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Lie Symmetries, One-Dimensional Optimal System and Optimal Reduction of (2 + 1)-Coupled nonlinear Schrödinger Equations

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Abstract

For a class of (1 + 2)-dimensional nonlinear Schrödinger equations, the infinite dimensional Lie algebra of the classical symmetry group is found and the one-dimensional optimal system of an 8-dimensional subalgebra of the infinite Lie algebra is constructed. The reduced equations of the equations with respect to the optimal system are derived. Furthermore, the one-dimensional optimal systems of the Lie algebra admitted by the reduced equations are also constructed. Consequently, the classification of the twice optimal symmetry reductions of the equations with respect to the optimal systems is presented. The reductions show that the (1 + 2)-dimensional nonlinear Schrödinger equations can be reduced to a group of ordinary differential equations which is useful for solving the related problems of the equations.

Keywords

Nonlinear Schrödinger Equations, Lie Aymmetry Group, Lie algebra, Optimal System

1. Introduction

We plan to consider the (1 + 2)-dimensional coupled nonlinear Schrödinger (2D-CNLS) equations with cubic nonlinearity

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$$\begin{cases} i\dfrac{\partial u}{\partial t}-\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\rho|u|^2 u-2uv=0, \\ \dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}-\delta\dfrac{\partial^2}{\partial x^2}\left(|u|^2\right)=0, \quad \delta,\rho=\text{constants}, \end{cases} \tag{1}$$

where u, v are complex-valued functions. The 2D-CNLS equations which describes the evolution of the wave packet on a two-dimensional water surface under gravity was derived by Benny and Roskes [1] and Davey and Stewartson [2]. The solutions of the equation have been studied by several authors [3]-[12]. The multi-soliton solutions were obtained by Anker and Freeman [8]. They showed that the two-soliton resonant interaction occurs and a triple soliton structure is produced. A similarity reductions of the 2D-CNLS equation is also studied in [9]. Nakamura [10] found explode-decay mode solutions by using the bilinear method. However, the algebra properties of the Lie algebra admitted by (1) has not been studied so far. The optimal system of the Lie algebra yields the optimal classification of the invariant solutions set to the 2D-CNLS which is essential to distinguish the inequivalent classes of the invariant solutions of the equation.

In this paper, we show the optimal reduction classifications of the 2D-CNLS equations (1) through studying one-dimensional optimal system of the Lie algebra of the equations.

Outline of the paper is following. In §2, the complete infinite-dimensional Lie algebra $\mathcal{L}^\infty$ of the Lie symmetry group of the 2D-CNLS equations is derived which covered the results obtained in [9]. In §3, the one-dimensional optimal system of an 8-dimensional subalgebra $\mathcal{L}^8$, presented in [9], of the $\mathcal{L}^\infty$ is constructed. In §4 the first reductions of the 2D-CNLS Equation (1) with respect to the optimal system obtained in §3 are given. In §5 we construct one-dimensional optimal systems of Lie algebras of the reduced equations obtained in §4 which yields the second reductions of (1). Consequently, the 2D-CNLS Equation (1) can be reduced to a group of scale ordinary differential equations, which is essential to solve different exact solutions of the 2D-NLS Equation (1).

2. The Lie Algebra of the 2D-CNLS Equations (1)

In this section, we present the Lie algebra of point symmetries of 2D-CNLS (1). To obtain the Lie algebra, we consider the one parameter Lie symmetry group of infinitesimal transformations in (t,x,y,u,v) given by

$$\begin{cases} t^* = t+\varepsilon\tau(t,x,y,u,v)+O(\varepsilon^2), \\ x^* = x+\varepsilon\xi(t,x,y,u,v)+O(\varepsilon^2), \\ y^* = y+\varepsilon\zeta(t,x,y,u,v)+O(\varepsilon^2), \\ u^* = u+\varepsilon\eta(t,x,y,u,v)+O(\varepsilon^2), \\ v^* = v+\varepsilon\phi(t,x,y,u,v)+O(\varepsilon^2), \end{cases} \tag{2}$$

where ε is the group parameter. Hence the corresponding generator of the Lie algebra of the symmetry group is

$$X=\tau(t,x,y,u,v)\frac{\partial}{\partial t}+\xi(t,x,y,u,v)\frac{\partial}{\partial x}+\zeta(t,x,y,u,v)\frac{\partial}{\partial y}+\eta(t,x,y,u,v)\frac{\partial}{\partial u}+\phi(t,x,y,u,v)\frac{\partial}{\partial v}.$$

Transforming 2D-CNLS equations (1) to real case by transformations $u=U+iu$, $v=V+iv$, where U, u, V and v are real functions, one has real form of the 2D-CNLS equations (1) in four unknown functions U,u,V and v.

Assuming that the 2D-CNLS equations (1) is invariant under the transformations (2), then its real form transformed system is invariant under the Lie symmetry group with generator

$X=\tau\partial_t+\xi\partial_x+\zeta\partial_y+\eta\partial_U+\phi\partial_u+\psi\partial_V+\omega\partial_v$, in which $\tau=\tau(t,x,y,U,u,V,v)$, $\xi=\xi(t,x,y,U,u,V,v)$,

$$\zeta=\zeta(t,x,y,U,u,V,v),\quad \eta=\eta(t,x,y,U,u,V,v),\quad \phi=\phi(t,x,y,U,u,V,v),\quad \psi=\psi(t,x,y,U,u,V,v),$$

$\omega=\omega(t,x,y,U,u,V,v)$. By invariance criterion in [11]-[13], we have the DTEs of the Lie symmetry group as follows

$$\eta_v=0,\eta_V=0,\phi_v=0,\phi_V=0,\psi_v=0,\psi_u=0,\psi_U=0,\psi_{xy}=0,\xi_v=0,\xi_V=0,\xi_u=0,\xi_U=0,\xi_y=0,$$

$$\zeta_v=0,\zeta_V=0,\zeta_u=0,\zeta_U=0,\zeta_x=0,\tau_v=0,\tau_V=0,\tau_u=0,\tau_U=0,\tau_y=0,\tau_x=0,$$

$$U\phi-u\eta+\left(U^2+u^2\right)\eta_u=0, U\phi-u\eta-\left(U^2+u^2\right)\phi_U=0,$$

$$u\phi+U\eta-\left(U^2+u^2\right)\eta_U=0, u\phi+U\eta-\left(U^2+u^2\right)\phi_u=0,$$

$$\left(u\phi+U\eta\right)+\left(U^2+u^2\right)\xi_x=0, u\phi+U\eta+\left(U^2+u^2\right)\zeta_y=0,$$

$$2\left(u\phi+U\eta\right)-\left(U^2+u^2\right)\psi_V=0,\ 2\left(u\phi+U\eta\right)+\left(U^2+u^2\right)\tau_t=0,$$

$$\left(U^2+u^2\right)\omega-2v\left(u\phi+U\eta\right)=0, U\phi_t-4V\left(u\phi+U\eta\right)-u\eta_t+2\left(U^2+u^2\right)\psi=0,$$

$$2\phi_x+U\xi_t=0,\ 2\eta_x-u\xi_t=0,\ 2\phi_y-U\zeta_t=0,\ 2\eta_y+u\zeta_t=0,\ 4\psi_x-\xi_{tt}=0,\ \zeta_{tt}+4\psi_y=0,$$

$$16u^2V^2\left(u\phi+U\eta\right)+4uV\left(U^2+u^2\right)\eta_t+U\left(U^2+u^2\right)\eta_{tt}-8u^2V\left(U^2+u^2\right)\psi-2Uu\left(U^2+u^2\right)\psi_t-4U^2\left(U^2+u^2\right)\psi_{yy}=0,$$

$$16u^2V^2\left(u\phi+U\eta\right)+4uV\left(U^2+u^2\right)\eta_t+U\left(U^2+u^2\right)\eta_{tt}-8u^2V\left(U^2+u^2\right)\psi-2Uu\left(U^2+u^2\right)\psi_t-4U^2\left(U^2+u^2\right)\psi_{xx}=0,$$

for functions $\tau,\xi,\zeta,\eta,\phi,\psi,\omega$. Solving this system by characteristic set algorithm given in [14] [15], we obtain the infinitesimal functions of generator X, *i.e.*,

$$\begin{cases}\tau=\tau(t),\\ \xi=\frac{1}{2}x\tau'(t)+\xi(t),\\ \zeta=\frac{1}{2}y\tau'(t)+\zeta(t),\\ \eta=\frac{1}{8}\left[-8u\phi(t)+4ux\xi'(t)-4uy\zeta'(t)-4U\tau'(t)+ux^2\tau''(t)-uy^2\tau''(t)\right],\\ \phi=\frac{1}{8}\left[8U\phi(t)-4Ux\xi'(t)+4Uy\zeta'(t)-4u\tau'(t)-Ux^2\tau''(t)+Uy^2\tau''(t)\right],\\ \psi=\frac{1}{16}\left[-16V\tau'(t)+x^2\tau^{(3)}(t)-y^2\tau^{(3)}(t)+4x\xi''(t)-4y\zeta''(t)-8\phi'(t)\right],\\ \omega=-v\tau'(t),\end{cases}\tag{3}$$

where $\tau(t)$, $\xi(t)$, $\zeta(t)$, $\phi(t)$ are arbitrary functions of their argument. Hence the 2D-CNLS (1) admits infinite dimensional Lie algebra $\mathcal{L}^\infty$. It is notice that in [9] only a special subset of (3) were found. Namely, if taking here a linear independent representatives of the vectors $\left(\tau(t),\zeta(t),\phi(t),\xi(t)\right)$ as

$$\begin{cases}\tau=\zeta=\phi=0,\ \xi=1;\\ \tau=\zeta=\phi=0,\ \xi=-t;\\ \tau=\xi=\phi=0,\ \zeta=1;\\ \tau=\xi=\phi=0,\ \zeta=t;\\ \tau=\xi=\zeta=0,\ \phi=1;\\ \xi=\zeta=\phi=0,\ \tau=1;\\ \xi=\zeta=\phi=0,\ \tau=2t;\\ \phi=\xi=\zeta=0,\ \tau=t^2;\end{cases}$$

respectively and by transforming $U+iu\to u$, $\partial_U\to\partial_u$, $\partial_u\to i\partial_u$, $V+iv\to v$, $\partial_V\to\partial_v$, $\partial_v\to i\partial_v$, we recover the basis of the 8-dimensional Lie algebra $\mathcal{L}^8$ given in [9] as follows

$$X_1=\partial_x,\ X_2=\partial_y,\ X_3=\partial_t,\ X_4=2t\partial_t+x\partial_x+y\partial_y-u\partial_u-2v\partial_v,$$
$$X_5=-t\partial_x+\frac{1}{2}ixu\partial_u,\ X_6=t\partial_y+\frac{1}{2}iyu\partial_u, \tag{4}$$
$$X_7=t^2\partial_t+tx\partial_x+ty\partial_y-\left(t+\frac{1}{4}ix^2-\frac{1}{4}iy^2\right)u\partial_u-2tv\partial_v,\ X_8=iu\partial_u.$$

If taking other linear independents case of vector $\left(\tau(t),\zeta(t),\phi(t),\xi(t)\right)$, we obtain other subalgebras of $\mathcal{L}^\infty$. In this paper, we take the case (4) as example to show the investigation procedure for finite sub-algebras properties of the infinite dimensional algebra $\mathcal{L}^\infty$.

The commutators of the generators (4) are given in the **Table 1**, where the entry in the i^{th} row and j^{th} column is defined as $\left[X_i,X_j\right]=X_iX_j-X_jX_i,\ (i,j=1,\cdots,8)$.

The table is fundamental for our constructing the optimal system of the $\mathcal{L}^8$ with basis (4).

3. One-Dimensional Optimal System of $\mathcal{L}^8$

In this section, we give an one-dimensional optimal system of the Lie algebra $\mathcal{L}^8$ spaned by (4). Finding one-dimensional optimal system of one-dimensional subalgebras of a Lie algebra is a subalgebra classification problem. It is essentially the same as the problem of classifying the orbit of the adjoint representation, since each one-dimensional subalgebra is determined by nonzero vector in the Lie algebra. Hence it is equivalent to classification of subalgebras under the adjoint representation of the Lie algebra. The adjoint representation is given by the Lie series

$$Ad\left(\exp(\varepsilon X_i)\right)X_j=X_j+\varepsilon\left[X_j,X_i\right]+\frac{1}{2!}\varepsilon^2\left[\left[X_j,X_i\right],X_i\right]+\cdots,$$

where $\left[X_i,X_j\right]$ is the commutator given in **Table 1**, ε is a parameter, and $i,j=1,2,\cdots,8$. This yields following adjoint commutator **Table 2** for (4) in which the (i,j) entry gives $Ad\left(\exp(\varepsilon X_i)\right)X_j$.

The following is the deduction procedure of one-dimensional optimal system of (4) by using the method given in [15]-[20].

Let $X=k_1X_1+k_2X_2+\cdots+k_8X_8$ be an element of $\mathcal{L}^8$ spanned by (4), which we shall try to simplify using suitable adjoint maps and find its equivalent representative. A key observation here is that the function $\eta(X)=(k_4)^2-k_3k_7$ is an invariant of the full adjoint action, that means $\eta\left(Ad(g)X\right)=\eta(X),\ X\in\mathcal{L}^8, g\in G$ (the corresponding symmetry group of the Lie algebra $\mathcal{L}^8$). The detection of such an invariant is important since it places restrictions on how far we can expect to simplify X. For example, if $\eta(X)\neq 0$, then we cannot simultaneously make k_3,k_7 and k_4 all zero through adjoint maps; if $\eta(X)<0$, we cannot make either k_3 or k_7 zero!

To begin the classification process, we first concentrate on the coefficients k_3,k_4,k_7 of X. Acting simultaneously adjoints of X_3 and X_7, one has

$$\tilde{X}=Ad\left(\exp(\alpha X_7)\right)\circ Ad\left(\exp\left(\frac{\beta}{2}X_3\right)\right)X=\sum_{i=1}^{8}\tilde{k}_iX_i$$

with coefficients

$$\tilde{k}_3=k_3-\beta k_4+\frac{\beta^2}{4}k_7,$$
$$\tilde{k}_4=k_4-\frac{\beta}{2}k_7+\alpha\left(k_3-\beta k_4+\frac{\beta^2}{4}k_7\right), \tag{5}$$
$$\tilde{k}_7=k_7+2\alpha\left(k_4-\frac{\beta}{2}k_7\right)+\alpha^2\left(k_3-\beta k_4+\frac{\beta^2}{4}k_7\right).$$

Table 1. The commutators of (4).

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	0	X_1	$\frac{1}{2}X_8$	0	$-X_5$	0
X_2	0	0	0	X_2	0	$\frac{1}{2}X_8$	X_6	0
X_3	0	0	0	$2X_3$	$-X_1$	X_2	X_4	0
X_4	$-X_1$	$-X_2$	$-2X_3$	0	X_5	X_6	$2X_7$	0
X_5	$-\frac{1}{2}X_8$	0	X_1	$-X_5$	0	0	0	0
X_6	0	$-\frac{1}{2}X_8$	$-X_2$	$-X_6$	0	0	0	0
X_7	X_5	$-X_6$	$-X_4$	$-2X_7$	0	0	0	0
X_8	0	0	0	0	0	0	0	0

Table 2. The adjoint commutator of (4).

Ad	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	X_1	X_2	X_3	$X_4-\varepsilon X_1$	$X_5-\frac{1}{2}\varepsilon X_8$	X_6	$X_7+\varepsilon X_5-\varepsilon^2\frac{1}{4}X_8$	X_8
X_2	X_1	X_2	X_3	$X_4-\varepsilon X_2$	X_5	$X_6-\frac{1}{2}\varepsilon X_8$	$X_7-\varepsilon X_6+\frac{1}{4}\varepsilon^2 X_8$	X_8
X_3	X_1	X_2	X_3	$X_4-2\varepsilon X_3$	$X_5+\varepsilon X_1$	$X_6-\varepsilon X_2$	$X_7-\varepsilon X_4+\varepsilon^2 X_3$	X_8
X_4	$e^{\varepsilon}X_1$	$e^{\varepsilon}X_2$	$e^{2\varepsilon}X_3$	X_4	$e^{-\varepsilon}X_5$	$e^{-\varepsilon}X_6$	$e^{-2\varepsilon}X_7$	X_8
X_5	$X_1+\frac{1}{2}\varepsilon X_8$	X_2	$X_3-\varepsilon X_1-\frac{1}{4}\varepsilon^2 X_8$	$X_4+\varepsilon X_5$	X_5	X_6	X_7	X_8
X_6	X_1	$X_2+\frac{1}{2}\varepsilon X_8$	$X_3+\varepsilon X_2+\frac{1}{4}\varepsilon^2 X_8$	$X_4+\varepsilon X_6$	X_5	X_6	X_7	X_8
X_7	$X_1-\varepsilon X_5$	$X_2+\varepsilon X_6$	$X_3+\varepsilon X_4+\varepsilon^2 X_7$	$X_4+2\varepsilon X_7$	X_5	X_6	X_7	X_8
X_8	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8

There are now three cases, depending on the sign of the invariant η.

Case 1. If $\eta(X)>0$, then we choose β to be either real root of the quadratic equation $\frac{k_7}{4}\beta^2-k_4\beta+k_3=0$ and $\alpha=k_7/(k_7\beta-2k_4)$ (which is always well defined). Then $\tilde{k}_3=\tilde{k}_7=0$, while $\tilde{k}_4=\sqrt{\eta(X)}\neq 0$, so X is equivalent to a multiple of $\tilde{X}=X_4+\tilde{k}_1X_1+\tilde{k}_2X_2+\tilde{k}_5X_5+\tilde{k}_6X_6+\tilde{k}_8X_8$. Acting further by adjoint maps generated respectively by X_1,X_2,X_5 and X_6 we can arrange that the coefficients of X_1,X_2,X_5 and X_6 in $\tilde{X}$ vanish. Therefore, every element X with $\eta(X)>0$ is equivalent to a multiple of X_4+aX_8 for some $a\in\mathbb{R}$. No further simplifications are possible.

Case 2. If $\eta(X)<0$ (implies $k_3\neq 0$), set $\beta=0,\alpha=-k_4/k_3$ to make $\tilde{k}_4=0$. Acting on X by the group generated by X_4, we can make the coefficients of X_3 and X_7 agree, so X is equivalent to a scalar multiple of $\tilde{X}=(X_3+X_7)+\tilde{k}_1X_1+\tilde{k}_2X_2+\tilde{k}_5X_5+\tilde{k}_6X_6+\tilde{k}_8X_8$. Further use of the groups generated by X_5, X_6, X_1 and X_2 show that $\tilde{X}$ is equivalent to a scalar multiple of $X_3+X_7+aX_8$ for some $a\in\mathbb{R}$.

Case 3. If $\eta(X)=0$, there are two subcases. If not all of the coefficients k_3,k_4,k_7 vanish, then we can choose α and β in (5) so that $\tilde{k}_3\neq 0$, but $\tilde{k}_4=\tilde{k}_7=0$, so X is equivalent to a multiple of $\tilde{X}=X_3+\tilde{k}_1X_1+\tilde{k}_2X_2+\tilde{k}_5X_5+\tilde{k}_6X_6+\tilde{k}_8X_8$. Suppose $\tilde{k}_6\neq 0$. Then we can make the coefficients of X_1,X_2 and X_8 zero using the groups generated by X_5,X_6 and X_2, while the group generated by X_4 independently scales the coefficients of X_3 and X_5. Thus such a X is equivalent to a multiple of either $X_3+\varepsilon X_6+aX_5$ for some $a\in\mathbb{R}$, $\varepsilon=\pm 1$. If $\tilde{k}_6=0$, so X is equivalent to a multiple of $\tilde{X}=X_3+\tilde{k}_1X_1+\tilde{k}_2X_2+\tilde{k}_5X_5+\tilde{k}_8X_8$, suppose $\tilde{k}_5\neq 0$. Then we can make the coefficients of X_1,X_2 and X_8 zero using the groups generated by X_5,X_6, and X_1, while the group generated by X_4 independently scales the coefficients of X_3 and X_5. Thus such a X is equivalent to a multiple of either $X_3+\varepsilon X_5, \varepsilon\pm 1$. If $\tilde{k}_6=\tilde{k}_5=0$, then the group generated by X_5 and X_6 can be reduce X to a vector of the form X_3+aX_8, for some $a\in\mathbb{R}$.

The last remaining case occurs when $k_3=k_4=k_7=0$, for which our earlier simplifications were unnecessary. If $k_1\neq 0$, then using groups generated by X_5 and X_7 we can arrange X to become a multiple of $X_1+aX_2+bX_6$ for some $a,b\in\mathbb{R}$. If $k_1=0$, but $k_2\neq 0$, $X=k_2X_2+k_5X_5+k_6X_6+k_8X_8$, then we can make the coefficients of X_8 and X_6 zero using the groups generated by X_6 and X_7, while X is equivalent to a multiple of X_2+aX_5 for some $a\in\mathbb{R}$. If $k_1=k_2=0$, but $k_5\neq 0$, we can first act by $Ad(\exp(\varepsilon X_3))$ and get a nonzero coefficients in front of X_1,X_2 which is reduced to the previous case. If $k_1=k_2=k_5=0$, but $k_8\neq 0$, $X=k_8X_8+k_6X_6$ then we can arrange X to become multiple of X_8+aX_6 for some $a\in\mathbb{R}$. The only remaining vectors are the multiple of X_6.

In summary, an optimal system of one-dimensional subalgebras of $\mathcal{L}^8$ with base (4) is provided by generators

$$
\begin{aligned}
&X^1=X_4+aX_8, && \eta>0, && a\in R,\\
&X^2=X_3+X_7+aX_8, && \eta<0, && a\in R,\\
&X^3=X_3+\varepsilon X_6+aX_5, && \eta=0, && a\in R, \varepsilon=\pm 1,\\
&X^4=X_3+\varepsilon X_5, && \eta=0, && \varepsilon=\pm 1,\\
&X^5=X_3+aX_8, && \eta=0, && a\in R,\\
&X^6=X_1+aX_2+bX_6, && \eta=0, && a,b\in R,\\
&X^7=X_2+aX_5, && \eta=0, && a\in R,\\
&X^8=X_8+aX_6, && \eta=0, && a\in R,\\
&X^9=X_6, && \eta=0.
\end{aligned}
\tag{6}
$$

4. First Optimal Reductions of 2D-CNLS (1) with (6)

In this section, we give a classification of symmetry reductions of 2D-CNLS (1) by using optimal system (6). Since the similarity, we will introduce the details of computation for $X^2=X_3+X_7+aX_8$ in (6) and directly give the computation results without showing the details of the procedure for the remaining cases in (6).

The differential invariants (and hence the similarity variables) for the generator X^2 can be obtained by solving the characteristic system

$$\frac{\mathrm{d}t}{1+t^2}=\frac{\mathrm{d}x}{tx}=\frac{\mathrm{d}y}{ty}=\frac{\mathrm{d}u}{\left(ai-t-\frac{1}{4}ix^2+\frac{1}{4}iy^2\right)u}=\frac{\mathrm{d}v}{-2tv}. \tag{7}$$

The system yields the similarity variables as follows

$$z_1(t,x,y)=x(1+t^2)^{-\frac{1}{2}},$$

$$z_2(t,x,y)=y(1+t^2)^{-\frac{1}{2}},$$

$$w^1(z_1,z_2)=(1+t^2)^{\frac{1}{2}}\exp\left(-\frac{i}{4}(1+t^2)^{-1}\left(4a(1+t^2)\arctan t-t(x^2-y^2)\right)\right)u,$$

$$w^2(z_1,z_2)=(1+t^2)v.$$

Hence we let

$$\begin{aligned}&u=(1+t^2)^{-\frac{1}{2}}w^1(z_1,z_2)\exp\left(\frac{i}{4}(1+t^2)^{-1}\left(4a(1+t^2)\arctan t-t(x^2-y^2)\right)\right),\\&v=(1+t^2)^{-1}w^2(z_1,z_2),\end{aligned}\tag{8}$$

and substitute them into the Equations (1), then the equations are reduced to

$$\begin{cases}\dfrac{d^2w^1}{dz_1^2}-\dfrac{d^2w^1}{dz_2^2}+\left(a+\dfrac{1}{4}(z_2^2-z_1^2)-\rho|w^1|^2+2w^2\right)w^1=0,\\ \dfrac{d^2w^2}{dz_1^2}+\dfrac{d^2w^2}{dz_2^2}-\delta\dfrac{\partial^2}{\partial(z_1)^2}\left(|w^1|^2\right)=0.\end{cases}\tag{9}$$

Using the rest elements in (6), we can obtain the rest reductions of 2D-CNLS Equations (1) presented in following Table 3. Here

5. Further Optimal Reductions of (1) through Reductions of the Equations in Table 3

In fact, the equations in Table 3 can be reduced further in the similar way which results in the second time reductions of 2D-CNLS (1). We take the second case in Table 3 as example to show the procedure of the second time reduction of the Equation (1).

Using characteristic set algorithm given in [14] [15], the symmetry algebra generators of the second Equations (9) with similarity variables of case B in Table 3 is determined as follows

$$\begin{aligned}&Y_1=z_1\partial_{z_1}+z_2\partial_{z_2}-w^1\partial_{w^1}+\frac{1}{2}(z_1^2-z_2^2-2a-4w^2)\partial_{w^2},\\&Y_2=\partial_{z_1}+\frac{1}{4}z_1\partial_{w^2},\\&Y_3=\partial_{z_2}-\frac{1}{4}z_2\partial_{w^2},\\&Y_4=-iw^1\partial_{w^1},\\&Y_5=\partial_{w^1},\\&Y_6=i\partial_{w^1}.\end{aligned}\tag{10}$$

Using the same procedure in last section, we can also find an one-dimensional optimal system of one-dimensional subalgebras of the Lie algebra spanned by (10). The optimal system consists of

$$\begin{aligned}&Y_1^2=Y_1+cY_4,\quad Y_2^2=Y_4+cY_2+dY_3,\quad Y_3^2=Y_5+cY_2+dY_3,\\&Y_4^2=Y_6+cY_2+dY_3,\quad Y_5^2=Y_2+cY_3,\quad Y_6^2=Y_3,\end{aligned}\tag{11}$$

where c, d are arbitrary constants. We take $Y_2^2=Y_4+cY_2+dY_3\ (cd\neq 0)$ as example to show the further

Table 3. The first reductions of the 2D-CNLS (1) by optimal system (6).

No.	Generators in (6)	The first reductions	Invariance variables
1	$X^1 = X_4 + aX_8$	$\frac{\partial^2 w^1}{\partial z_1^2} - \frac{\partial^2 w^1}{\partial z_2^2} + \frac{i}{2}\left(z_1 \frac{\partial w^1}{\partial z_1} + z_2 \frac{\partial w^1}{\partial z_2}\right) + \left(\frac{a+i}{2} - \rho\left\|w^1\right\|^2 + 2w^2\right)w^1 = 0,$ $\frac{\partial^2 w^2}{\partial z_1^2} + \frac{\partial^2 w^2}{\partial z_2^2} - \delta \frac{\partial^2}{\partial z_1^2}\left(\left\|w^1\right\|^2\right) = 0.$	A
2	$X^2 = X_3 + X_7 + aX_8$	$\frac{\partial^2 w^1}{\partial z_1^2} - \frac{\partial^2 w^1}{\partial z_2^2} + \left[a + \frac{1}{4}\left(z_2^2 - z_1^2\right) - \rho\left\|w^1\right\|^2 + 2w^2\right]w^1 = 0,$ $\frac{\partial^2 w^2}{\partial z_1^2} + \frac{\partial^2 w^2}{\partial z_2^2} - \delta \frac{\partial^2}{\partial z_1^2}\left(\left\|w^1\right\|^2\right) = 0.$	B
3	$X^3 = X_3 + \varepsilon X_6 + aX_5$	$\frac{\partial^2 w^1}{\partial z_1^2} - \frac{\partial^2 w^1}{\partial z_2^2} + \left(\frac{a}{2} z_1 + \frac{\varepsilon}{2} z_2 - \rho\left\|w^1\right\|^2 + 2w^2\right)w^1 = 0,$ $\frac{\partial^2 w^2}{\partial z_1^2} + \frac{\partial^2 w^2}{\partial z_2^2} - \delta \frac{\partial^2}{\partial z_1^2}\left(\left\|w^1\right\|^2\right) = 0.$	C
4	$X^4 = X_3 + \varepsilon X_5$	$\frac{\partial^2 w^1}{\partial z_1^2} - \frac{\partial^2 w^1}{\partial z_2^2} + \left(\frac{\varepsilon}{2} z_1 - \rho\left\|w^1\right\|^2 + 2w^2\right)w^1 = 0,$ $\frac{\partial^2 w^2}{\partial z_1^2} + \frac{\partial^2 w^2}{\partial z_2^2} - \delta \frac{\partial^2}{\partial z_1^2}\left(\left\|w^1\right\|^2\right) = 0.$	D
5	$X^5 = X_3 + aX_8$	$\frac{\partial^2 w^1}{\partial z_1^2} - \frac{\partial^2 w^1}{\partial z_2^2} + \left(a - \rho\left\|w^1\right\|^2 + 2w^2\right)w^1 = 0,$ $\frac{\partial^2 w^2}{\partial z_1^2} + \frac{\partial^2 w^2}{\partial z_2^2} - \delta \frac{\partial^2}{\partial z_1^2}\left(\left\|w^1\right\|^2\right) = 0.$	E
6	$X^6 = X_1 + aX_2 + bX_6$	$\left(1 - \left(a + bz_1\right)^2\right)\frac{\partial^2 w^1}{\partial z_2^2} + i\left(\frac{\partial w^1}{\partial z_1} + b\left(a + bz_1\right)z_2 \frac{\partial w^1}{\partial z_2}\right)$ $+\left(\frac{ib}{2}\left(a + bz_1\right) + \frac{b^2}{4} z_2^2 + \rho\left\|w^1\right\|^2 - 2w^2\right)w^1 = 0,$ $w^2 = \frac{\delta\left(a + bz_1\right)^2}{1 + \left(a + bz_1\right)^2}\left\|w^1\right\|^2 + k_1 z_2 + k_2.$	F
7	$X^7 = X_2 + aX_5$	$i\frac{\partial w^1}{\partial z_1} + \left(1 - \frac{1}{a^2 z_1^2}\right)\frac{\partial^2 w^1}{\partial z_2^2} - \left(\frac{i}{2z_1} - \rho\left\|w^1\right\|^2 + 2w^2\right)w^1 = 0,$ $w^2 = \frac{\delta}{1 + a^2}\left\|w^1\right\|^2 + k_1 z_2 + k_2.$	G
8	$X^8 = X_8 + aX_6$	$i\frac{\partial w^1}{\partial z_1} - \frac{\partial^2 w^1}{\partial z_2^2} + \left(\frac{i}{2z_1} - \frac{1}{a^2 z_1^2} + \rho\left\|w^1\right\|^2 - 2w^2\right)w^1 = 0,$ $\frac{\partial^2 w^2}{\partial z_2^2} - \delta \frac{\partial^2}{\partial z_2^2}\left(\left\|w^1\right\|^2\right) = 0.$	H
9	$X^9 = X_6$	$i\frac{\partial w^1}{\partial z_1} - \frac{\partial^2 w^1}{\partial z_2^2} + \left(\frac{i}{2z_1} w^1 + \rho\left\|w^1\right\|^2 - 2w^2\right)w^1 = 0,$ $\frac{\partial^2 w^2}{\partial z_2^2} - \delta \frac{\partial^2}{\partial z_2^2}\left(\left\|w^1\right\|^2\right) = 0.$	I

$$A: u = w^1(z_1,z_2)t^{\frac{ai-1}{2}},\ v = w^1(z_1,z_2)t^{-1},\ z_1 = xt^{-\frac{1}{2}},\ z_2 = yt^{-\frac{1}{2}}.$$

$$B: u = (1+t^2)^{-\frac{1}{2}} w^1(z_1,z_2)\exp\left(\frac{i}{4}(1+t^2)^{-1}\left(4a(1+t^2)\arctan t - t(x^2-y^2)\right)\right),$$
$$v = (1+t^2)^{-1} w^2(z_1,z_2),\quad z_1 = x(1+t^2)^{-\frac{1}{2}},\ z_2 = y(1+t^2)^{-\frac{1}{2}}.$$

$$C: u = w^1(z_1,z_2)\exp\left(i\left(\frac{a^2-\varepsilon^2}{6}t^3 + \frac{ax+\varepsilon y}{2}t\right)\right),\ v = w^2(z_1,z_2),\ z_1 = x+\frac{a}{2}t^2,\ z_2 = y-\frac{\varepsilon}{2}t^2.$$

$$D: u = w^1(z_1,z_2)\exp\left(i\left(\frac{\varepsilon^2 t^3}{6} + \frac{\varepsilon tx}{2}\right)\right),\ v = w^2(z_1,z_2),\ z_1 = x+\frac{\varepsilon}{2}t^2,\ z_2 = y.$$

$$E: u = w^1(z_1,z_2)\exp(ati),\ v = w^2(z_1,z_2),\ z_1 = x,\ z_2 = y.$$

$$F: u = w^1(z_1,z_2)\exp\left(i\left(\frac{bxy}{2} - \frac{bx^2(a+bt)}{4}\right)\right),\ v = w^2(z_1,z_2),\ z_1 = t,\ z_2 = y-(a+bt)x.$$

$$G: u = w^1(z_1,z_2)\exp\left(i\left(\frac{a^2y^2t}{4} + \frac{axy}{2}\right)\right),\ v = w^2(z_1,z_2),\ z_1 = t,\ z_2 = x+aty.$$

$$H: u = w^1(z_1,z_2)\exp\left(i\left(\frac{y^2}{4t} + \frac{y}{at}\right)\right),\ v = w^2(z_1,z_2), z_1 = t,\ z_2 = x.$$

$$I: u = w^1(z_1,z_2)\exp\left(i\left(\frac{y^2}{4t}\right)\right),\ v = w^2(z_1,z_2),\ z_1 = t,\ z_2 = x.$$

reduction procedure.

The characteristic system

$$\frac{dz_1}{c} = \frac{dz_2}{d} = \frac{dw^1}{-iw^1} = \frac{dw^2}{\frac{1}{4}(cz_1 - dz_2)}, \tag{12}$$

yields the corresponding similarity variables

$$z = z_2 - \frac{d}{c}z_1,\quad F(z) = w^1 e^{\frac{i}{c}z_1},\quad G(z) = w^2 - \frac{z_1^2}{8} + \frac{d^2z_1^2}{8c^2} + \frac{(cz_2 - dz_1)dz_1}{4c^2}.$$

Hence we let

$$w^1 = F(z)e^{-\frac{i}{c}z_1},\quad w^2 = G(z) + \frac{1}{8}\left(1+\frac{d^2}{c^2}\right)z_1^2 - \frac{d}{4c}z_1z_2, \tag{13}$$

and substitute them into the underline equations, then the second equation in **Table 3** is reduced to

$$\begin{cases} \left(\frac{d^2}{c^2}-1\right)\frac{d^2F}{dz^2} + \frac{2di}{c^2}\frac{dF}{dz} + \left(a - \frac{1}{c^2} + \frac{1}{4}z^2 - \rho|F|^2 + 2G\right)F = 0, \\ (c^2+d^2)\left(1+4\frac{d^2G}{dz^2}\right) - 4\delta\frac{d^2}{dz^2}\left(|F|^2\right) = 0. \end{cases} \tag{14}$$

This is a result of twice reductions of (1) by $X^2 = X_3 + X_7 + aX_8$ and $Y_2^2 = Y_4 + cY_2 + dY_3\ (cd \neq 0)$ successively. In the same manner, we can obtain the other reductions of the equation with using the other elements in (11) which are listed in the following **Table 4**. In fact, (13) and (14) are listed as second case in **Table 4**.

Solving the second equation in (14), we have

$$G = \frac{d^2\delta}{c^2+d^2}|F|^2 - \frac{1}{8}z^2 + k_1z + k_2,$$

where k_1, k_2 are arbitrary constants. Substituting this into the first equation of (14), we get a scale reduction of 2D-CNLS (1) as follows

$$\left(\frac{d^2}{c^2}-1\right)\frac{d^2F}{dz^2} + \frac{2di}{c^2}\frac{dF}{dz} + \left(a - \frac{1}{c^2} + 2k_1z + 2k_2 + \left(\frac{2\delta}{c^2+d^2} - \rho\right)|F|^2\right)F = 0,$$

Table 4. The second reductions of the 2D-CNLS (1) with X^2.

No.	Generators in (11)	The second time reductions of 2D-CNLS (1)	Invariance variables
1	$Y_1^2 = Y_1 + cY_4$	$(z^2-1)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + (4+2ci)z\frac{\mathrm{d}F}{\mathrm{d}z} + \left(2-c^2+3ci-\rho\lvert F\rvert^2+2G\right)F = 0,$ $(z^2+1)\frac{\mathrm{d}^2G}{\mathrm{d}z^2} + 6z\frac{\mathrm{d}G}{\mathrm{d}z} + 6G - \delta\left(6+6z\frac{\mathrm{d}}{\mathrm{d}z}+z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)\lvert F\rvert^2 = 0.$	A
2	$Y_2^2 = Y_4 + cY_2 + dY_3$	$\left(\frac{d^2}{c^2}-1\right)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \frac{2di}{c^2}\frac{\mathrm{d}F}{\mathrm{d}z} + \left(a-\frac{1}{c^2}+\frac{1}{4}z^2-\rho\lvert F\rvert^2+2G\right)F = 0,$ $G = \frac{d^2\delta}{c^2+d^2}\left(\lvert F\rvert^2\right) - \frac{1}{8c^2}z^2 + k_1z + k_2, \quad k_1, k_2 = \text{constants}.$	B
3	$Y_5^2 = Y_2 + cY_3$	$(c^2-1)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \left(a+\frac{1}{4}z^2-\rho\lvert F\rvert^2+2G\right)F = 0,$ $G = \frac{c^2\delta}{c^2+1}\left(\lvert F\rvert^2\right) - \frac{1}{8}z^2 + k_1z + k_2, \quad k_1, k_2 = \text{constants}.$	C
4	$4Y_6^2 = Y_3$	$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \left(a-\frac{1}{4}z^2-\rho\lvert F\rvert^2+2G\right)F = 0,$ $G = \delta\lvert F\rvert^2 + \frac{1}{8}z^2 + k_1z + k_2, \quad k_1, k_2 = \text{constants}.$	D

where

$$Y_1 = z_1\partial_{z_1} + z_2\partial_{z_2} - w^1\partial_{w^1} - \frac{1}{4}\left(3az_1 + 3\varepsilon z_2 + 8w^2\right)\partial_{w^2}, Y_2 = \partial_{z_1} - \frac{1}{4}a\partial_{w^2},$$

$$Y_3 = \partial_{z_2} - \frac{1}{4}\varepsilon\partial_{w^2},\ Y_4 = -iw^1\partial_{w^1},\ Y_5 = \partial_{w^1},\ Y_6 = i\partial_{w^1}.$$

$$A: w^1 = F(z)z_1^{-(1+ci)},\ w^2 = z_1^{-2}G(z) - \frac{a}{4}z_1 - \frac{\varepsilon}{4}z_2,\ z = z_1^{-1}z_2.$$

$$B: w^1 = F(z)\exp\left(-\frac{iz_1}{c}\right),\ w^2 = G(z) - \frac{1}{4c}(ac+\varepsilon d)z_1,\ z = cz_2 - dz_1.$$

$$C: w^1 = F(z),\ w^2 = G(z) - \frac{1}{4}(a+\varepsilon c)z_1,\ z = z_2 - cz_1.$$

$$D: w^1 = F(z),\ w^2 = G(z) - \frac{\varepsilon}{4}z_2,\ z = z_1.$$

where c, d, k_1 and k_2 are arbitrary constants.

For X^3, X^4 and X^5, we also have optimal systems and the corresponding reductions which are given in **Table 5**, **Table 6**, **Table 7** and **Table 8** respectively.

6. Conclusion

In this paper, the infinite dimensional Lie algebra of 2D-NLS Equations (1) is determined. The optimal system of a sub-algebra $\mathcal{L}^8$ of the infinite dimensional Lie algebra is constructed using method given in [12]-[14]. As a result, the first reductions of the 2D-NLS Equation (1) is presented by infinitesimal invariant method [14] [15]. The corresponding optimal systems of the Lie algebras admitted by the first reduced equations are also constructed. Consequently, the second time reductions classifications of the 2D-NLS Equations (1) are obtained by these optimal systems. The twice reduction procedure shows that the 2D-NLS Equation (1) can be reduced to a group of ordinary differential equations, which is helpful to explicitly solve the 2D-NLS Equations (1).

Table 5. The second reductions of 2D-CNLS (1) with X^3.

No.	Generators in (11)	The second time reductions of 2D-CNLS (1)	Invariance variables
1	$Y_1^3 = Y_1 + cY_4$	$(z^2-1)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + (4+2ci)z\frac{\mathrm{d}F}{\mathrm{d}z} + \left(2-c^2+3ci-\rho\lvert F\rvert^2+2G\right)F = 0,$ $(z^2+1)\frac{\mathrm{d}^2G}{\mathrm{d}z^2} + 6z\frac{\mathrm{d}G}{\mathrm{d}z} + 6G - \delta\left(6+6z\frac{d}{\mathrm{d}z}+z^2\frac{d^2}{\mathrm{d}z^2}\right)\lvert F\rvert^2 = 0.$	A
2	$Y_2^3 = Y_4 + cY_2 + dY_3$	$(d^2-c^2)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + 2i\frac{d}{c}\frac{\mathrm{d}F}{\mathrm{d}z} + \left(\frac{\varepsilon}{2c}z - \frac{1}{c^2} - \rho\lvert F\rvert^2 + 2G\right)F = 0,$ $G = \frac{d^2\delta}{c^2+d^2}\lvert F\rvert^2 + k_1z + k_2,\ c \neq 0, k_1, k_2 = \text{constants}.$	B
3	$Y_5^3 = Y_2 + cY_3$	$(c^2-1)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \left(\frac{\varepsilon}{2}z - \rho\lvert F\rvert^2 + 2G\right)F = 0,\ G = \frac{c^2\delta}{c^2+1}\lvert F\rvert^2 + k_1z + k_2,\quad k_1, k_2 = \text{constants}.$	C
4	$Y_6^3 = Y_3$	$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \left(a\frac{z}{2} - \rho\lvert F\rvert^2 + 2G\right)F = 0,\ G = \delta\lvert F\rvert^2 + k_1z + k_2,\quad k_1, k_2 = \text{constants}.$	D

where

$$Y_1 = z_1\partial_{z_1} + z_2\partial_{z_2} - w^1\partial_{w^1} - \frac{1}{4}\left(3az_1 + 3\varepsilon z_2 + 8w^2\right)\partial_{w^2}, Y_2 = \partial_{z_1} - \frac{1}{4}a\partial_{w^2}, Y_3 = \partial_{z_2} - \frac{1}{4}\varepsilon\partial_{w^2},\ Y_4 = -iw^1\partial_{w^1},\ Y_5 = \partial_{w^1},\ Y_6 = i\partial_{w^1}.$$

$$A: w^1 = F(z)z_1^{-(1+ci)},\ w^2 = z_1^{-2}G(z) - \frac{a}{4}z_1 - \frac{\varepsilon}{4}z_2,\ z = z_1^{-1}z_2.$$

$$B: w^1 = F(z)\exp\left(-\frac{iz_1}{c}\right),\ w^2 = G(z) - \frac{1}{4c}(ac + \varepsilon d)z_1,\ z = cz_2 - dz_1.$$

$$C: w^1 = F(z),\ w^2 = G(z) - \frac{1}{4}(a + \varepsilon c)z_1,\ z = z_2 - cz_1.\ D: w^1 = F(z),\ w^2 = G(z) - \frac{\varepsilon}{4}z_2,\ z = z_1.$$

Table 6. The second reductions of 2D-CNLS (1) with X^4.

No.	Generators in (11)	The second time reductions of 2D-CNLS (1)	Invariance variables
1	$Y_1^4 = Y_1 + cY_4$	$(z^2-1)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + (4+2ci)z\frac{\mathrm{d}F}{\mathrm{d}z} + \left(2-c^2+3ci-\rho\lvert F\rvert^2+2G\right)F = 0,$ $(z^2+1)\frac{\mathrm{d}^2G}{\mathrm{d}z^2} + 6z\frac{\mathrm{d}G}{\mathrm{d}z} + 6G - \delta\left(6+6z\frac{\mathrm{d}}{\mathrm{d}z}+z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)\lvert F\rvert^2 = 0.$	A
2	$Y_2^4 = Y_4 + cY_2 + dY_3$	$(d^2-c^2)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + 2i\frac{d}{c}\frac{\mathrm{d}F}{\mathrm{d}z} - \left(\frac{1}{c^2} + \rho\lvert F\rvert^2 - 2G\right)F = 0,$ $G = \frac{d^2\delta}{c^2+d^2}\lvert F\rvert^2 + k_1z + k_2,\ c \neq 0, k_1, k_2 = \text{constants}.$	B
3	$Y_5^4 = Y_2 + cY_3$	$(c^2-1)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} - \left(\rho\lvert F\rvert^2 - 2G\right)F = 0,\ G = \frac{c^2\delta}{c^2+1}\lvert F\rvert^2 + k_1z + k_2,\quad k_1, k_2 = \text{constants}.$	C
4	$Y_6^4 = Y_3$	$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \left(\varepsilon\frac{z}{2} - \rho\lvert F\rvert^2 + 2G\right)F = 0,\ G = \delta\lvert F\rvert^2 + k_1z + k_2,\quad k_1, k_2 = \text{constants}.$	D

where

$$Y_1 = z_1\partial_{z_1} + z_2\partial_{z_2} - w^1\partial_{w^1} - \frac{1}{4}\left(3\varepsilon z_1 + 8w^2\right)\partial_{w^2},\ Y_2 = \partial_{z_1} - \frac{1}{4}\varepsilon\partial_{w^2}, Y_3 = \partial_{z_2},\ Y_4 = -iw^1\partial_{w^1},\ Y_5 = \partial_{w^1},\ Y_6 = i\partial_{w^1}.$$

$$A: w^1 = F(z)z_1^{-(1+ci)},\ w^2 = z_1^{-2}G(z) - \frac{\varepsilon}{4}z_1,\ z = z_1^{-1}z_2.$$

$$B: w^1 = F(z)\exp\left(-\frac{iz_1}{c}\right),\ w^2 = G(z) - \frac{\varepsilon}{4}z_1,\ z = cz_2 - dz_1.$$

$$C: w^1 = F(z),\ w^2 = G(z) - \frac{\varepsilon}{4}z_1,\ z = z_2 - cz_1.$$

$$D: w^1 = F(z),\ w^2 = G(z),\ z = z_1.$$

Table 7. The second reductions of 2D-CNLS (1) with X^5.

No.	Generators in (11)	The second time reductions of 2D-CNLS (1)	Invariance variables
1	$Y_1^5 = Y_1 + cY_4$	$(z^2-1)\frac{d^2F}{dz^2}+(4+2ci)z\frac{dF}{dz}+(2-c^2+3ci-\rho\|F\|^2+2G)F=0,$ $(z^2+1)\frac{d^2G}{dz^2}+6z\frac{dG}{dz}+6G-\delta\left(6+6z\frac{d}{dz}+z^2\frac{d^2}{dz^2}\right)\|F\|^2=0.$	A
2	$Y_2^5 = Y_4 + cY_2 + dY_3$	$(d^2-c^2)\frac{d^2F}{dz^2}-2i\frac{d}{c}\frac{dF}{dz}+\left(a-\frac{1}{c^2}-\rho\|F\|^2+2G\right)F=0,$ $G=\frac{d^2\delta}{c^2+d^2}\|F\|^2+k_1z+k_2,\ c\neq 0,\ k_1,k_2=\text{constants}.$	B
3	$Y_5^5 = Y_2 + cY_3$	$(c^2-1)\frac{d^2F}{dz^2}+(a-\rho\|F\|^2+2G)F=0,\ G=\frac{c^2\delta}{c^2+1}\|F\|^2+k_1z+k_2,\ k_1,k_2=\text{constants}.$	C
4	$Y_6^5 = Y_3$	$\frac{d^2F}{dz^2}+(a-\rho\|F\|^2+2G)F=0,\ G=\delta\|F\|^2+k_1z+k_2,\ k_1,k_2=\text{constants}.$	D

where

$$Y_1 = z_1\partial_{z_1}+z_2\partial_{z_2}-w^1\partial_{w^1}-(a+2w^2)\partial_{w^2},\ Y_2=\partial_{z_1}, Y_3=\partial_{z_2},\ Y_4=-iw^1\partial_{w^1},\ Y_5=\partial_{w^2},\ Y_6=i\partial_{w^1}.$$

$$A: w^1=F(z)z_1^{-(1+ci)},\ w^2=z_1^{-2}G(z)-\frac{a}{2},\ z=z_1^{-1}z_2.\ B: w^1=F(z)\exp\left(-\frac{iz_1}{c}\right),\ w^2=G(z),\ z=cz_2-dz_1.$$

$$C: w^1=F(z),\ w^2=G(z),\ z=z_2-cz_1.\quad D: w^1=F(z),\ w^2=G(z),\ z=z_1.$$

Table 8. The second reductions of 2D-CNLS (1) with X^1, X^6, X^7, X^8 and X^9.

No.	Generators in (6)	Generators of the first reduced eqs.	The second time reductions of 2D-CNLS (1)	Invariance variables
1	$X^1 = X_4 + aX_8$	$Y_1^1 = Y_1 + cY_2$	$\frac{d^2F}{dz^2}+\frac{ci}{2}\frac{dF}{dz}+\left(\frac{a}{2}+\frac{i}{4}-\frac{c^2}{16}-\rho\|F\|^2+2G\right)F=0,$ $G=\delta\|F\|^2+\frac{1}{32}z^2+k_1z+k_2,\ k_1,k_2=\text{constants}.$	A
2	$X^6 = X_1 + aX_2 + bX_6$	infinite dimensional		
3	$X^7 = X_2 + aX_5$	$Z_1^7 = Z_1$	$i\frac{dF}{dz}+\left(\frac{i}{2z}+\rho\|F\|^2-2G\right)F=0,\ G$ is arbitrary function.	B
4	$X^8 = X_8 + aX_6$	$W_1^8 = W_1$	$\frac{d^2F}{dz^2}-(1-\rho\|F\|^2+2G)F=0,\ G=\delta\|F\|^2+k_1z+k_2,\ k_1,k_2=\text{constants}.$	C
5	$X^9 = X_6$	$V_1^9 = V_3 + V_1$	$\frac{d^2F}{dz^2}-\left(\frac{1}{4}z^2+\rho\|F\|^2-2G\right)F=0,\ G=\delta\|F\|^2+k_1z+k_2,\ k_1,k_2=\text{constants}.$	D
		$V_2^9 = V_3 - V_1$	$\frac{d^2F}{dz^2}+\left(\frac{1}{4}z^2-\rho\|F\|^2+2G\right)F=0,\ G=\delta\|F\|^2+k_1z+k_2,\ k_1,k_2=\text{constants}.$	E
		$V_3^9 = V_2$	$\frac{d^2F}{dz^2}+\frac{i}{2}z\frac{dF}{dz}-\rho\|F\|^2F+2FG=0,\ G=\delta\|F\|^2+k_1z+k_2,\ k_1,k_2=\text{constants}.$	F
		$V_4^9 = V_1$	$\frac{d^2F}{dz^2}-\rho\|F\|^2F+2FG=0,\ G=\delta\|F\|^2+k_1z+k_2,\ k_1,k_2=\text{constants}.$	G

where

$$Y_1 = \partial_{z_1} - \frac{i}{4} z_1 w^1 \partial_{w^1} - \frac{1}{16} z_1 \partial_{w^2},\ Y_2 = -i w^1 \partial_{w^1},\ Z_1 = \partial_{z_2},$$

$$W_1 = z_1^2 \partial_{z_1} + z_1 z_2 \partial_{z_2} - \left(z_1 + \frac{i}{4} z_2^2 \right) w^1 \partial_{w^1} - 2 z_1 w^2 \partial_{w^2},$$

$$V_1 = z_1^2 \partial_{z_1} + z_1 z_2 \partial_{z_2} - \left(z_1 + \frac{i}{4} z_2^2 \right) w^1 \partial_{w^1} - 2 z_1 w^2 \partial_{w^2},$$

$$V_2 = z_1 \partial_{z_1} + \frac{1}{2} z_2 \partial_{z_2} - \frac{1}{2} w^1 \partial_{w^1} - w^2 \partial_{w^2},\ V_3 = \partial_{z_1} - \frac{i}{4 z_1^2} \partial_{w^2}.$$

$$A: w^1 = F(z) \exp\left(-i \left(\frac{z_1^2}{8} + \frac{c z_1}{4} \right) \right),\ w^2 = G(z) - \frac{1}{32} z_1^2,\ z = z_1.$$

$$B: w^1 = F(z),\ w^2 = G(z),\ z = z_1.$$

$$C: w^1 = z_1^{-1} F(z) \exp\left(-\frac{i}{4} z_1 z_2^2 \right),\ w^2 = z_1^{-2} G(z),\ z = z_1^{-1} z_2.$$

$$D: w^1 = \left(1 + z_1^2\right)^{-\frac{1}{2}} F(z) \exp\left(-\frac{i z_1 z_2^2}{4\left(1 + z_1^2\right)} \right),\ w^2 = \left(1 + z_1^2\right)^{-1} G(z) + \frac{i}{4} z_1^{-1} \left(1 + z_1^2\right)^{-1},\ z = \left(1 + z_1^2\right)^{-\frac{1}{2}} z_2.$$

$$E: w^1 = \left(z_1^2 - 1\right)^{-\frac{1}{2}} F(z) \exp\left(-\frac{i z_1 z_2^2}{4\left(z_1^2 - 1\right)} \right),\ w^2 = \left(z_1^2 - 1\right)^{-1} G(z) - \frac{i}{4} z_1^{-1} \left(z_1^2 - 1\right)^{-1},\ z = \left(z_1^2 - 1\right)^{-\frac{1}{2}} z_2.$$

$$F: w^1 = z_1^{-\frac{1}{2}} F(z),\ w^2 = z_1^{-1} G(z),\ z = z_1^{-\frac{1}{2}} z_2.$$

$$F: w^1 = z_1^{-1} F(z) \exp\left(-\frac{i z_2^2}{4 z_1} \right),\ w^2 = z_1^{-2} G(z),\ z = z_1^{-1} z_2.$$

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