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A Generalization of Ince's Equation

Ridha Moussa

University of Wisconsin, Waukesha, USA
Email: rmoussa@uwm.edu

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Abstract

We investigate the Hill differential equation $(1+A(t))y''(t)+B(t)y'(t)+(\lambda+D(t))y(t)=0$, where $A(t)$, $B(t)$, and $D(t)$ are trigonometric polynomials. We are interested in solutions that are even or odd, and have period π or semi-period π. The above equation with one of the above conditions constitutes a regular Sturm-Liouville eigenvalue problem. We investigate the representation of the four Sturm-Liouville operators by infinite banded matrices.

Keywords

Hill Equation, Ince Equation, Sturm-Liouville Problem, Infinite Banded Matrix, Eigenvalues, Eigenfunctions

1. Introduction

The first known appearance of the Ince equation,

$$(1+a\cos(2t))y''(t)+(b\sin(2t))y'(t)+(\lambda+d\cos(2t))y(t)=0,$$

is in Whittaker's paper ([1], Equation (5)) on integral equations. Whittaker emphasized the special case $a=0$, and this special case was later investigated in more detail by Ince [2] [3]. Magnus and Winkler's book [4] contains a chapter dealing with the coexistence problem for the Ince equation. Also Arscott [5] has a chapter on the Ince equation with $a=0$.

One of the important features of the Ince equation is that the corresponding Ince differential operator when applied to Fourier series can be represented by an infinite tridiagonal matrix. It is this part of the theory that makes the Ince equation particularly interesting. For instance, the coexistence problem which has no simple solution for the general Hill equation has a complete solution for the Ince equation (see [6]).

When studying the Ince equation, it became apparent that many of its properties carry over to a more general class of equations "the generalized Ince equation". These linear second order differential equations describe

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important physical phenomena which exhibit a pronounced oscillatory character; behavior of pendulum-like systems, vibrations, resonances and wave propagation are all phenomena of this type in classical mechanics, (see for example [7]), while the same is true for the typical behavior of quantum particles (Schrödinger's equation with periodic potential [8]).

2. The Differential Equation

We consider the Hill differential equation

$$\left(1+A(t)\right)y''(t)+B(t)y'(t)+\left(\lambda+D(t)\right)y(t)=0, \tag{2.1}$$

where

$$A(t)=\sum_{j=1}^{\eta}a_j\cos(2jt),$$

$$B(t)=\sum_{j=1}^{\eta}b_j\sin(2jt),$$

$$D(t)=\sum_{j=1}^{\eta}d_j\cos(2jt).$$

Here η is a positive integer, the coefficients a_j, b_j, d_j, for $j=1,2,\cdots,\eta$ are specified real numbers. The real number λ is regarded as a spectral parameter. We further assume that $\sum_{j=1}^{\eta}\left|a_j\right|<1$. Unless stated otherwise solutions $y(t)$ are defined for $t\in\mathbb{R}$. We will at times represent the coefficients a_j, b_j, d_j, for $j=1,2,\cdots,\eta$ in the vector form: $\boldsymbol{a}=\left[a_1,a_2,\cdots,a_\eta\right]$, $\boldsymbol{b}=\left[b_1,b_2,\cdots,b_\eta\right]$, $\boldsymbol{d}=\left[d_1,d_2,\cdots,d_\eta\right]$.

The polynomials

$$Q_j(\mu):=2a_j\mu^2-b_j\mu-\frac{d_j}{2},\quad j=1,2,\cdots,\eta, \tag{2.2}$$

will play an important role in the analysis of (2.1). For ease of notation we also introduce the polynomials

$$Q_j^{\dagger}(\mu):=Q_j(\mu-1/2),\quad j=1,2,\cdots,\eta. \tag{2.3}$$

Equation (2.1) is a natural generalization to the original Ince equation

$$\left(1+a\cos(2t)\right)y''(t)+\left(b\sin(2t)\right)y'(t)+\left(\lambda+d\cos(2t)\right)y(t)=0. \tag{2.4}$$

Ince's equation by itself includes some important particular cases, if we choose for example $a=b=0$, $d=-2q$ we obtain the famous Mathieu's equation

$$y''(t)+\left(\lambda-2q\cos(2t)\right)y(t)=0, \tag{2.5}$$

with associated pzlynomial

$$Q(\mu)=q. \tag{2.6}$$

If we choose $a=0$, $b=-4q$, and $d=4q(\nu-1)$, where q, ν are real numbers, Ince's equation becomes Whittaker-Hill equation

$$y''(t)-4q(\sin 2t)y'(t)+\left(\lambda+4q(\nu-1)\cos 2t\right)y(t)=0, \tag{2.7}$$

with associated polynomial

$$Q(\mu)=2q(2\mu-\nu+1). \tag{2.8}$$

Equation (2.1) can be brought to algebraic form by applying the transformation $\xi=\cos^2 t$. For example when $\eta=2$, and $\boldsymbol{a}=\boldsymbol{b}=0$, we obtain

$$\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}+\frac{1}{2}\left(\frac{1-2\xi}{\xi(1-\xi)}\right)\frac{\mathrm{d}y}{\mathrm{d}\xi}+\frac{1}{4}\left(\frac{8d_2\xi^2+(2d_1-8d_2)\xi-d_1+d_2+\lambda}{\xi(1-\xi)}\right)y=0. \tag{2.9}$$

3. Eigenvalues

Equation (2.1) is an even Hill equation with period π. We are interested in solutions which are even or odd and have period π or semi period π *i.e.* $y(t+\pi)=\pm y(t)$. We know that $y(t)$ is a solution to (2.1) then $y(t+\pi)$, and $y(-t)$ are also solutions. From the general theory of Hill equation (see [9], Theorem 1.3.4); we obtain the following lemmas:

Lemma 3.1. *Let* $y(t)$ *be a solution of* (2.1), *then* $y(t)$ *is even with period* π *if and only if*

$$y'(0)=y'(\pi/2)=0; \tag{3.1}$$

$y(t)$ *is even with semi period* π *if and only if*

$$y'(0)=y(\pi/2)=0; \tag{3.2}$$

$y(t)$ *is odd with semi period* π *if and only if*

$$y(0)=y'(\pi/2)=0; \tag{3.3}$$

$y(t)$ *is odd with period* π *if and only if*

$$y(0)=y(\pi/2)=0. \tag{3.4}$$

Equation (2.1) *can be written in the self adjoint form*

$$-\left(\left(1+A(t)\right)\omega(t)y'(t)\right)'-D(t)\omega(t)y(t)=\lambda\omega(t)y(t), \tag{3.5}$$

where

$$\omega(t)=\exp\left(\int\frac{B(t)-A'(t)}{1+A(t)}\mathrm{d}t\right). \tag{3.6}$$

Note that $\omega(t)$ is even and π-periodic since the function $\dfrac{B(t)-A'(t)}{1+A(t)}$ is continuous, odd, and π-periodic.

Proof. Let $r(t)=\left(1+A(t)\right)\omega(t)$. (3.5) can be written as,

$$\left(-r(t)y'(t)\right)'-D(t)\omega(t)y(t)=\lambda\omega(t)y(t), \tag{3.7}$$

which is equivalent to

$$-r'(t)y'(t)-r(t)y''(t)-D(t)\omega(t)y(t)=\lambda\omega(t)y(t). \tag{3.8}$$

Noting that

$$r'(t)=\left(1+A(t)\right)\omega'(t)+A'(t)\omega(t),$$

and

$$\omega'(t)=\frac{B(t)-A'(t)}{1+A(t)}\omega(t),$$

we see that

$$r'(t)=B(t)\omega(t).$$

Therefore, (3.8) can be written as

$$-B(t)\omega(t)y'(t)-\left(1+A(t)\right)\omega(t)y''(t)-D(t)\omega(t)y(t)=\lambda\omega(t)y(t). \tag{3.9}$$

Since $\omega(t)$ is strictly positive, the lemma follows. □

In the case of Ince's Equation (2.4), we have the following formula for the function ω

$$\omega(t) := \begin{cases} (1+a\cos 2t)^{-1-b/2a} & \text{if } a \neq 0, \\ \exp\left(\frac{-b}{2}\cos 2t\right) & \text{if } a = 0. \end{cases} \tag{3.10}$$

When $\eta \geq 2$, the function can be computed explicitly using *Maple*. For example, let us consider the case $\eta = 2$, with $\boldsymbol{a} = \left[\frac{1}{4}, \frac{1}{8}\right]$, $\boldsymbol{b} = [1,1]$. Applying (3.6), we obtain

$$\omega(t) = \frac{1}{\left(7 + 2\cos 2t + 2(\cos 2t)^2\right)^3}.$$

Equation (2.1) with one of the boundary conditions in lemma 3.1 is a regular Sturm-Liouville problem. From the theory of Sturm-Liouville ordinary differential equations it is known that such an eigenvalue problem has a sequence of eigenvalues that converge to infinity. These eigen values are denoted by α_{2m}, α_{2m+1}, β_{2m+1}, and β_{2m+2}, $m = 0,1,2,\cdots$ to correspond to the boundary conditions in lemma 3.1 respectively. This notation is consistent with the theory of Mathieu and Ince's equations (see [4] [10]). Lemma 3.1 implies the following theorem.

Theorem 3.2. *The generalized Ince equation admits a nontrivial even solution with period* π *if and only if* $\lambda = \alpha_{2m}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$ *for some* $m \in \mathbb{N}_0$; *it admits a nontrivial even solution with semi-period* π *if and only if* $\lambda = \alpha_{2m+1}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$ *for some* $m \in \mathbb{N}_0$; *it admits a nontrivial odd solution with semi-period* π *if and only if* $\lambda = \beta_{2m+1}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$ *for some* $m \in \mathbb{N}_0$; *it admits a nontrivial odd solution with period* π *if and only if* $\lambda = \beta_{2m+2}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$ *for some* $m \in \mathbb{N}_0$.

Example 3.3. *To gain some understanding about the notation we consider the almost trivial completely solvable example, the so called Cauchy boundary value problem*

$$y''(t) + \lambda y(t) = 0, \tag{3.11}$$

subject to the boundary conditions of lemma 3.1. *We have the following for the eigenvalues* λ *in terms of* $m = 0,1,2,\cdots$.

1) Even with period π we have $\lambda = \alpha_{2m} = (2m)^2$.

2) Even with semi-period π we have $\lambda = \alpha_{2m+1} = (2m+1)^2$.

3) Odd with semi-period π we have $\lambda = \beta_{2m+1} = (2m+1)^2$.

4) Odd with semi-period π we have $\lambda = \beta_{2m+2} = (2m+2)^2$.

The formal adjoint of the generalized Ince equation is

$$\left((1+A(t))y(t)\right)'' - \left(B(t)y(t)\right)' + \left(\lambda + D(t)\right)y(t) = 0. \tag{3.12}$$

By introducing the functions

$$B^*(t) = 2A'(t) - B(t) = \sum_{j=1}^{\eta} -\left(2ja_j + b_j\right)\sin(2jt),$$

$$D^*(t) = D(t) + A'(t) - B''(t) = \sum_{j=1}^{\eta} -\left(4j^2a_j + 2jb_j - d_j\right)\cos(2jt),$$

we note that the adjoint of (2.1) has the same form and can be written in the following form:

$$\left(1+A(t)\right)y''(t) + B^*(t)y'(t) + \left(\lambda + D^*(t)\right)y(t) = 0. \tag{3.13}$$

Lemma 3.4. *If* $y(t)$ *is twice differentiable defined on* $\mathbb{R}$, *then,* $y(t)$ *is a solution to the generalized Ince equation if and only if* $\omega(t)y(t)$ *is a solution to its adjoint.*

Proof. We Know that

$$B^* = 2A' - B, \quad D^* = D + A'' - B', \quad \omega' = \frac{B - A'}{1 + A}\omega,$$

and

$$\omega'' = \frac{(B' - A'')(1 + A) - A'(B - A') + (B - A')^2}{(1 + A)^2}\omega.$$

For ease of notation, let

$$p = \frac{B - A'}{1 + A}, \quad q = \frac{(B' - A'')(1 + A) - A'(B - A') + (B - A')^2}{(1 + A)^2},$$

then

$$\begin{aligned}
&(1 + A)(\omega y)'' + B^*(\omega y)' + (\lambda + D^*)(\omega y) \\
&= (1 + A)(\omega'' y + 2\omega' y' + \omega y'') + B^*(\omega' y + \omega y') + (\lambda + D^*)(\omega y) \\
&= (1 + A)(q\omega y + 2p\omega y' + \omega y'') + B^*(p\omega y + \omega y') + (\lambda + D^*)(\omega y).
\end{aligned}$$

Substituting for p, q, B^*, and D^* and simplifying we obtain

$$(1 + A)(\omega y)'' + B^*(\omega y)' + (\lambda + D^*)(\omega y) = \omega\big((1 + A)y'' + By' + (\lambda + D)y\big). \quad \square$$

From lemma 3.4 we know that if y is twice differentiable, y is a solution to the generalized Ince's equation with parameters λ, $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{d}$ if and only if ωy is a solution to its formal adjoint. Since the function ω is even with period π, the boundary condition for y and ωy are the same. Therefore we have the following theorem.

Theorem 3.5. *We have for* $m \in \mathbb{N}_0$,

$$\alpha_m(a_j, b_j, d_j) = \alpha_m(a_j, -4ja_j - b_j, d_j - 4j^2 a_j - 2jb_j), \quad j = 1, 2, \cdots, \eta, \tag{3.14}$$

$$\beta_{m+1}(a_j, b_j, d_j) = \beta_{m+1}(a_j, -4ja_j - b_j, d_j - 4j^2 a_j - 2jb_j), \quad j = 1, 2, \cdots, \eta. \tag{3.15}$$

From Sturm-Liouville theory we obtain the following statement on the distribution of eigenvalues.

Theorem 3.6. *The eigenvalues of the generalized Ince equation satisfy the inequalities*

$$\alpha_0 < \begin{Bmatrix}\alpha_1 \\ \beta_1\end{Bmatrix} < \begin{Bmatrix}\alpha_2 \\ \beta_2\end{Bmatrix} < \begin{Bmatrix}\alpha_3 \\ \beta_3\end{Bmatrix} < \cdots \tag{3.16}$$

The theory of Hill equation [4] gives the following results.

Theorem 3.7. *If* $\lambda \le \alpha_0$ *or* λ *belongs to one of the closed intervals with distinct endpoints* α_m, β_m, $m = 0, 1, 2, \cdots$, *then the generalized Ince equation is unstable. For all other real values of* λ *the equation is stable. In the case*

$$\alpha_m(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}) = \beta_m(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}), \tag{3.17}$$

for some positive integer m *and the parameters* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{d}$ *the degenerate interval* $[\alpha_m, \beta_m]$ *is not an instability interval: The generalized Ince equation is stable if*

$$\lambda = \alpha_m(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}) = \beta_m(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}).$$

4. Eigenfunctions

By theorem 3.2, the generalized Ince's equation with $\lambda = \alpha_{2m}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})$ admits a non trivial even solution with period π. It is uniquely determined up to a constant factor. We denote this Ince function by $Ic_{2m}(t) = Ic_{2m}(t; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})$ when it is normalized by the conditions $Ic_{2m}(0) > 0$ and

$$\int_0^{\pi/2} \left(Ic_{2m}(t)\right)^2 \mathrm{d}t = \frac{\pi}{4}. \tag{4.1}$$

The generalized Ince's equation with $\lambda = \alpha_{2m+1}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$ admits a non trivial even solution with semi-period π. It is uniquely determined up to a constant factor. We denote this Ince function by $Ic_{2m+1}(t) = Ic_{2m+1}(t;\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$ when it is normalized by the conditions $Ic_{2m+1}(0) > 0$ and

$$\int_0^{\pi/2} \left(Ic_{2m+1}(t)\right)^2 \mathrm{d}t = \frac{\pi}{4}. \tag{4.2}$$

The generalized Ince equation with $\lambda = \beta_{2m+1}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$ admits a non trivial odd solution with semi-period π. It is uniquely determined up to a constant factor. We denote this Ince function by $Is_{2m+1}(t) = Is_{2m+1}(t;\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$ when it is normalized by the conditions $\frac{\mathrm{d}}{\mathrm{d}t} Is_{2m+1}(0) > 0$ and

$$\int_0^{\pi/2} \left(Is_{2m+1}(t)\right)^2 \mathrm{d}t = \frac{\pi}{4}. \tag{4.3}$$

The generalized Ince equation with $\lambda = \beta_{2m+2}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$ admits a non trivial odd solution with period π. It is uniquely determined up to a constant factor. We denote this Ince function by $Is_{2m+2}(t) = Is_{2m+2}(t;\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$ when it is normalized by the conditions $\frac{\mathrm{d}}{\mathrm{d}t} Is_{2m+2}(0) > 0$ and

$$\int_0^{\pi/2} \left(Is_{2m+2}(t)\right)^2 \mathrm{d}t = \frac{\pi}{4}. \tag{4.4}$$

From Sturm-Liouville theory ([11] Chapter 8, Theorem 2.1) we obtain the following oscillation properties.

Theorem 4.1. *Each of the function systems*

$$\left\{Ic_{2m}\right\}_{m=0}^{\infty}, \tag{4.5}$$

$$\left\{Ic_{2m+1}\right\}_{m=0}^{\infty}, \tag{4.6}$$

$$\left\{Is_{2m+1}\right\}_{m=0}^{\infty}, \tag{4.7}$$

$$\left\{Is_{2m+2}\right\}_{m=0}^{\infty}, \tag{4.8}$$

is orthogonal over $[0,\pi/2]$ *with respect to the weight* $\omega(t)$, *that is, for* $m \neq n$,

$$\int_0^{\pi/2} \omega(t) Ic_{2m}(t) Ic_{2m}(t) \mathrm{d}t = 0, \tag{4.9}$$

$$\int_0^{\pi/2} \omega(t) Ic_{2m+1}(t) Ic_{2m+1}(t) \mathrm{d}t = 0, \tag{4.10}$$

$$\int_0^{\pi/2} \omega(t) Is_{2m+1}(t) Is_{2m+1}(t) \mathrm{d}t = 0, \tag{4.11}$$

$$\int_0^{\pi/2} \omega(t) Is_{2m+2}(t) Is_{2m+2}(t) \mathrm{d}t = 0. \tag{4.12}$$

Moreover, each of the previous system is complete over $[0,\pi/2]$.

Using the transformations that led to Theorem 3.5, we obtain the following result.

Theorem 4.2. *We have*

$$Ic_m\left(t;\boldsymbol{a},\boldsymbol{b}^*,\boldsymbol{d}^*\right) = c_m(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})\,\omega(t;\boldsymbol{a},\boldsymbol{b})\, Ic_m(t;\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}), \tag{4.13}$$

$$Is_m\left(t;\boldsymbol{a},\boldsymbol{b}^*,\boldsymbol{d}^*\right) = s_m\left(\boldsymbol{a},\boldsymbol{b}^*,\boldsymbol{d}^*\right)\omega(t;\boldsymbol{a},\boldsymbol{b})\, Is_m(t;\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}), \tag{4.14}$$

where $c_m(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$ *and* $s_m(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$ *are positive and independent of* t, *and*

$$\boldsymbol{b}^* = \left[b_1^*, b_2^*, \cdots, b_\eta^*\right], \qquad \boldsymbol{d}^* = \left[d_1^*, d_2^*, \cdots, d_\eta^*\right],$$

with

$$b_j^* = -4ja_j - b_j, \quad d_j^* = d_j - 4j^2 a_j - 2jb_j, \quad j = 1, 2, \cdots, \eta.$$

The adopted normalization of Ince functions is easily expressible in terms of the Fourier coefficients of Ince functions and so is well suited for numerical computations [6]; However, it has the disadvantage that Equations (4.13) and (4.14) require coefficients c_m and s_m which are not explicitly known.

Of course, once the generalized Ince functions Ic_m and Is_m, are known we can express c_m and s_m in the form

$$c_m(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}) = \frac{1}{\omega(0; \boldsymbol{a}, \boldsymbol{b})} \frac{Ic_m(0; \boldsymbol{a}, \boldsymbol{b}^*, \boldsymbol{d}^*)}{Ic_m(0; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})}, \tag{4.15}$$

$$s_m(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}) = \frac{1}{\omega(0; \boldsymbol{a}, \boldsymbol{b})} \frac{Is_m(0; \boldsymbol{a}, \boldsymbol{b}^*, \boldsymbol{d}^*)}{Is_m(0; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})}. \tag{4.16}$$

If we square both sides of (4.13) and (4.14) and integrate, we find that

$$c_m^2 \int_0^{\pi/2} \left(\omega(t; \boldsymbol{a}, \boldsymbol{b}) Ic_m(t; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})\right)^2 \mathrm{d}t = \pi/4, \tag{4.17}$$

$$s_m^2 \int_0^{\pi/2} \left(\omega(t; \boldsymbol{a}, \boldsymbol{b}) Is_m(t; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})\right)^2 \mathrm{d}t = \pi/4. \tag{4.18}$$

If $\omega(t; \boldsymbol{a}, \boldsymbol{b})$ is very simple, then it is possible to evaluate the integrals in (4.17), (4.18) in terms of the Fourier coefficients of the generalized Ince functions. This provides another way to to calculate c_m and s_m.

Once we know c_m and s_m, we can evaluate the integrals on the left-hand sides of the following equations

$$c_m \int_0^{\pi/2} \omega(t; \boldsymbol{a}, \boldsymbol{b}) \left(Ic_m(t; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})\right)^2 \mathrm{d}t = \int Ic_m(t; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}) Ic_m(t; \boldsymbol{a}, \boldsymbol{b}^*, \boldsymbol{d}^*) \mathrm{d}t. \tag{4.19}$$

$$s_m \int_0^{\pi/2} \omega(t; \boldsymbol{a}, \boldsymbol{b}) \left(Is_m(t; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})\right)^2 \mathrm{d}t = \int Is_m(t; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}) Is_m(t; \boldsymbol{a}, \boldsymbol{b}^*, \boldsymbol{d}^*) \mathrm{d}t. \tag{4.20}$$

The integrals on the right-hand sides of (4.19) and (4.20) are easy to calculate once we know the Fourier series of Ince functions.

5. Operators and Banded Matrices

In this section we introduce four linear operators associated with Equation (2.1), and represent them by banded matrices of width $2\eta + 1$. It is this simple representation that is fundamental in the theory of the generalized Ince equation. We assume known some basic notions from spectral theory of operators in Hilbert space.

Let H_1 be the Hilbert space consisting of even, locally square-summable functions $f: \mathbb{R} \to \mathbb{C}$ with period π. The inner product is given by

$$\langle f, g \rangle = \int_0^{\pi/2} f(t) \overline{g(t)} \mathrm{d}t. \tag{5.1}$$

By restricting functions to $[0, \pi/2]$, H_1 is isometrically isomorphic to the standard $L^2(0, \pi/2)$. We also consider a second inner product

$$\langle f, g \rangle_\omega = \int_0^{\pi/2} \omega(t) f(t) \overline{g(t)} \mathrm{d}t. \tag{5.2}$$

We consider the differential operator

$$(S_1 y)(t) = -(1 + A(t)) y''(t) - B(t) y'(t) - D(t) y(t). \tag{5.3}$$

The domain $D(S_1)$ of definition of consists of all functions $y \in H_1$ for which y and y' are absolutely continuous and $y'' \in H_1$, by restricting functions to $[0, \pi/2]$, this corresponds to the usual domain of a Sturm-Liouville operator associated with the boundary conditions (3.1). It is known ([12] Chapter V, Section 3.6) that

S_1 is self-adjoint with compact resolvent when considered as an operator in $\left(H_1,\langle,\rangle_\omega\right)$, and its eigenvalues are $\alpha_{2m}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$, $m=0,1,2,\cdots$. All eigenvalues of S_1 are simple. If we consider S_1 as an operator in the Hilbert space $\left(H_1,\langle,\rangle\right)$, then its adjoint S_1^* is given by the operator

$$y \to -\left(\left(1+A(t)\right)y(t)\right)'' + \left(B(t)y(t)\right)' - D(t)y(t),$$

on the same domain $D(S_1)$; see ([12], Chapter III, Example 5.32). The adjoint S_1^* is of the same form as S_1 but with $\boldsymbol{b}$, $\boldsymbol{d}$ replaced by $\boldsymbol{b}^*$, $\boldsymbol{d}^*$, respectively. By Theorem 3.5, we see that S_1^* has the same eigenvalues as S_1. Let $\ell^2(\mathbb{N}_0)$ be the space of square-summable sequences $x=\{x_n\}_{n=0}^{\infty}$ with its standard inner product $\langle,\rangle$. Then

$$(T_1 x) := \frac{x_0}{\sqrt{2}} + \sum_{n=1}^{\infty} x_n \cos(2nt),$$

defines a bijective linear map $T_1:\ell^2(\mathbb{N}_0)\to H_1$. Consider the operator $M_1 := T_1^{-1}S_1T_1$ defined on

$$D(M_1) = T_1^{-1}\left(D(S_1)\right) = \left\{x\in\ell^2(\mathbb{N}_0): \sum_{n=0}^{\infty} n^4 |x_n|^2 < \infty\right\}. \tag{5.4}$$

Let e_n denotes the sequence with a 1 in the n^{th} position and 0's in all other positions, we also define $u_n(t) := (T_1 e_n)(t)$, *i.e.* $u_0(t) = \frac{1}{\sqrt{2}}$ and $u_n(t) = \cos(2nt)$ for $n=1,2,\cdots$. We find that the operator M_1 can be represented in the following way,

$$M_1 e_n = \begin{cases} \sum_{j=1}^{\eta} \sqrt{2} q_0^j e_j & \text{if } n=0, \\ r_n e_n + \sum_{j=1}^{\eta} q_{-n}^j \delta_{n-j} e_{|n-j|} + \sum_{j=1}^{\eta} q_n^j e_{n+j} & \text{if } n\geq 1, \end{cases} \tag{5.5}$$

where $\delta_0=\sqrt{2}$ and $\delta_k=0$ if $k\neq 0$, and $r_n=4n^2$, $n\in\mathbb{N}$. Note that the factor $\sqrt{2}$ should appear only with e_0.

M_1 is self-adjoint with compact resolvent in $\ell^2(\mathbb{N}_0)$ equipped with the inner product $\langle T_1x, T_1y\rangle_\omega$. This inner product generates a norm that is equivalent to the usual $\ell^2(\mathbb{N}_0)$. The operator M_1 has the eigenvalues $\alpha_{2m}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$ and the corresponding eigenvectors form sequences of Fourier coefficients for the functions Ic_{2m}.

Now consider the operator S_2 that is defined as S_1 in (5.3) but in the Hilbert space H_2 consisting of even functions with semi-period π. This operator has eigenvalues $\alpha_{2m+1}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})$, with eigenfunctions $Ic_{2m+1}(t)$, $m=0,1,2,\cdots$. Using the basis $\cos(2n+1)t$, $n\in\mathbb{N}_0$, then,

$$(T_2 x)(t) := \sum_{n=0}^{\infty} x_n \cos(2n+1)t,$$

defines a bijective linear map $T_2:\ell^2(\mathbb{N}_0)\to H_2$. Consider the operator $M_2 := T_2^{-1}S_2T_2$ defined on

$$D(M_2) = T_2^{-1}\left(D(S_2)\right) = \left\{x\in\ell^2(\mathbb{N}_0): \sum_{n=0}^{\infty} n^4 |x_n|^2 < \infty\right\}.$$

Let $u_n(t) := (T_2 e_n)(t) = \cos(2n+1)t$, for $n=0,1,2,\cdots$, we get the following formula for M_2

$$M_2 e_n = r_n e_n + \sum_{j=1}^{\eta} q_{-n}^{\dagger j} e_{\left|n-j+\frac{1}{2}\right|-\frac{1}{2}} + \sum_{j=1}^{\eta} q_{n+1}^{\dagger j} e_{n+j}, \quad n\geq 0, \tag{5.6}$$

where

$$q_n^{\dagger j} = Q_j\left(n - \frac{1}{2}\right), \quad j = 1, 2, \cdots, \eta, \quad r_n = \begin{cases} 1 + q_0^{\dagger j} & \text{if } n = 0, \\ (2n+1)^2 & \text{if } n \geq 1. \end{cases}$$

Now consider the operator S_3 that is defined as S_1 but in the Hilbert space H_3 consisting of odd functions with semi-period π. This operator has the eigenvalues β_{2m+1} with eigenfunctions $Is_{2m+1}(t)$, $m = 0, 1, 2, \cdots$. Using the basis functions $\sin(2n+1)t$, $n \in \mathbb{N}_0$.

$$(T_3 x)(t) := \sum_{n=0}^{\infty} x_n \sin(2n+1)t,$$

defines a bijective linear map $T_3 : \ell^2(\mathbb{N}_0) \to H_3$. Consider the operator $M_3 := T_3^{-1} S_3 T_3$ defined on

$$D(M_3) = T_3^{-1}(D(S_3)) = \left\{ x \in \ell^2(\mathbb{N}_0) : \sum_{n=0}^{\infty} n^4 |x_n|^2 < \infty \right\}.$$

Let $u_n(t) := (T_3 e_n)(t) = \sin(2n+1)t$, for $n = 0, 1, 2, \cdots$, we have the following formula for M_3,

$$M_3 e_n = r_n^{\dagger} e_n + \sum_{j=1}^{\eta} q_{-n}^{\dagger j} \varepsilon_j e_{\left|n-j+\frac{1}{2}\right|-\frac{1}{2}} + \sum_{j=1}^{\eta} q_{n+1}^{\dagger j} e_{n+j}, \tag{5.7}$$

where

$$q_n^{\dagger j} = Q_j\left(n - \frac{1}{2}\right), j = 1, 2, \cdots, \eta, \quad r_n^{\dagger} = \begin{cases} 1 - q_0^{\dagger 1} & \text{if } n = 0, \\ (2n+1)^2 & \text{if } n \geq 1, \end{cases}$$

and

$$\varepsilon_j = \begin{cases} 1 & \text{if } n \geq j \\ -1 & \text{if } n < j \end{cases}.$$

Finally, consider the operator S_4 that is defined as S_1 but in the Hilbert space H_4 consisting of odd functions with period π. This operator has the eigenvalues β_{2m+2} with eigenfunctions Is_{2m+2}, $m = 0, 1, 2, \cdots$. Using the basis $\sin(2n+2)t$, $n \in \mathbb{N}_0$,

$$(T_4 x)(t) := \sum_{n=0}^{\infty} x_n \sin(2n+2)t,$$

defines a bijective linear map $T_4 : \ell^2(\mathbb{N}_0) \to H_4$. Consider the operator $M_4 := T_4^{-1} S_4 T_4$ defined on

$$D(M_4) = T_4^{-1}(D(S_4)) = \left\{ x \in \ell^2(\mathbb{N}_0) : \sum_{n=0}^{\infty} n^4 |x_n|^2 < \infty \right\}.$$

Let $u_n(t) := (T_4 e_n)(t) = \sin(2n+2)t$, for $n = 0, 1, 2, \cdots$. Then, the formula for M_4 is

$$M_4 e_n = r_n e_n + \sum_{j=1}^{\min(n,\eta)} q_{-n-1}^{j} \varepsilon_j e_{n-j} - \sum_{j=n+2}^{\eta} q_{-n-1}^{j} e_{j-n-2} + \sum_{j=1}^{\eta} q_{n+1}^{j} e_{n+j}, \tag{5.8}$$

where

$$r_n = (2n+2)^2, \quad n = 0, 1, 2, \cdots.$$

Example 5.1. For the Whittaker-Hill Equation (2.7) in the following form [8]

$$y'' + \left(\lambda + 4\alpha s \cos 2t + 2\alpha^2 \cos 4t\right) y = 0, \quad \alpha \in \mathbb{R}, s \in \mathbb{N}, \tag{5.9}$$

the function $\omega(t)$ from (3.6) is equal to 1, therefore the operators S_j, $j = 1, 2, 3, 4$, are self-adjoint on the

Hilbert spaces $(H_1,\langle,\rangle)$, $j=1,2,3,4$, respectively. Hence the infinite matrices S_j, $j=1,2,3,4$, are symmetric. They are represented by

$$M_1=\begin{pmatrix} 0 & -2\sqrt{2}\alpha s & -\sqrt{2}\alpha^2 & 0 & \cdots \\ -2\sqrt{2}\alpha s & 4-\alpha^2 & -2\alpha s & -\alpha^2 & \cdots \\ -\sqrt{2}\alpha^2 & -2\alpha s & 16 & -2\alpha s & \cdots \\ 0 & -\alpha^2 & -2\alpha s & 36 & \cdots \\ 0 & 0 & -\alpha^2 & -2\alpha s & \cdots \\ 0 & 0 & 0 & -\alpha^2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{5.10}$$

$$M_2=\begin{pmatrix} 1-2\alpha s & -\alpha(2s+\alpha) & -\alpha^2 & 0 & \cdots \\ -\alpha(2s+\alpha) & 9 & -2\alpha s & -\alpha^2 & \cdots \\ -\alpha^2 & -2\alpha s & 25 & -2\alpha s & \cdots \\ 0 & -\alpha^2 & -2\alpha s & 49 & \cdots \\ 0 & 0 & -\alpha^2 & -2\alpha s & \cdots \\ 0 & 0 & 0 & -\alpha^2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{5.11}$$

$$M_3=\begin{pmatrix} 1+2\alpha s & -\alpha(2s-\alpha) & -\alpha^2 & 0 & \cdots \\ -\alpha(2s-\alpha) & 9 & -2\alpha s & -\alpha^2 & \cdots \\ -\alpha^2 & -2\alpha s & 25 & -2\alpha s & \cdots \\ 0 & -\alpha^2 & -2\alpha s & 49 & \cdots \\ 0 & 0 & -\alpha^2 & -2\alpha s & \cdots \\ 0 & 0 & 0 & -\alpha^2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{5.12}$$

$$M_4=\begin{pmatrix} 4-\alpha^2 & -2\alpha s & -\alpha^2 & 0 & \cdots \\ -2\alpha s & 16 & -2\alpha s & -\alpha^2 & \cdots \\ -\alpha^2 & -2\alpha s & 36 & -2\alpha s & \cdots \\ 0 & -\alpha^2 & -2\alpha s & 64 & \cdots \\ 0 & 0 & -\alpha^2 & -2\alpha s & \cdots \\ 0 & 0 & 0 & -\alpha^2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{5.13}$$

6. Fourier Series

The generalized Ince functions admit the following Fourier series expansions

$$Ic_{2m}(t)=\frac{A_0}{\sqrt{2}}+\sum_{n=1}^{\infty}A_{2n}\cos(2nt), \tag{6.1}$$

$$Ic_{2m+1}(t)=\sum_{n=0}^{\infty}A_{2n}\cos(2n+1)t, \tag{6.2}$$

$$Is_{2m+1}(t) = \sum_{n=0}^{\infty} B_{2n+1} \sin(2n+1)t, \tag{6.3}$$

$$Is_{2m+2}(t) = \sum_{n=0}^{\infty} B_{2n+2} \sin(2n+2)t. \tag{6.4}$$

We did not indicate the dependence of the Fourier coefficients on m, $\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}$. The normalization of Ince functions implies

$$\sum_{n=1}^{\infty} A_{2n}^2 = 1, \qquad \frac{A_0}{\sqrt{2}} + \sum_{n=1}^{\infty} A_{2n} > 0, \tag{6.5}$$

$$\sum_{n=1}^{\infty} A_{2n+1}^2 = 1, \qquad \sum_{n=0}^{\infty} A_{2n+1} > 0, \tag{6.6}$$

$$\sum_{n=1}^{\infty} B_{2n+1}^2 = 1, \qquad \sum_{n=0}^{\infty} (2n+1) B_{2n+1} > 0, \tag{6.7}$$

$$\sum_{n=1}^{\infty} B_{2n+2}^2 = 1, \qquad \sum_{n=0}^{\infty} (2n+1) B_{2n+2} > 0. \tag{6.8}$$

Using relations (4.13) and (4.14), we can represent the generalized functions in a different way

$$Ic_m(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}) = \left(\omega(t;\boldsymbol{a},\boldsymbol{b}) c_m(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})\right)^{-1} Ic_m\left(\boldsymbol{a},\boldsymbol{b}^*,\boldsymbol{d}^*\right), \tag{6.9}$$

$$Is_m(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}) = \left(\omega(t;\boldsymbol{a},\boldsymbol{b}) s_m(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})\right)^{-1} Is_m\left(\boldsymbol{a},\boldsymbol{b}^*,\boldsymbol{d}^*\right), \tag{6.10}$$

where

$$b_j^* = -4ja_j - b_j, \quad d_j^* = d_j - 4j^2 a_j - 2jb_j, \quad j = 1,2,\cdots,\eta.$$

Therefore, we can write

$$Ic_{2m}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}) = \left(\omega(t;\boldsymbol{a},\boldsymbol{b})\right)^{-1} \left(\frac{C_0}{\sqrt{2}} + \sum_{n=1}^{\infty} C_{2n} \cos(2nt)\right), \tag{6.11}$$

$$Ic_{2m+1}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}) = \left(\omega(t;\boldsymbol{a},\boldsymbol{b})\right)^{-1} \left(\sum_{n=0}^{\infty} C_{2n+1} \cos(2nt)\right), \tag{6.12}$$

$$Is_{2m+1}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}) = \left(\omega(t;\boldsymbol{a},\boldsymbol{b})\right)^{-1} \left(\sum_{n=0}^{\infty} D_{2n+1} \sin(2nt)\right), \tag{6.13}$$

$$Is_{2m+2}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}) = \left(\omega(t;\boldsymbol{a},\boldsymbol{b})\right)^{-1} \left(\sum_{n=0}^{\infty} D_{2n+2} \sin(2nt)\right), \tag{6.14}$$

where

$$C_m = \left(c_m(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})\right)^{-1} A_m, \qquad D_m = \left(s_m(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})\right)^{-1} B_m,$$

and the Fourier coefficients A_n and B_n belong to the parameters $\boldsymbol{a},\boldsymbol{b}^*,\boldsymbol{d}^*$. Properties of the coefficients C_n and D_n follow from those of A_n and B_n.

A generalized Ince function is called a generalized Ince polynomial of the first kind if its Fourier series (6.1), (6.2), (6.3), or (6.4) terminate. It is called a generalized Ince polynomial of the second kind if its expansion (6.11), (6.12), (6.13), or (6.14) terminate. If they exist, these generalized Ince polynomials and their corresponding eigenvalues can be computed from the finite subsections of the matrices M_j, $j = 1,2,3,4$ of Section 5.

Example 6.1. Consider the equation

$$(1 + \cos 2t + \cos 4t) y'' + (\sin 2t + \sin 4t) y' + \lambda y = 0, \tag{6.15}$$

one can check that if we set $\lambda = 0$, any constant function y is an eigenfunction corresponding to the eigenvalue $\alpha_0 = 0$. The adopted normalization of Section 4 implies that $Ic_0(t) = \frac{1}{\sqrt{2}}$. It is a generalized Ince polynomial (even with period π).

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