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A Survey of the Implementation of Numerical Schemes for the Heat Equation Using Forward Euler in Time

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Abstract

We establish the conditions for the compute of the Global Truncation Error (GTE), stability restriction on the time step and we prove the consistency using forward Euler in time and a fourth order discretization in space for Heat Equation with smooth initial conditions and Dirichlet boundary conditions.

Keywords

Global Truncation, Forward Euler, Heat Equation

1. Introduction

In this paper we have considered the heat equation $u_t = Cu_{xx}$ on $[0,1]\times[0,T]$ with smooth initial conditions and Dirichlet boundary conditions $\left(C \in \mathbb{R}^+\right)$. Using forward Euler in time and fourth order discretization in space, we compute the Global Truncation Error (GTE), the stability restriction on the time step Δt, also we prove consistency and finally we prove the convergence for this scheme.

Much attention has been paid to the development, analysis and implementation of accurate methods for the numerical solution of this problem in the literature. Many problems are modeled by smooth initial conditions and Dirichlet boundary conditions. A number of procedures have been suggested (see, for instance [1]-[3]). We can say that three classes of solution techniques have emerged for solution of PDE: the finite difference techniques, the finite element methods and the spectral techniques. The last one has the advantage of high accuracy attained by the resulting discretization for a given number of nodes [4]-[7]. Let Δx denote the grid-size in the spatial direction and Δt the gridsize in the time direction. By using forward Euler in time, and the fourth order

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discretization from the previous problem in space, the heat equation reads:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=C\frac{-u_{i-2}^n+16u_{i-1}^n-30u_i^n+16u_{i+1}^n-u_{i+2}^n}{12\Delta x^2} \tag{1}$$

We'll assume that the discretizations used near the boundaries have the same order [8] and [9].

2. Global Truncation Error (GTE)

There are three equivalent ways of computing the Global Truncation Error for this case.

Way 1. We can always go back to the definition of the GTE. Let U^n be the true solution at stage n, and v^n be the solution returned by the scheme at stage n. Therefore

$$E^n=v^n-U^n \tag{2}$$

We consider de LTE

$$\begin{aligned}\tau=&\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}\\&-\frac{C}{12(\Delta x)^2}\left[-u(x-2\Delta x,t)+16u(x-\Delta x,t)-30u(x,t)+16u(x+\Delta x,t)-u(x+2\Delta x,t)\right]\end{aligned} \tag{3}$$

$$\tau_i^n=\frac{u_i^{n+1}-u_i^n}{\Delta t}-\frac{C}{12(\Delta x)^2}\left[-u_{i-2}^n+16u_{i-1}^n-30u_i^n+16u_{i+1}^n-u_{i+2}^n\right] \tag{4}$$

$$u_i^{n+1}=u_i^n+\frac{C\Delta t}{12(\Delta x)^2}\left[u_{i-2}^n+16u_{i-1}^n-30u_i^n+16u_{i+1}^n-u_{i+2}^n\right]+\Delta t\tau_i^n \tag{5}$$

$$u_i^{n+1}=\frac{C\Delta t}{12(\Delta x)^2}\left[u_{i-2}^n+16u_{i-1}^n+\left(\frac{12(\Delta x)^2}{C\Delta t}-30\right)u_i^n+16u_{i+1}^n-u_{i+2}^n\right]+\Delta t\tau_i^n \tag{6}$$

$$u_i^{n+1}=\frac{C\Delta t}{12(\Delta x)^2}\left(-1,16,\frac{12(\Delta x)^2}{C\Delta t}-30,16,-1\right)\begin{pmatrix}u_{i-2}^n\\u_{i-1}^n\\u_i^n\\u_{i+1}^n\\u_{i+2}^n\end{pmatrix}+\Delta t\tau_i^n \tag{7}$$

So that at stage n, we have

$$U^{n+1}=\frac{C\Delta t}{12(\Delta x)^2}B(\Delta t,\Delta x)U^n+\Delta t\tau^n+\beta^n \tag{8}$$

where

$$U^n=\begin{pmatrix}u_0^n\\u_1^n\\\vdots\\u_{M-1}^n\\u_M^n\end{pmatrix}\quad\text{and}\quad\tau^n=\begin{pmatrix}\tau_0^n\\\tau_1^n\\\vdots\\\tau_{M-1}^n\\\tau_M^n\end{pmatrix} \tag{9}$$

β^n is a vector taking care of the boundary conditions and $B(\Delta t,\Delta x)$ is a matrix. Since

$$v^{n+1}=\frac{C\Delta t}{12(\Delta x)^2}B(\Delta t,\Delta x)v^n+\beta^n \tag{10}$$

we get at stage N

$$E^N = v^N - U^N = \frac{C\Delta t}{12(\Delta x)^2} B(\Delta t, \Delta x)\left(v^{N-1} - U^{N-1}\right) - \Delta t \tau^N \tag{11}$$

$$\begin{aligned} &= \frac{C\Delta t}{12(\Delta x)^2} B(\Delta t, \Delta x)\left(\frac{C\Delta t}{12(\Delta x)^2} B(\Delta t, \Delta x)\left(v^{N-2} - U^{N-2}\right) - \Delta t \tau^{N-2}\right) - \Delta t \tau^{N-1} \\ &= \left(\frac{C\Delta t}{12(\Delta x)^2}\right)^2 B^2(\Delta t, \Delta x)\left(v^{N-2} - U^{N-2}\right) - \Delta t\left(\frac{C\Delta t}{12(\Delta x)^2} B(\Delta t, \Delta x)\tau^{N-2} - \tau^{N-1}\right) \end{aligned} \tag{12}$$

$$\cdots$$

$$= \left(\frac{C\Delta t}{12(\Delta x)^2}\right)^N B^N(\Delta t, \Delta x)(v^0 - U^0) - \Delta t \sum_{n=1}^{N}\left(\frac{C\Delta t}{12(\Delta x)^2}\right)^{N-n} B^{N-n}(\Delta t, \Delta x)\tau^{N-n} \tag{13}$$

$$= \left(\frac{C\Delta t}{12(\Delta x)^2}\right)^N B^N(\Delta t, \Delta x)E^0 - \Delta t \sum_{n=1}^{N}\left(\frac{C\Delta t}{12(\Delta x)^2}\right)^{N-n} B^{N-n}(\Delta t, \Delta x)\tau^{N-n} \tag{14}$$

We now wish to estimate this quantity: first using the triangle inequality, we get

$$\text{GTE} = \left\|E^N\right\| \le \left\|\left(\frac{C\Delta t}{12(\Delta x)^2}\right)^N B^N(\Delta t, \Delta x)E^0\right\| + \Delta t\left\|\sum_{n=1}^{N}\left(\frac{C\Delta t}{12(\Delta x)^2}\right)^{N-n} B^{N-n}(\Delta t, \Delta x)\tau^{N-n}\right\| \tag{15}$$

Now, taking stability into account, we can see that $\left\|B\left((\Delta x)^2, \Delta x\right)\right\| = O(1)$. Letting $T = N\Delta t$ we get

$$\left\|E^N\right\| \le \left\|\left(\frac{C\Delta t}{12(\Delta x)^2}\right)^N B^N(\Delta t, \Delta x)E^0\right\| + \Delta t\left\|\sum_{n=1}^{N}\left(\frac{C\Delta t}{12(\Delta x)^2}\right)^{N-n} B^{N-n}(\Delta t, \Delta x)\tau^{N-n}\right\| \tag{16}$$

$$= \left\|O(1)\Delta O(1)E^0\right\| + \Delta t\left\|\sum_{n=1}^{N} O(1)\Delta O(1)\tau^{N-n}\right\| \tag{17}$$

$$\le \left\|E^0\right\| + \Delta t\sum_{n=1}^{N}\left\|\tau^{N-n}\right\| \le \left\|E^0\right\| + \Delta t\ N\left\|\tau^{N-n}\right\| \le \left\|E^0\right\| + T\left\|\tau^{N-n}\right\| \tag{18}$$

Now, assuming that initial error is not too large, we have

$$\text{GTE} \le \left\|\tau^{N-n}(\Delta t, \Delta x)\right\| \underbrace{=}_{\text{(by stability)}} \left\|\tau^{N-n}\left((\Delta x)^2, \Delta x\right)\right\| = O\left((\Delta x)^2 + (\Delta x)^4\right) = O\left((\Delta x)^2\right) \tag{19}$$

Finally, we can conclude that the $\text{GTE} \propto O\left((\Delta x)^2\right)$

Way 2. The GTE can be estimated by computing the LTE $\tau_i^n(\Delta t, \Delta x)$ and imposing stability to it

$$\text{GTE} \le \tau_i^n(\Delta t, \Delta x) \underbrace{=}_{\text{(by stability)}} \tau_i^n\left((\Delta x)^2, \Delta x\right) = O\left((\Delta x)^2 + (\Delta x)^4\right) = O\left((\Delta x)^2\right) \tag{20}$$

Way 3. We can also compute the *one-step-error* for the scheme. This quantity is basically equal to $\Delta t \cdot \tau_i^n$ since it is computed as follows

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = C\frac{-u_{i-2}^n + 16u_{i-1}^n - 30u_i^n + 16u_{i+1}^n - u_{i+2}^n}{12(\Delta x)^2} \tag{21}$$

$$\Leftrightarrow u_i^{n+1} = u_i^n + C(\Delta t)\frac{-u_{i-2}^n + 16u_{i-1}^n - 30u_i^n + 16u_{i+1}^n - u_{i+2}^n}{12(\Delta x)^2} \tag{22}$$

then substitute the true solution and compute the difference of the two sides

$$\alpha_i^n = u(x,t) + C(\Delta t)\left(\frac{-u(x-2\Delta x)+16u(x-\Delta x)-30u(x)+16u(x+\Delta x)-u(x+2\Delta x)}{12(\Delta x)^2}\right) - u(x,t+\Delta t)$$
$$= O\left(\Delta t(\Delta x)^4 + (\Delta x)^2\right). \tag{24}$$

We can then estimate the GTE by summing up the one-step error at each stage

$$\text{GTE} \le \left\|\sum_{n=1}^{N}\alpha^n\right\| \le N\left\|\alpha^n\right\| = \frac{T}{\Delta t}O\left(\Delta t(\Delta x)^4 + (\Delta x)^2\right) \tag{25}$$

$$\underbrace{=}_{\text{(by stability)}} \frac{T}{(\Delta x)^2}O\left((\Delta x)^2(\Delta x)^4 + (\Delta x)^2\right) = O\left((\Delta x)^2\right) \tag{26}$$

3. Stability Restriction

We start by computing the stability restriction one has to impose on Δt. We apply *Von Neumann* stability analysis to the scheme: Letting k denote the wave number, we get

$$\frac{G-1}{\Delta t} = C\frac{-e^{-2ik\Delta x} + 16e^{-ik\Delta x} - 30 + 16e^{ik\Delta x} - e^{2ik\Delta x}}{12(\Delta x)^2} \tag{27}$$

$$= \frac{C}{3(\Delta x)^2}\frac{1}{4}\left(-e^{-2ik\Delta x} + 16e^{-ik\Delta x} - 30 + 16e^{ik\Delta x} - e^{2ik\Delta x}\right) \tag{28}$$

$$= -\frac{C}{3(\Delta x)^2}\frac{1}{4}\left(e^{-2ik\Delta x} - 16e^{-ik\Delta x} + 30 - 16e^{ik\Delta x} + e^{2ik\Delta x}\right) \tag{29}$$

$$= -\frac{C}{3(\Delta x)^2}\frac{1}{4}\left(28 + e^{2ik\Delta x} + 2 + e^{-2ik\Delta x} - 16e^{-ik\Delta x} - 16e^{ik\Delta x}\right) \tag{30}$$

$$= -\frac{C}{3(\Delta x)^2}\frac{1}{4}\left(7 + \frac{e^{2ik\Delta x} + 2 + e^{-2ik\Delta x}}{4} - 8\frac{e^{ik\Delta x} + e^{-ik\Delta x}}{2}\right) \tag{31}$$

$$= -\frac{C}{3(\Delta x)^2}\left(7 + \cos^2(k\Delta x) - 8\cos(k\Delta x)\right) \tag{32}$$

then

$$G(k) = 1 - \frac{C\Delta t}{3(\Delta x)^2}\left(\cos^2(k\Delta x) - 8\cos(k\Delta x) + 7\right) \tag{33}$$

Now, let $y = \cos(k\Delta x)$ and

$$w(y) = y^2 - 8y + 7 = (y-7)(y-1) \tag{34}$$

So that $w \ge 0$ when $y = \cos(k\Delta x) \in [-1,1]$. This guarantees that $G(k) < 1$. Now, in order to make sure that $|G(k)| \le 1$, we must have

$$-1 \le G(k) \Leftrightarrow -1 \le 1 - \frac{C\Delta t}{3(\Delta x)^2}\left(\cos^2(k\Delta x) - 8\cos(k\Delta x) + 7\right) \tag{35}$$

$$\Leftrightarrow \frac{C\Delta t}{3(\Delta x)^2}\left(\cos^2(k\Delta x)-8\cos(k\Delta x)+7\right)\le 2 \tag{36}$$

$$\Leftrightarrow \frac{C\Delta t}{3(\Delta x)^2}\max\left(\cos^2(k\Delta x)-8\cos(k\Delta x)+7\right)\le 2 \tag{37}$$

$$\Leftrightarrow \frac{C\Delta t}{3(\Delta x)^2}\left((-1)^2-8(-1)+7\right)\le 2 \tag{38}$$

$$\Leftrightarrow \frac{16C\Delta t}{3(\Delta x)^2}\le 2 \tag{39}$$

$$\Leftrightarrow \Delta t\le \frac{3}{8}\frac{(\Delta x)^2}{C} \qquad (\text{stability criterion}) \tag{40}$$

4. Consistency and Convergence

We know that a discretization scheme [10] for a PDE is *consistent* provided that $\tau(\Delta x,\Delta t)\to 0$ as $\Delta x,\Delta t\to 0$, where τ is the LTE. We compute it by substituting the true solution in the scheme and by using Taylor expansions

$$\begin{aligned}\tau(\Delta x,\Delta t)=&\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}\\&-C\frac{-u(x-2\Delta x,t)+16u(x-\Delta x,t)-30u(x,t)+16u(x+\Delta x,t)-u(x+2\Delta x,t)}{12(\Delta x)^2}\end{aligned} \tag{41}$$

$$\begin{aligned}&=u_t+\frac{\Delta t}{2}u_{tt}+\cdots-C\left(u_{xx}+O\left((\Delta x)^4\right)\right)\\&=\underbrace{u_t-Cu_{xx}}_{=0}+O\left(\Delta t+(\Delta x)^4\right)=O\left(\Delta t+(\Delta x)^4\right)\end{aligned} \tag{42}$$

Thus, τ obviously goes to 0 as Δx and Δt go to 0. Therefore, we can say that the scheme is *consistent*.

Lastly, since we proved that the scheme is consistent and stable, by Lax equivalence theorem, we prove that the scheme is *convergent*. (By the above, since the GTE is $O\left((\Delta x)^2\right)$, it goes to 0 as $\Delta x\to 0$). We can see that Lax Equivalence Theorem for PDEs holds provided the scheme is linear (which is the case here). It may not hold for non-linear schemes.

Another way to get the one-step error for the scheme is to combine the LTE for the temporal and spatial discretization, as follows.

LTE for forward Euler is $O\left((\Delta t)^2\right)$ and the LTE for the spatial discretization is

$$\tau(\Delta x)=-u(x-2\Delta x)+16u(x-\Delta x)-30u(x)+16u(x+\Delta x)-u(x+2\Delta x)-1(\Delta x)^2u_{xx} \tag{43}$$

$$=12(\Delta x)^2\left(\frac{-u(x-2\Delta x)+16u(x-\Delta x)-30u(x)+16u(v)-u(x+2\Delta x)}{12(\Delta x)^2}-u_{xx}\right) \tag{44}$$

$$=12(\Delta x)^2O\left((\Delta x)^4\right)=O\left((\Delta x)^6\right) \tag{45}$$

This is equivalent to the previous method for getting the one-step error.

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