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Degeneration of the Superintegrable System with Potentials Described by the Sixth Painlevé Transcendents

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Abstract

This article concerns the quantum superintegrable system obtained by Tremblay and Winternitz, which allows the separation of variables in polar coordinates and possesses three conserved quantities with the potential described by the sixth Painlevé equation. The degeneration procedure from the sixth Painlvé equation to the fifth one yields another new superintegrable system; however, the Hermitian nature is broken.

Keywords

Superintegrable System, Painlevé Equation, Degeneration

1. Introduction

1.1. Superintegrable Systems Separating in Polar Coordinates

Consider the quantum superintegrable system which allows the separation of variables in polar coordinates and possesses three conserved quantities as follows:

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(r,\theta),$$

$$X = L_3^2 + 2S(\theta),$$

$$Y = \sum_{k+l+m=3} A_{klm}\left\{L_3^k, p_1^l p_2^m\right\} + \left\{g_1(\boldsymbol{x}), p_1\right\} + \left\{g_2(\boldsymbol{x}), p_2\right\},$$

where $\boldsymbol{x} = (x_1, x_2) = (r\cos\theta, r\sin\theta)$, $p_k = -i\hbar\partial/\partial x_k$ $(j=1,2)$, $L_3 = x_1 p_2 - x_2 p_1 = -i\hbar\partial/\partial\theta$, $V(r,\theta) = R(r) + S(\theta)/r^2$, $\{A,B\} = AB + BA$. Here $R(r)$, $S(\theta)$, $g_1(\boldsymbol{x})$, $g_2(\boldsymbol{x})$ are arbitrary functions, and A_{klm}'s are real constants. Note that the Hermitian nature of the operators causes the anti-commutator $\{\ ,\ \}$ and the parity. In the followings, we use such the notation as $\partial_\theta = \partial/\partial\theta$ for brevity.

Tremblay and Winternitz [1] classified the cases where the above system is superintegrable, *i.e.* it allows the

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third conserved quantity Y, and obtained $S(\theta)$ which is written by the solution of the sixth Painlevé equation. If $V=S(\theta)/r^2$, $T=T(\theta)$, $T'=S(\theta)$,

$$H=-\frac{\hbar^2}{2}\left\{\partial_r^2+r^{-1}\partial_r+r^{-2}\partial_\theta^2\right\}+S(\theta)/r^2,$$

$$X=-\hbar^2\partial_\theta^2+2S(\theta),$$

$$\begin{aligned}Y&=\left\{L_3^2,p_1\right\}+\left\{G_1\cos\theta-G_2 r\sin\theta,p_1\right\}+\left\{G_1\sin\theta+G_2 r\cos\theta,p_2\right\}\\&=-\hbar^2\left\{\partial_\theta^2,\cos\theta\partial_r-\frac{1}{r}\sin\theta\partial_\theta\right\}+\left(\left\{\frac{1}{r}G_1,\partial_r\right\}+\left\{G_2,\frac{1}{r}\partial_\theta\right\}\right)r,\end{aligned}$$

$$G_1=G_1(r,\theta)=\beta(\theta),$$

$$G_2=G_2(r,\theta)=\frac{1}{r}\beta'(\theta)-2S'(\theta)\cos\theta,$$

$$\beta(\theta)=\beta_0(\theta)+(-T\sin\theta+2T'\cos\theta),$$

$$\beta_0(\theta)=\beta_1\cos\theta+\beta_2\sin\theta;\quad \beta_1,\ \beta_2:\ \text{const}.$$

The commutation $[H,Y]=0$ is reduced to

$$\begin{aligned}&\hbar^2\left(-T''''\sin\theta-4T'''\cos\theta+6T''\sin\theta+4T'\cos\theta\right)+4T''\left\{3T'\sin\theta+T\cos\theta-\beta_0'(\theta)\right\}\\&+8T'\left\{2T'\cos\theta-T\sin\theta+\beta_0(\theta)\right\}=0.\end{aligned}$$

By change of variables $(\theta,T(\theta))\mapsto(t,w(t))$ s.t.

$$\tan\theta=2\sqrt{t(1-t)}/(1-2t),\quad T=\beta_2+\left\{\hbar^2 w+\frac{1}{8}\left(\hbar^2+4\beta_1\right)(1-2t)\right\}\Big/\sqrt{t(1-t)},$$

the above equation is reduced to F-VII $\left(-\frac{1}{4}-2\beta_1/\hbar^2,0\right)$. Here, F-VII (A_0,A_2) is a 4th order ODE defined by

$$\begin{aligned}&-2t^2(t-1)^2w''''-6t(2t-1)(t-1)w'''-24t(t-1)w'w''+8(2t-1)ww''+\left\{2A_0-2-12t(t-1)\right\}w''-4(2t-1)w'^2\\&+8ww'+A_2=0,\end{aligned}$$

with an independent variable t, a dependent variable $w=w(t)$, constant parameters A_0 and A_2 (See [1] [2]). This equation can be integrated twice, and reduced to SD-I.a [1] [3] [4] with constants of integration B_3 and A_4:

$$-t^2(t-1)^2w''^2-4w'(tw'-w)^2+4w'^2(tw'-w)+A_0w'^2+A_2(tw'-w)+B_3w''+A_4=0.$$

Then, by the Bäcklund correspondence

$$\begin{aligned}w(t)&=\left\{t^2(t-1)^2/4u(u-1)(u-t)\right\}\left\{u'-u(u-1)/t(t-1)\right\}^2+\frac{1}{8}\Theta_\infty^2(1-2u)+\frac{1}{8}\theta_0^2(1-2t/u)\\&\quad-\frac{1}{8}\theta_1^2\left\{1-2(t-1)/(u-1)\right\}+\frac{1}{8}\theta_t^2\left\{1-2t(u-1)/(u-t)\right\},\\w'(t)&=-\left\{t(t-1)/4u(u-1)\right\}\left\{u'\pm\theta_\infty u(u-1)/t(t-1)\right\}^2+\frac{1}{4}\theta_0^2(u-t)/u(t-1)-\frac{1}{4}\theta_1^2(u-t)/t(u-1),\end{aligned}$$

SD-I.a is reduced to the sixth Painlevé equation [3]:

$$\begin{aligned}u''&=\frac{1}{2}\left\{1/u+1/(u-1)+1/(u-t)\right\}u'^2-\left\{1/t+1/(t-1)+1/(u-t)\right\}u'\\&\quad+\left\{u(u-1)(u-t)/t^2(t-1)^2\right\}\left[\theta_\infty^2/2-\theta_0^2t/2u^2+\theta_1^2(t-1)/2(u-1)^2-\left(\theta_t^2-1\right)t(t-1)/2(u-t)^2\right],\end{aligned}$$

where the correspondence of the parameters is given by $\Theta_\infty = \theta_\infty + 1$,

$$A_0 = \left(\Theta_\infty^2 + \theta_0^2 + \theta_1^2 + \theta_t^2\right)/2,\ A_1 = \left(\Theta_\infty^2 - \theta_0^2\right)\left(\theta_t^2 - \theta_1^2\right)/4,\ A_2 = \left(\Theta_\infty^2 - \theta_t^2\right)\left(\theta_1^2 - \theta_0^2\right)/4,$$

$$A_3 = \left(\Theta_\infty^2 - \theta_1^2\right)\left(\theta_0^2 - \theta_t^2\right)/4,\ A_4 = \left\{\left(\Theta_\infty^2 + \theta_t^2\right)\left(\theta_0^2 - \theta_1^2\right)^2 + \left(\Theta_\infty^2 - \theta_t^2\right)\left(\theta_0^2 + \theta_1^2\right)\right\}/32,$$

$$B_3 = A_3 + A_0^2/4.$$

F-VII $\left(-\frac{1}{4} - 2\beta_1/\hbar^2, 0\right)$ is reduced to the sixth Painlevé equation with $\theta_1^2 = \theta_0^2$ because of the symmetry of F-VII (A_0, A_2).

1.2. Degeneration Scheme of the Painlevé Equations

Six Painlevé equations are the nonlinear ODEs which define the special functions containing Gauss' hypergeometric function, Bessel functions, Airy functions, etc., and yields elliptic functions and trigonometric functions as the autonomous limits [5] [6]. Solutions to the Painlevé equations are called the Painlevé transcendents. So, the Painlevé transcendents are the ancestors of all classical special functions satisfying ODEs. And, all of the Painlevé equations are derived from the sixth Painlevé equation by some limitation which is called *the degeneration scheme* [5] [7]. For example, the fifth Painlevé equation:

$$u'' = \left\{1/2u + 1/(u-1)\right\}u'^2 - u'/t + \left\{(u-1)^2/t^2\right\}\left[\left(\theta_\infty^2/2\right)u + \left(-\theta_0^2/2\right)/u\right]$$
$$+ \eta(1+\kappa)u/t + \left(-\eta^2/2\right)u(u+1)/(u-1)$$

is derived from the sixth Painlevé equation as follows: replace $(u, t; \theta_\infty, \theta_0, \theta_1, \theta_t)$ by $(u, 1+\varepsilon t; \theta_\infty, \theta_0, \eta/\varepsilon + \kappa, -\eta/\varepsilon)$, and then take limitation $\varepsilon \to 0$.

In this article, we lift-up the degeneration scheme of the Painlevé equation to the superintegrable system, and get the system with potential described by the fifth Painlevé transcendents. The degenerated system should break one or more rules for classification set up by Tremblay and Winternitz [1].

2. Results

Theorem 1. By change of variables $t = s/(s-1)$, $w = \left(v + \frac{1}{4}A_0 s\right)/(1-s)$, the superintegrable system obtained by Tremblay and Winternitz [1] is reduced into the system

$$H = -\frac{\hbar^2}{2}\left[\partial_r^2 + r^{-1}\partial_r + r^{-2}\left(\sqrt{-s}(s-1)\partial_s\right)^2\right] + S(s)/r^2,$$

$$X = -\hbar^2\left(\sqrt{-s}(s-1)\partial_s\right)^2 + 2S(s),$$

$$Y/(-i\hbar) = -\hbar^2\left\{\left(\sqrt{-s}(s-1)\partial_s\right)^2, \partial_{x1}\right\} + \left\{g_1, \partial_{x1}\right\} + \left\{g_2, \partial_{x2}\right\},$$

where $S = \sqrt{-s}(s-1)\partial_s T$, $\partial_\theta = \sqrt{-s}(s-1)\partial_s$, $\partial_{x1} = \frac{1+s}{1-s}\partial_r - \frac{2s}{r}\partial_s$ $\partial_{x2} = \frac{2\sqrt{-s}}{1-s}\partial_r - \frac{\sqrt{-s}(1+s)}{r}\partial_s$.

Moreover, F-VII $(A_0, 0)$ is reduced to F-VII $\left(A_0, -A_0^2/4\right)$, *i.e.* if $w = w(t)$ solves F-VII $(A_0, 0)$, then $v = v(s)$ solves F-VII $\left(A_0, -A_0^2/4\right)$.

Theorem 2. By the degeneration scheme from the six Painlevé equation to the fifth Painlevé equation, the system is reduced into the one

$$H = -\frac{\hbar^2}{2}\left[\partial_r^2 + r^{-1}\partial_r + r^{-2}\left(is\partial_s\right)^2\right] + S(s)/r^2,$$

$$X = -\hbar^2\left(is\partial_s\right)^2 + 2S(s),$$

$$Y/(-i\hbar) = -\hbar^2\left\{(is\partial_s)^2, \partial_{x1}\right\} + \{g_1, \partial_{x1}\} + \{g_2, \partial_{x2}\},$$

where $S = is\partial_s T$, $\partial_\theta = is\partial_s$, $\partial_{x1} = -\frac{2}{s}\partial_r - \frac{2}{r}\partial_s$ $\partial_{x2} = -\frac{2i}{s}\partial_r - \frac{2i}{r}\partial_s$.

Theorem 3. By change of the independent variable $s = \exp(i\sigma)$ and $(x, y) = (r\cos\sigma, r\sin\sigma)$, the system is reduced into the one

$$H = -\frac{\hbar^2}{2}\left[\partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\sigma^2\right] + S(\sigma)/r^2,$$

$$X = -\hbar^2\partial_\sigma^2 + 2S(\sigma),$$

$$Y/(-i\hbar) = -\hbar^2\left\{\partial_\sigma^2, \partial_{x1}\right\} + \{g_1, \partial_{x1}\} + \{g_2, \partial_{x2}\},$$

where $S = \partial_\sigma T$, $\partial_\sigma = is\partial_s$, $\partial_{x1} = -2\partial_x + 2i\partial_y$, $\partial_{x2} = -2\partial_y - 2i\partial_x$.

Each theorem is obtained by a straight-forward computation.

3. Discussion

The degeneration scheme broke the reality of the coordinates, which is not a surprising conclusion. The fact says that, if the assumption of the Tremblay and Winternitz [1] is made looser, then another superintegrable system may appear. So, the author thinks that the assumption of the Hermitian nature is too strong to get the sixth Painlevé equation with full-parameter or other Painlevé equations.

Marquette and Winternitz [8] also obtained other superintegrable systems with potentials described by the first, second and fourth Painlevé equations. But it is uncertain if the system above degenerates into the system obtained in [8].

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