right)\nabla u\right\|_{L^{q'}\left((-T,T),L^{r'}\left(\mathbb{R}^d\right)\right)}\ge\varepsilon_0. \tag{44}$$

By using (37) and possibly extracting a subsequence, we may assume that $u_k\to u$ a.e. on $(-T,T)\times\mathbb{R}^d$ and that there exists $v\in L^q\left((-T,T),L^r\left(\mathbb{R}^d\right)\right)$ such that $|u_k|\le v$ a.e. on $(-T,T)\times\mathbb{R}^d$. In particular, both $\left(f_1'(u_k)-f_1'(u)\right)\nabla u$ and $\left(f_2'(u_k)-f_2'(u)\right)\nabla u$ converge to 0 a.e. on $(-T,T)\times\mathbb{R}^d$. Since

$$\left|\left(f_1'(u_k)-f_1'(u)\right)\nabla u\right|\le 2C_3|\nabla u|\in L^1\left((-T,T),L^2\left(\mathbb{R}^d\right)\right),$$

and

$$\left|\left(f_2'(u_k)-f_2'(u)\right)\nabla u\right|\lesssim_\alpha\left(|u_k|^\alpha+|u|^\alpha\right)|\nabla u|\lesssim_\alpha\left(v^\alpha+|u|^\alpha\right)|\nabla u|\in L^{q'}\left((-T,T),L^{r'}\left(\mathbb{R}^d\right)\right),$$

we obtain from the dominated convergence a contradiction with (44).

Step 3. Proof of (iii). Consider $u_0\in H^1\left(\mathbb{R}^d\right)$ and let

$$T_{\max}(u_0)=\sup\{T>0:\text{there exists a solution of }(1)\text{ on }[0,T]\},$$

$$T_{\min}(u_0)=\sup\{T>0:\text{there exists a solution of }(1)\text{ on }[-T,0]\}.$$

It follows from part (i) there exists a solution

$$u\in C\left(\left(-T_{\min},T_{\max}\right),H^1\left(\mathbb{R}^d\right)\right),$$

of (1).

Step 4. Proof of (iv). Suppose now that $T_{\max}<\infty$, and assume that there exist $M<\infty$ and a sequence $t_j\nearrow T_{\max}$ such that $\left\|u(t_j)\right\|_{H^1\left(\mathbb{R}^d\right)}\le M$. Let k be such that $t_k+T(d,\alpha,M)>T_{\max}(u_0)$. By part (i), from the initial data $u(t_k)$, one can extend u up to $t_k+T(d,\alpha,M)$, which contradicts maximality. Therefore,

$$\|u(t)\|_{H^1\left(\mathbb{R}^d\right)}\to\infty,\ \text{as } t\nearrow T_{\max}.$$

One shows by the same argument that if $T_{\min}<\infty$, then

$$\|u(t)\|_{H^1\left(\mathbb{R}^d\right)}\to\infty,\ \text{as } t\searrow -T_{\min x}.$$

This completes the proof. □

Acknowledegments

We are grateful to the anonymous referee for many helpful comments and suggestions, which have been incorporated into this version of the paper. C. Liu was supported in part by the NSFC under Grants No. 11101171, 11071095 and the Fundamental Research Funds for the Central Universities. And M. Liu was supported by science research foundation of Wuhan Institute of Technology under grants No. k201422.

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