

# Lie Symmetries, 1-Dimensional Optimal System and Optimal Reductions of (1 + 2)-Dimensional Nonlinear Schrödinger Equation

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## Abstract

For a class of (1 + 2)-dimensional nonlinear Schrödinger equations, classical symmetry algebra is found and 1-dimensional optimal system, up to conjugacy, is constructed. Its symmetry reductions are performed for each class, and some examples of exact invariant solutions are given.

## Keywords

Nonlinear Schrödinger Equation, Classical Symmetry, Optimal System, Symmetry Reductions, Invariant Solutions

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## 1. Introduction

Inspired by Galois' theory, Sophus Lie developed an analogous theory of symmetry for differential equations. Lie's theory led to an algorithmic way to find special explicit solutions to differential equation with its admitted symmetry. These special solutions are called group invariant solutions and they constitute practically every known explicit solution to the systems of non-linear partial differential equations (PDEs) arising in mathematical physics, differential geometry and other areas. These group-invariant solutions are found by solving a reduced system of differential equations involving fewer independent variables than the original system. The reduced system is obtained through reducing the underline system by using its admitted Lie symmetry. Hence, it is critical to solve the invariant solutions that one finds all the symmetries of the original system. On the other

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hand, the algebra properties of the Lie algebra corresponding to the Lie symmetry group of a PDE show some essential nature of the solutions of the PDE. For example, from the optimal structure of the Lie algebra, one can solve optimal set of invariant solution to the PDE, which is critical to distinguish different classes of invariant solutions of the PDE.

Generally, an optimal system of  $s$ -parameter subgroups of a Lie algebra is a list of conjugacyin equivalent  $s$ -parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Correspondingly, a list of  $s$ -parameter subalgebras forms an optimal system if every  $s$ -parameter subalgebra of the Lie algebra is equivalent to unique member of the list under some element of the adjoint representation set. The optimal system of a Lie algebra provide the optimal classification of various different dimensional subalgebras. For a PDE, the optimal system of the Lie algebra admitted by the PDE provides the optimal classification of invariant solutions of the PDE under its Lie symmetry group.

In this paper, we plan to consider the  $(1 + 2)$ -dimensional nonlinear Schrödinger equation (NLSE) with cubic nonlinearity [1]

$$u_{xx} + u_{yy} + r|u|^2 u = iu_t \quad (1)$$

where  $u = u(t, x, y)$  is a complex function and  $r$  is a non-zero real parameter. This equation occurs in various chapters of physics, including nonlinear optics, superconductivity, quantum mechanics and plasma physics [2]. The cubic nonlinearity is the most common nonlinearity in applications. It arises as a simplified model for studying Bose-Einstein condensates, Kerr media in nonlinear optics freak waves in the ocean (see [3] and references therein).

For the NLSE (1), many researchers' work mostly concentrated on obtaining exact or approximate solutions. The classical symmetry reductions and some similarity solutions of the (1) are given in [1] by Lie symmetry method. Its some approximate solutions are obtained in [3] with applying the differential transform method. The 1-soliton solution of (1) has been obtained in [4]. Reliable analysis for the  $(1 + 1)$ -dimensional NLSE with power law nonlinearity has been investigated by Wazwaz in [5]. Some soliton and periodic solutions of  $(1 + 2)$ -dimensional NLSE (1) are constructed in [6]. However, the algebra properties of the Lie algebra admitted by the NLSE (1) such as optimal system of the Lie algebra, have not been studied so far.

We will construct optimal systems of the Lie algebra of the both NLSE (1) and its first reduction systems under their admitted Lie symmetry groups respectively. Furthermore, we derive twice reductions of NLSE (1) with respect to the obtained optimal system. Consequently, we show the NLSE (1) can be reduced to ordinary differential equations which yields exact solutions of the equation. The outline of the article is following. In Section 2, the 9-dimensional classical Lie algebra  $L^9$  of (1) is given by characteristic set algorithm proposed in In Section 3, 1-dimensional optimal system of the Lie algebra derived by using the algorithm given in [7]. In Section 4, the first reductions with respect to the obtained optimal system of the equation are studied by means of Lie's method of infinitesimal transformation. In Section 5, the optimal system of 6-dimensional Lie algebra  $L^6$  of a first reduced equations of the NLSE (1) obtained in Section 4 are given. Consequently, we obtain a twice reduction of the original Equation (1). In Section 6, some exact invariant solutions of the NLSE (1) are obtained by combining the twice reductions procedure using invariant method given in [8]. Concluding remarks are given in Section 7.

## 2. Lie Symmetry of the Equation (1)

In this section we present the generators of the Lie algebra corresponding to classical symmetries of the NLSE (1). Let the one parameter Lie group of infinitesimal transformations in  $(t, x, y, u)$  given by

$$\begin{aligned} t^* &= t + \varepsilon\tau(t, x, y, u) + O(\varepsilon^2) \\ x^* &= x + \varepsilon\xi(t, x, y, u) + O(\varepsilon^2) \\ y^* &= y + \varepsilon\zeta(t, x, y, u) + O(\varepsilon^2) \\ u^* &= u + \varepsilon\eta(t, x, y, u) + O(\varepsilon^2) \end{aligned} \quad (2)$$

where  $\varepsilon$  is the group parameter. The corresponding generator of the Lie algebra is

$$X = \tau(t, x, y, u)\partial_t + \xi(t, x, y, u)\partial_x + \zeta(t, x, y, u)\partial_y + \eta(t, x, y, u)\partial_u \tag{3}$$

For fitting the algorithm and software in [9], transform the NLSE (1) to real case by transforming  $u \rightarrow u + iv$ , where  $u$  and  $v$  are real function. For this transformed system, using characteristic set algorithm given in [9], we find the simplified determining equations for generator

$$X = \tau(t, x, y, u)\partial_t + \xi(t, x, y, u)\partial_x + \zeta(t, x, y, u)\partial_y + \eta(t, x, y, u)\partial_u$$

are as follows

$$\begin{aligned} u\varphi_t - v\eta_t &= 0, \quad u\varphi + (v^2 + u^2)\eta_v - v\eta = 0, \quad u\eta + v\varphi - (v^2 + u^2)\eta_u = 0, \\ u\eta + v\varphi - (v^2 + u^2)\varphi_v &= 0, \quad u\varphi - v\eta - (v^2 + u^2)\varphi_u = 0, \quad \xi_v = 0, \quad \xi_u = 0, \\ u\xi_t + 2\varphi_x &= 0, \quad 2\eta_x - v\xi_t = 0, \quad \xi_u = 0, \quad \xi_{yt} = 0, \quad \xi_{yy} = 0, \quad u\zeta_t + 2\varphi_y = 0, \\ u\eta + v\varphi + (v^2 + u^2)\xi_x &= 0, \quad \zeta_v = 0, \quad \zeta_u = 0, \quad 2\eta_y - v\zeta_t = 0, \quad \zeta_x + \xi_y = 0, \\ u\eta + v\varphi + (v^2 + u^2)\zeta_y &= 0, \quad \tau_v = 0, \quad \tau_u = 0, \quad 2(u\eta + v\varphi) + (v^2 + u^2)\tau_t = 0, \\ \tau_x &= 0, \quad \tau_y = 0, \quad \eta_u = 0, \quad \zeta_u = 0. \end{aligned}$$

Solving the above system of PDEs, we obtain:

$$\begin{aligned} \tau(t, x, y, u, v) &= c_8 t^2 + 2c_4 t + c_3 \\ \xi(t, x, y, u, v) &= c_4 x + c_5 y + (c_6 + c_8 x)t + c_1 \\ \zeta(t, x, y, u, v) &= c_4 y - c_5 x + (c_7 + c_8 y)t + c_2 \\ \eta(t, x, y, u, v) &= \frac{1}{4}c_8(x^2 + y^2)v + \frac{1}{2}(c_6 x + c_7 y)v - (c_4 + c_8 t)u + \frac{1}{2}c_9 v \\ \varphi(t, x, y, u, v) &= -\frac{1}{4}c_8(x^2 + y^2)u - \frac{1}{2}(c_6 x + c_7 y)u - (c_4 + c_8 t)v - \frac{1}{2}c_9 u \end{aligned}$$

where  $c_i (i = 1, 2, \dots, 9)$  are arbitrary constants.

Therefore the infinitesimal generators of the transformed real form equations are given by

$$\begin{aligned} X_1 &= \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_t, \quad X_4 = x\partial_x + y\partial_y + 2t\partial_t - u\partial_u - v\partial_v, \\ X_5 &= y\partial_x - x\partial_y, \quad X_6 = t\partial_x + \frac{1}{2}vx\partial_u - \frac{1}{2}ux\partial_v, \quad X_7 = t\partial_y + \frac{1}{2}vy\partial_u - \frac{1}{2}uy\partial_v, \\ X_8 &= tx\partial_x + ty\partial_y + t^2\partial_t + \left[\frac{1}{4}v(x^2 + y^2) - ut\right]\partial_u - \left[\frac{1}{4}u(x^2 + y^2) + vt\right]\partial_v, \\ X_9 &= \frac{1}{2}v\partial_u - \frac{1}{2}u\partial_v. \end{aligned}$$

These are equivalent to generators for Lie algebra of NLSE (1) obtained by transforming  $u + iv \rightarrow u$ ,  $\partial u \rightarrow \partial u$ ,  $\partial v \rightarrow i\partial u$ . It spans 9-dimensional Lie algebra  $L^9$  of admitted by NLSE (1). The commutation relations of (4) are given in the following Table 1. It is fundamental to constructing the optimal system of  $L^9$  spanned by (4).

$$\begin{aligned} X_1 &= \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_t, \quad X_4 = x\partial_x + y\partial_y + 2t\partial_t - u\partial_u, \\ X_5 &= y\partial_x - x\partial_y, \quad X_6 = t\partial_x - \frac{i}{2}xu\partial_u, \quad X_7 = t\partial_y - \frac{i}{2}yu\partial_u, \\ X_8 &= tx\partial_x + ty\partial_y + t^2\partial_t - \left[\frac{i}{4}(x^2 + y^2) + t\right]u\partial_u, \quad X_9 = -\frac{i}{2}u\partial_u. \end{aligned} \tag{4}$$

The commutation relations of the basis is given in the following **Table 1**.

### 3. 1-Dimensional Optimal System of $L^9$

In this section we present a 1-dimensional optimal system of the algebra  $L^9$  obtained in Section 2 spanned by (4).

The problem of finding an optimal system of subalgebra of a Lie algebra is a subalgebra classification problem. Unfortunately, this problem can still be quite complicated and has no efficient method to use. For 1-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each 1-dimensional subalgebra is determined by a nonzero vector in the algebra. Essentially, this problem is attacked by the naive approach of taking a general element  $X$  in the algebra and subjecting it to various adjoint transformations so as to “simplify” it as much as possible.

Now, in our case, given a nonzero vector  $X = k^1 X_1 + k^2 X_2 + \dots + k^9 X_9$  in algebra spanned by (4). The vector corresponds to the vector  $k = (k^1, k^2, \dots, k^9)$ . Our task is to simplify as many of the coefficients  $k^i$  being zero as possible through application of adjoint maps on  $X$  and find its equivalent one.

Here we use the matrix method given in [7] to determine an optimal system of 1-dimensional subalgebra of  $L^9$ . The algorithm is given by following steps.

**Step 1.** Determine structure constants matrix  $C(j)$  by formula

$$(C(j))_i^k = c_{ij}^k$$

where  $c_{ij}^k$  is structure constants given in commutator **Table 1**.

**Step 2.** Calculate adjoint matrix  $A(j, \varepsilon)$  by definition.

$$A(j, \varepsilon) = \exp(\varepsilon C(j)) = \sum_{n=0}^{\infty} C(j)^n \frac{\varepsilon^n}{n!}$$

**Step 3.** Simplify the vector  $k = (k^1, k^2, \dots, k^9)$  as much as possible by applying  $A(j, \varepsilon)$  on  $k$  and obtain equivalent one of  $X$ .

In the Following, we construct the optimal system of  $L^9$  by following above the steps.

As the commutator table 1 showing, we have the structure constants matrices for the  $L^9$  areas follows

**Table 1.** Commutation table for the generators of the Lie algebra  $L^9$ .

| [ ]   | $X_1$      | $X_2$       | $X_3$    | $X_4$    | $X_5$   | $X_6$  | $X_7$   | $X_8$ | $X_9$ |
|-------|------------|-------------|----------|----------|---------|--------|---------|-------|-------|
| $X_1$ | 000        | $X_1 - X_2$ | $X_3$    | $X_4$    | $X_5$   | $X_6$  | $X_7$   | $X_8$ | $X_9$ |
| $X_2$ | 000        | $X_2$       | $X_1$    | $X_3$    | $X_4$   | $X_5$  | $X_6$   | $X_7$ | $X_8$ |
| $X_3$ | 000        | $2 X_3$     | $X_1$    | $X_2$    | $X_4$   | $X_5$  | $X_6$   | $X_7$ | $X_8$ |
| $X_4$ | $-X_1$     | $-X_2$      | $-2 X_3$ | $00$     | $X_6$   | $X_7$  | $2 X_8$ | $0$   | $0$   |
| $X_5$ | $X_2$      | $-X_1$      | $000$    | $X_7$    | $-X_6$  | $00$   | $0$     | $0$   | $0$   |
| $X_6$ | $-X_9$     | $0$         | $-X_1$   | $-X_6$   | $-X_7$  | $0000$ | $0$     | $0$   | $0$   |
| $X_7$ | $0$        | $-X_9$      | $-X_2$   | $-X_7$   | $X_6$   | $0000$ | $0$     | $0$   | $0$   |
| $X_8$ | $-X_6$     | $-X_7$      | $-X_4$   | $-2 X_8$ | $00000$ | $0$    | $0$     | $0$   | $0$   |
| $X_9$ | $00000000$ | $0$         | $0$      | $0$      | $0$     | $0$    | $0$     | $0$   | $0$   |





$$\begin{aligned}
A(6, \varepsilon) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\varepsilon^2 \\ 0 & 0 & 0 & 1 & 0 & \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & A(7, \varepsilon) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon \\ 0 & \varepsilon & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\varepsilon^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
A(8, \varepsilon) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 1 & \varepsilon & 0 & 0 & 0 & \varepsilon^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & A(9, \varepsilon) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Hence, the vector  $k = (k^1, k^2, \dots, k^9)$  is transformed by the matrices  $A(j, \varepsilon)$  is as follows

$$\begin{aligned}
kA(1, \varepsilon) &= \left( k^1 - \varepsilon k^4, k^2 + \varepsilon k^5, k^3, k^4, k^5, k^6 - \varepsilon k^8, k^7, k^8, k^9 - \varepsilon k^6 + \frac{1}{2}\varepsilon^2 k^8 \right), \\
kA(2, \varepsilon) &= \left( k^1 - \varepsilon k^5, k^2 - \varepsilon k^4, k^3, k^4, k^5, k^6, k^7 - \varepsilon k^8, k^8, k^9 - \varepsilon k^7 + \frac{1}{2}\varepsilon^2 k^8 \right), \\
kA(3, \varepsilon) &= \left( k^1 - \varepsilon k^6, k^2 - \varepsilon k^7, k^3 - 2\varepsilon k^4 + \varepsilon^2 k^8, k^4 - \varepsilon k^8, k^5, k^6, k^7, k^8, k^9 \right), \\
kA(4, \varepsilon) &= \left( e^\varepsilon k^1, e^\varepsilon k^2, e^{2\varepsilon} k^3, k^4, k^5, e^{-\varepsilon} k^6, e^{-\varepsilon} k^7, e^{-2\varepsilon} k^8, k^9 \right), \\
kA(5, \varepsilon) &= \left( k^1 \cos \varepsilon + k^2 \sin \varepsilon, -k^1 \sin \varepsilon + k^2 \cos \varepsilon, k^3, k^4, k^5, k^6 \cos \varepsilon + k^7 \sin \varepsilon, -k^6 \sin \varepsilon + k^7 \cos \varepsilon, k^8, k^9 \right) \\
kA(6, \varepsilon) &= \left( k^1 + \varepsilon k^3, k^2, k^3, k^4, k^5, k^6 + \varepsilon k^4, k^7 + \varepsilon k^5, k^8, k^9 + \varepsilon k^1 + \frac{1}{2}\varepsilon^2 k^3 \right), \\
kA(7, \varepsilon) &= \left( k^1, k^2 + \varepsilon k^3, k^3, k^4, k^5, k^6 - \varepsilon k^5, k^7 + \varepsilon k^4, k^8, k^9 + \varepsilon k^2 + \frac{1}{2}\varepsilon^2 k^3 \right), \\
kA(8, \varepsilon) &= \left( k^1, k^2, k^3, k^4 + \varepsilon k^3, k^5, k^6 + \varepsilon k^1, k^7 + \varepsilon k^2, k^8 + 2\varepsilon k^4 + \varepsilon^2 k^3, k^9 \right), \\
kA(9, \varepsilon) &= \left( k^1, k^2, k^3, k^4, k^5, k^6, k^7, k^8, k^9 \right).
\end{aligned}$$

In order to obtain the optimal system, we solve  $k$  as simple as possible from the above transformations.

If  $k^5 \neq 0$ , by setting  $\varepsilon_2 = \frac{k^1}{k^5}$ ,  $\varepsilon_1 = -\frac{k^2}{k^5}$ ,  $\varepsilon_7 = -\frac{k^6}{k^5}$ ,  $\varepsilon_6 = -\frac{k^7}{k^5}$ ,  $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_8 = 0$ , the coefficients  $k^1, k^2, k^6$  and  $k^7$  are vanished. By scaling, we can suppose  $k^5 = 1$ . Hence  $X$  is equivalent to  $X_5 + aX_3 + bX_4 + cX_8 + qX_9$ , where we use  $a, b, c$  and  $q$  to denote the arbitrary constants.

If  $k^5 = 0$  and  $k^4 \neq 0$ , by setting  $\varepsilon_1 = \frac{k^1}{k^4}$ ,  $\varepsilon_2 = \frac{k^2}{k^4}$ ,  $\varepsilon_6 = -\frac{k^6}{k^4}$ ,  $\varepsilon_7 = -\frac{k^7}{k^4}$ ,  $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_8 = 0$ , coefficients  $k^1, k^2, k^6$  and  $k^7$  are vanished. By scaling, we can suppose  $k^4 = 1$ . Hence  $X$  is equivalent to  $X_4 + aX_3 + bX_8 + qX_9$ .

If  $k^4 = k^5 = 0$  and  $k^8 \neq 0$ , by setting  $\varepsilon_1 = \frac{k^6}{k^8}$ ,  $\varepsilon_2 = \frac{k^7}{k^8}$ ,  $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \varepsilon_8 = 0$ , coefficients  $k^6$  and  $k^7$  are vanished. By scaling, we can suppose  $k^8 = 1$ . Hence  $X$  is equivalent to  $X_8 + aX_1 + bX_2 + cX_3 + qX_9$ .

If  $k^4 = k^5 = k^8 = 0$  and  $k^3 \neq 0$ , by setting  $\varepsilon_6 = -\frac{k^1}{k^3}$ ,  $\varepsilon_7 = -\frac{k^2}{k^3}$ ,  $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_8 = 0$ , coefficients  $k^1$  and  $k^2$  are vanished. By scaling, we can suppose  $k^3 = 1$ . Hence  $X$  is equivalent to  $X_3 + aX_6 + bX_7 + qX_9$ .

If  $k^3 = k^4 = k^5 = k^8 = 0$  and  $k^6 \neq 0$ , by setting  $\varepsilon_3 = -\frac{k^1}{k^6}$ ,  $\varepsilon_5 = \arctan \frac{k^7}{k^6}$ ,  $\varepsilon_1 = \arctan \frac{k^9}{k^6}$ ,  $\varepsilon_2 = \varepsilon_4 = \varepsilon_6 = \varepsilon_7 = \varepsilon_8 = 0$ , coefficients  $k^1, k^7, k^9$  are vanished. By scaling, we can suppose  $k^6 = 1$ . Hence  $X$  is equivalent to  $X_6 + aX_2$ .

If  $k^3 = k^4 = k^5 = k^6 = k^8 = 0$  and  $k^7 \neq 0$ , by setting  $\varepsilon_3 = \frac{k^2}{k^7}$ ,  $\varepsilon_2 = \frac{k^9}{k^7}$ ,  $\varepsilon_1 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \varepsilon_8 = 0$ , coefficients  $k^2, k^9$  are vanished. By scaling, we can suppose  $k^7 = 1$ . Hence  $X$  is equivalent to  $X_7 + aX_1$ .

If  $k^3 = k^4 = k^5 = k^6 = k^7 = k^8 = 0$  and  $k^2 \neq 0$  by setting  $\varepsilon_5 = \arctan -\frac{k^1}{k^2}$ ,  $\varepsilon_7 = -\frac{k^9}{k^2}$ ,  $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_6 = \varepsilon_8 = 0$ , coefficients  $k^1, k^9$  are vanished. By scaling, we can suppose  $k^2 = 1$ . Hence  $X$  is equivalent to  $X_2$ .

If  $k^2 = k^3 = k^4 = k^5 = k^6 = k^7 = k^8 = 0$  and  $k^9 \neq 0$ , by scaling, we can suppose  $k^9 = 1$ . Hence  $X$  is equivalent to  $X_9 + aX_1$ .

If  $k^2 = k^3 = k^4 = k^5 = k^6 = k^7 = k^8 = k^9 = 0$ , by scaling, we can suppose  $k^1 = 1$ . Hence  $X$  is equivalent to  $X_1$ .

Therefore the set of generators

$$\begin{aligned} X^1 &= X_1, \quad X^2 = X_2, \quad X^3 = X_3 + aX_6 + bX_7 + qX_9, \quad X^4 = X_4 + aX_3 + bX_8 + qX_9, \\ X^5 &= X_5 + aX_3 + bX_4 + cX_8 + qX_9, \quad X^6 = X_6 + aX_2, \quad X^7 = X_7 + aX_1, \\ X^8 &= X_8 + aX_1 + bX_2 + cX_3 + qX_9, \quad X^9 = X_9 + aX_1. \end{aligned} \tag{5}$$

is consisted of the optimal system of 1-dimensional subalgebra of the algebra  $L^9$  of the NLSE (1), where  $a, b, c$  and  $q$  are arbitrary constants.

#### 4. Reductions of NLSE (1) with (5)

In this section we give a classification of symmetry reductions of NLSE (1) by using (5).

The reduction in this section require lengthy computations and do not follow a standard algorithmic procedure, so it would be difficult to reproduce them using software. We will introduce the details for the case  $X = X_3 + aX_6 + bX_7 + qX_9$  and the remaining ones can be reproduced in a similar manner.

The differential invariants (and hence the similarity variables) for the generator can be determined by solving the characteristic system

$$\frac{dt}{1} = \frac{dx}{at} = \frac{dy}{bt} = -\frac{du}{\frac{i}{2}(ax + by + q)u} \tag{6}$$



Solving this system, one obtains invariant (similarity) variables as follows

$$y^1 = x - \frac{1}{2}at^2,$$

$$y^2 = y - \frac{1}{2}bt^2,$$

$$v = e^{-\frac{i}{6}(a^2+b^2)t^3 - \frac{i}{2}(ax+by+q)t} u.$$

Hence let

$$u = e^{\frac{i}{6}(a^2+b^2)t^3 - \frac{i}{2}(ax+by+q)t} V(y^1, y^2) \tag{7}$$

and substitute (7) into the Equation (1) which yields the first reduced system of the NLSE (1)

$$V_{y^1 y^1} + V_{y^2 y^2} + r|V|^2 V - \frac{1}{2}(ay^1 + by^2 + q)V = 0 \tag{8}$$

In the same manner, we can obtain the additional reductions of (1) with using other subalgebras in (5) which are listed in **Table 2**. The last column shows the cases of similarity variables. Here

$$A : u = V(y^1, y^2); y^1 = t, y^2 = y;$$

$$B : u = e^{\frac{i}{6}(a^2+b^2)t^3 - \frac{i}{2}(ax+by+q)t} V(y^1, y^2); y^1 = x - \frac{1}{2}at^2, y^2 = y - \frac{1}{2}bt^2;$$

$$C : u = \frac{1}{\sqrt{bt^2 + 2t + a}} e^{-\frac{i}{4}bt \left[ \frac{x^2}{bt^2 + 2t + a} + \frac{(bt - cy)^2}{c^2(t^2 + c)} \right] + \frac{2q \arctan \frac{1+bt}{\sqrt{ab-1}}}{\sqrt{ab-1}}} V(y^1, y^2); y^1 = \frac{x}{\sqrt{bt^2 + 2t + a}}, y^2 = \frac{cy - bt}{c\sqrt{t^2 + c}};$$

**Table 2.** The first reductions of Equation (1) by optimal system (5).

| Generator                               | The rst reduced equations  | Cases |
|---|--|-------|
| $X^1 = X_1, X^2 = X_2$                  | $V_{y^2 y^2} - iV_{y^1} + r V ^2 V = 0$  | A     |
| $X^3 = X_3 + aX_6 + bX_7 + qX_9$        | $V_{y^1 y^1} + V_{y^2 y^2} + r V ^2 V - \frac{1}{2}(ay^1 + by^2 + q)V = 0$   | B     |
| $X^4 = X_4 + aX_3 + bX_8 + qX_9$        | $V_{y^1 y^1} + V_{y^2 y^2} + i(y^1 V_{y^1} + y^2 V_{y^2}) + [4i - 2q - ab((y^1)^2 + (y^2)^2)]V + r V ^2 V = 0$   | C     |
| $X^5 = X_5 + aX_3 + bX_4 + cX_8 + qX_9$ | $V_{y^1 y^1} + V_{y^2 y^2} + i(by^1 + y^2)V_{y^1} + i(-y^1 + by^2)V_{y^2} + \frac{1}{4}[2i(2b + iq) - ac((y^1)^2 + (y^2)^2)]V + \frac{1}{4}r V ^2 V = 0$ | D     |
| $X^6 = X_6 + aX_2, X^7 = X_7 + aX_1$    | $2[a^2 + (y^1)^2]V_{y^2 y^2} - 2i(y^1)^2 V_{y^1} + 2r(y^1)^2  V ^2 V - iy^1 V = 0$   | E     |
| $X^8 = X_8 + aX_1 + bX_2 + cX_3 + qX_9$ | $V_{y^1 y^1} + V_{y^2 y^2} + \frac{1}{4c}[a^2 + b^2 - 2cq + c^2((y^1)^2 + (y^2)^2)]V + r V ^2 V = 0$   | F     |
| $X^9 = X_9 + aX_1$                      | $V_{y^2 y^2} - iV_{y^1} - \frac{1}{4a^2}V + r V ^2 V = 0$  | G     |

$$\begin{aligned}
 D : u &= e^{-\frac{ict(x^2+y^2)}{4(a+2bt+ct^2)} + \frac{2q \arctan \frac{b+ct}{\sqrt{ac-b^2}}}{4\sqrt{ac-b^2}}} V(y^1, y^2); \\
 y^1 &= \frac{x \cos \frac{\arctan \frac{b+ct}{\sqrt{ac-b^2}}}{\sqrt{ac-b^2}} - y \sin \frac{\arctan \frac{b+ct}{\sqrt{ac-b^2}}}{\sqrt{ac-b^2}}}{\sqrt{a+2bt+ct^2}}, \\
 y^2 &= \frac{y \cos \frac{\arctan \frac{b+ct}{\sqrt{ac-b^2}}}{\sqrt{ac-b^2}} + x \sin \frac{\arctan \frac{b+ct}{\sqrt{ac-b^2}}}{\sqrt{ac-b^2}}}{\sqrt{a+2bt+ct^2}}; \\
 E : u &= e^{-\frac{ix^2}{4t}} V(y^1, y^2); y^1 = t, y^2 = y - \frac{ax}{t}; \\
 F : u &= \frac{1}{\sqrt{t^2+c}} e^{\frac{i[(a^2+b^2)t-2c(ax+by)-ct(x^2+y^2)]}{4c(t^2+c)} + \frac{i(a^2+b^2-2cq) \arctan \frac{t}{\sqrt{c}}}{4c^2}} V(y^1, y^2); y^1 = \frac{cx-at}{c\sqrt{t^2+c}}, y^2 = \frac{cy-bt}{c\sqrt{t^2+c}}; \\
 G : u &= e^{-\frac{ix}{2a}} V(y^1, y^2); y^1 = t, y^2 = y.
 \end{aligned}$$

### 5. Reductions of NLSE (1) with (5)

The equations obtained in Table 2 can be reduced further in the similar way in last section, consequently, we obtain second time reductions of NLSE (1). We take the first equation (the reduced one of (1) by  $X^1$  or  $X^2$ ) in Table 2

$$V_{y^2 y^2} - iV_{y^1} + r|V|^2 V = 0 \tag{9}$$

as example. This is a nonlinear “1 + 1” Schrödinger equation.

#### 5.1. Lie Symmetry of the Equation (9)

In this section we present all symmetries of Equation (9). To obtain the Lie group symmetries of Equation (9), we consider the one parameter Lie group of infinitesimal transformations in  $(y^1, y^2, v)$  given by

$$\begin{aligned}
 (y^1)^* &= y^1 + \varepsilon \tau(y^1, y^2, V) + O(\varepsilon^2) \\
 (y^2)^* &= y^2 + \varepsilon \xi(y^1, y^2, V) + O(\varepsilon^2) \\
 V^* &= V + \varepsilon \eta(y^1, y^2, V) + O(\varepsilon^2)
 \end{aligned}$$

where  $\varepsilon$  is the group parameter. The corresponding generator of the Lie algebra of the group symmetry is

$$Y = \tau(y^1, y^2, V) \partial_{y^1} + \xi(y^1, y^2, V) \partial_{y^2} + \eta(y^1, y^2, V) \partial_V$$

For fitting the algorithm and software in [9], we transform the (9) to real case by transformation  $V \rightarrow V + iV'$  where  $V$  and  $V'$  are real function. For this transformed system, we find the generator

$$Y = \tau(y^1, y^2, V, V') \partial_{y^1} + \xi(y^1, y^2, V, V') \partial_{y^2} + \eta(y^1, y^2, V, V') \partial_V + \varphi(y^1, y^2, V, V') \partial_{V'}$$

for its Lie symmetry group. The simplified determining equations are

$$\begin{aligned} \eta_{y^1} = 0, \quad \varphi_{y^1} = 0, \quad -V'\eta + V\varphi + (V'^2 + V^2)\eta_{V'} = 0, \quad V\eta + V'\varphi - (V'^2 + V^2)\eta_V = 0, \quad V\xi_{y^1} + 2\varphi_{y^2} = 0, \\ V\eta + V'\varphi - (V'^2 + V^2)\varphi_{V'} = 0, \quad V'\eta - V\varphi + (V'^2 + V^2)\varphi_V = 0, \quad \tau_{V'} = 0, \quad \tau_V = 0, \quad \tau_{y^2} = 0, \quad \xi_{y^1 y^1} = 0, \\ 2(V\eta + V'\varphi) + (V'^2 + V^2)\tau_{y^1} = 0, \quad \xi_V = 0, \quad V\eta + V'\varphi + (V'^2 + V^2)\xi_{y^2} = 0, \quad 2\eta_{y^2} - V'\xi_{y^1} = 0. \end{aligned}$$

After solving the above system of PDEs, we have

$$\begin{aligned} \tau(y^1, y^2, V, V') &= c_3(y^1)^2 + 2c_4 y^1 + c_1, \\ \xi(y^1, y^2, V, V') &= c_3 y^1 y^2 + c_5 y^1 + c_4 y^2 + c_2, \\ \eta(y^1, y^2, V, V') &= \frac{1}{4}c_3(y^2)^2 V' + \frac{1}{2}c_5 y^2 V' + \frac{i}{2}c_6 V, \\ \varphi(y^1, y^2, V, V') &= -\frac{1}{4}c_3(y^2)^2 V - \frac{1}{2}c_5 y^2 V + \frac{i}{2}c_6 V'. \end{aligned}$$

Here  $c_i (i=1, 2, \dots, 6)$  are constants.

Therefore the symmetry algebra generators are

$$\begin{aligned} Y_1 = \partial_{y^1}, \quad Y_2 = \partial_{y^2}, \quad Y_3 = (y^1)^2 \partial_{y^1} + y^1 y^2 \partial_{y^2} + \frac{1}{4}(y^2)^2 V' \partial_V - \frac{1}{4}(y^2)^2 V \partial_{V'}, \\ Y_4 = 2y^1 \partial_{y^1} + y^2 \partial_{y^2}, \quad Y_5 = y^1 \partial_{y^2} + \frac{1}{2}y^2 V' \partial_V, \quad Y_6 = \frac{i}{2}V \partial_V + \frac{i}{2}V' \partial_{V'}. \end{aligned}$$

By transformation  $V + iV' \rightarrow V$ ,  $\partial_V \rightarrow \partial_V$ ,  $\partial_{V'} \rightarrow i\partial_V$ , the 6-dimensional Lie algebra  $L^6$  of (9) is spanned by the set of generators

$$\begin{aligned} Y_1 = \partial_{y^1}, \quad Y_2 = \partial_{y^2}, \quad Y_3 = (y^1)^2 \partial_{y^1} + y^1 y^2 \partial_{y^2} - \frac{1}{4}(y^2)^2 V \partial_V, \\ Y_4 = 2y^1 \partial_{y^1} + y^2 \partial_{y^2}, \quad Y_5 = y^1 \partial_{y^2} - \frac{i}{2}y^2 V \partial_V, \quad Y_6 = \frac{i}{2}V \partial_V. \end{aligned} \tag{10}$$

The commutation relations of the basis is given in the following **Table 3**.

### 5.2. 1-Dimensional Optimal System of the $L^6$

This section presents a optimal system of 1-dimensional subalgebras of the symmetry algebras  $L^6$  obtained in Section 5.1. As above section, we have

$$\begin{aligned} C(1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C(2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C(3) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ C(4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C(5) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C(6) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The exponentiation of the matrices  $C(j)$  are obtained as follows

**Table 3.** Commutator for (10).

| $[ \ ] Y_1 Y_2 Y_3 Y_4 Y_5 Y_6$    |
|------------------------------------|
| $Y_1 0 0 \quad Y_4 2 Y_1 Y_2 0$    |
| $Y_2 0 0 Y_3 Y_2 - Y_6 0$          |
| $Y_3 - Y_4 - Y_5 0 \quad -2 Y_3 0$ |
| $Y_4 - 2 Y_1 - Y_2 2 Y_3 0 Y_5 0$  |
| $Y_5 - Y_2 Y_6 0 \quad - Y_5 0 0$  |
| $Y_6 0 0 0 0 0 0$                  |

$$A(1, \varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \varepsilon^2 & 0 & 1 & -\varepsilon & 0 & 0 \\ -2\varepsilon & 0 & 0 & 1 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A(2, \varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\varepsilon & -\frac{1}{2}\varepsilon^2 \\ 0 & -\varepsilon & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \varepsilon \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A(3) = \begin{pmatrix} 1 & 0 & \varepsilon^2 & \varepsilon & 0 & 0 \\ 0 & 1 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2\varepsilon & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A(4, \varepsilon) = \begin{pmatrix} e^\varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & e^\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-2\varepsilon} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\varepsilon} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A(5, \varepsilon) = \begin{pmatrix} 1 & \varepsilon & 0 & 0 & 0 & -\frac{1}{2}\varepsilon^2 \\ 0 & 1 & 0 & 0 & 0 & -\varepsilon \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A(6, \varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the nonzero generator  $Y = k^1 Y_1 + k^2 Y_2 + \dots + k^6 Y_6$ , the vector  $k = (k^1, k^2, \dots, k^6)$  is transformed by the matrices  $A(j, \varepsilon)$  as follows

$$\begin{aligned} kA(1, \varepsilon) &= (k^3 \varepsilon^2 - 2k^4 \varepsilon + k^1, k^2 - k^5 \varepsilon, k^3, k^4 - k^3 \varepsilon, k^5, k^6) \\ kA(2, \varepsilon) &= \left( k^1, k^2 - \varepsilon k^4, k^3, k^4, k^5 - \varepsilon k^3, -\frac{1}{2} k^3 \varepsilon^2 + k^5 \varepsilon + k^6 \right) \\ kA(3, \varepsilon) &= (k^1, k^2, k^1 \varepsilon^2 + 2k^4 \varepsilon + k^3, k^4 + \varepsilon k^1, k^5 + \varepsilon k^2, k^6) \\ kA(4, \varepsilon) &= (e^\varepsilon k^1, e^\varepsilon k^2, e^{-2\varepsilon} k^3, k^4, e^{-\varepsilon} k^5, k^6) \\ kA(5, \varepsilon) &= \left( k^1, k^2 + \varepsilon k^1, k^3, k^4, k^5 + \varepsilon k^4, -\frac{1}{2} k^1 \varepsilon^2 - k^2 \varepsilon + k^6 \right) \\ kA(6, \varepsilon) &= (k^1, k^2, k^3, k^4, k^5, k^6) \end{aligned}$$

In order to get the optimal system, we need to transform  $k$  as simple as possible. This is done by following deduction.

If  $k^3 \neq 0$  by setting  $\varepsilon_1 = \frac{k^4}{k^3}$ ,  $\varepsilon_2 = \frac{k^5}{k^3}$  and  $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 0$ , the coefficients  $k^4$  and  $k^5$  are vanished. By scaling, we can suppose  $k^3 = 1$ . Hence  $Y$  is equivalent to  $Y_3 + aY_1 + bY_2 + cY_6$ .

If  $k^3 = 0$  and  $k^4 \neq 0$  by setting  $\varepsilon_1 = \frac{k^1}{2k^4}$ ,  $\varepsilon_2 = \frac{k^2}{k^4}$ ,  $\varepsilon_5 = -\frac{k^5}{k^4}$  and  $\varepsilon_3 = \varepsilon_4 = \varepsilon_6 = 0$ , the coefficients  $k^1$ ,  $k^2$  and  $k^5$  are vanished. By scaling, we can suppose  $k^4 = 1$ . Hence  $Y$  is equivalent to  $Y_4 + aY_6$ .

If  $k^3 = k^4 = 0$  and  $k^5 \neq 0$  by setting  $\varepsilon_1 = \frac{k^2}{k^5}$ ,  $\varepsilon_2 = -\frac{k^6}{k^5}$ ,  $\varepsilon_4 = -\frac{1}{2} \ln \frac{k^1}{k^5}$  and  $\varepsilon_3 = \varepsilon_5 = \varepsilon_6 = 0$ , the coefficients  $k^2$  and  $k^5$  are vanished. By scaling, we can suppose  $k^5 = 1$ . Hence  $Y$  is equivalent to  $Y_5 + Y_1$ .

If  $k^3 = k^4 = k^5 = 0$  and  $k^1 \neq 0$  by setting  $\varepsilon_4 = \ln \frac{k^6}{k^1}$ ,  $\varepsilon_5 = -\frac{k^2}{k^1}$  and  $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_6 = 0$ , the coefficient  $k^2$  is vanished. By scaling, we can suppose  $k^1 = 1$ . Hence  $Y$  is equivalent to  $Y_1 + Y_6$ .

If  $k^1 = k^3 = k^4 = k^5 = 0$  and  $k^6 \neq 0$  by setting  $\varepsilon_4 = -\ln \frac{k^2}{k^6}$  and  $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_5 = \varepsilon_6 = 0$ . By scaling, we can suppose  $k^6 = 1$ . Hence  $Y$  is equivalent to  $Y_6 + Y_2$ .

If  $k^1 = k^3 = k^4 = k^5 = k^6 = 0$  and  $k^2 \neq 0$ , by scaling, we can suppose  $k^2 = 1$ . Hence  $Y$  is equivalent to  $Y_2$ .

Therefore the set of generators

$$Y_1^1 = Y_1 + Y_6, Y_2^1 = Y_2, Y_3^1 = Y_3 + aY_1 + bY_2 + cY_6, Y_4^1 = Y_4 + aY_6, Y_5^1 = Y_5 + Y_1, Y_6^1 = Y_6 + Y_2. \quad (11)$$

is consist of the optimal system of 1-dimensional subalgebra of the symmetry algebra of Equation (9), where  $a$ ,  $b$  and  $c$  are arbitrary constants.

### 5.3. 1-Dimensional Optimal System of the $L^6$

In this section we give a classification of symmetry reductions of PDE (9) with the optimal system of (11).

The Equation (9) admits optimal system (11) obtained in Section 5.2.

Taking  $Y_5^1 = Y_5 + Y_1$  as example to show the procedure of the reduction.

The differential invariants (and hence the similarity variables) for the generator can be determined by solving the characteristic system

$$\frac{dy^1}{1} = \frac{dy^2}{y^1} = -\frac{dV}{\frac{i}{2}y^2V} \quad (12)$$

Solving this system, one obtains invariant (similarity) variables as follows

$$z = y^2 - \frac{1}{2}(y^1)^2, \quad w = e^{-\frac{i}{6}(y^1)^3 + \frac{i}{2}y^1y^2} V.$$

Hence let

$$V = e^{\frac{i}{6}(y^1)^3 - \frac{i}{2}y^1y^2} W(z), \quad (13)$$

which yields reduction of (9)

$$W_{zz} - \frac{1}{2}zW + r|W|^2W = 0.$$

This is the second reductions of (1), which is an ordinary differential equation.

In the same manner, we can obtain the second reductions of Equation (9) with respect to other subalgebras of (11) which are listed in **Table 4**. For the equation reduced by  $X^9$  (the last equation in **Table 2**) where

$$\begin{aligned}
 A: V &= e^{\frac{i \left[ \sqrt{a} \left( b^2 y^1 - 2ab y^2 - a y^1 (y^2)^2 \right) + (b^2 + 2ac) \left( a + (y^1)^2 \right) \arctan \frac{y^1}{\sqrt{a}} \right]}{4a^{\frac{3}{2}} \left( a + (y^1)^2 \right)}} W(z), \quad z = \frac{a y^2 - b y^1}{a \sqrt{a + (y^1)^2}} \\
 B: V &= e^{\frac{i}{6} y^1 \left( (y^1)^2 - 3 y^2 \right)} W(z), \quad z = y^2 - \frac{1}{2} (y^1)^2 \\
 V_{y^2 y^2} - i V_{y^1} - \frac{1}{4a^2} V + r |V|^2 V &= 0
 \end{aligned} \tag{14}$$

we have the same reduction as (9).

For the equation reduced by  $X^6$  or  $X^7$  (the fifth equation in **Table 2**)

$$2 \left[ a^2 + (y^1)^2 \right] V_{y^2 y^2} - 2i (y^1)^2 V_{y^1} + 2r (y^1)^2 |V|^2 V - i y^1 V = 0 \tag{15}$$

admits optimal system  $Y_1^6, \dots, Y_4^6$  and leads to the reductions which are listed in **Table 5**. where

$$\begin{aligned}
 Z_1 &= \partial_{y^2}, \quad Z_2 = \frac{(y^1)^2 - a^2}{y^1} \partial_{y^2} - \frac{i}{2} y^2 V \partial_V, \\
 Z_3 &= \left( a^2 + (y^1)^2 \right) \partial_{y^1} + \frac{y^2}{y^1} \left( (y^1)^2 - a^2 \right) \partial_{y^2} - \frac{i}{4} \left( (y^2)^2 + y^1 \right) V \partial_V, \quad Z_4 = -\frac{i}{2} V \partial_V. \\
 A: V &= W(z), \quad z = y^1; \\
 B: V &= e^{\frac{i y^1 (y^2)^2}{4 \left( (y^1)^2 - a^2 \right)}} W(z), \quad z = y^1; \\
 C: V &= \left( a^2 + (y^1)^2 \right)^{\frac{1}{2}} e^{\frac{1}{4} \left[ \frac{i y^1 \left( (y^1)^2 - a^2 \right) (y^2)^2 + 2i \arctan \left( \frac{a}{y^1} \right)}{\left( a^2 + (y^1)^2 \right)^2} \right]} W(z), \quad z = \frac{y^1 y^2}{a^2 + (y^1)^2}; \\
 D: V &= e^{\frac{i}{2a} y^2} W(z), \quad z = y^1.
 \end{aligned}$$

Similarly, other equations in **Table 2** also can be further reduced. Consequently, we obtain the twice reductions of (1) shown in **Table 6**. where

$$\begin{aligned}
 W_1 &= \partial_{y^1}, \quad W_2 = \partial_{y^2}, \quad W_3 = y^2 \partial_{y^1} - y^1 \partial_{y^2}, \\
 W_4 &= y^1 \partial_{y^1} + y^2 \partial_{y^2} - V \partial_V, \quad W_5 = -i V \partial_V, \quad V_1 = y^2 \partial_{y^1} - y^1 \partial_{y^2}, \quad V_2 = -i V \partial_V. \\
 z &= y^2; \quad z = y^1; \quad z = \frac{a y^2 - b y^1}{a \sqrt{a + (y^1)^2}}; \quad z = \frac{y^1}{(y^2)^2}; \quad z = y^2 - \frac{1}{2} (y^1)^2; \quad z = y^1; \quad z = y^2; \\
 z &= y^1; \quad z = \frac{1}{2} \left( (y^1)^2 + (y^2)^2 \right); \quad z = (y^1)^2 + (y^2)^2; \quad z = y^1; \quad z = y^1; \quad z = \frac{y^1 y^2}{a^2 + (y^1)^2}; \quad z = y^1.
 \end{aligned}$$

**Table 4.** The further reductions of (9) with (11).

| Generator the second reductions similarity variables   |
|--|
| $Y_1^1 = Y_1 + Y_6 \quad W_{zz} + \frac{1}{2}aW + r W ^2 W = 0 \quad V = e^{\frac{i}{2}y^1} W(z), \quad z = y^2$         |
| $Y_2^1 = Y_2 \quad iW_z - r W ^2 W = 0 \quad V = W(z), \quad z = y^1$  |
| $Y_3^1 = Y_3 + aY_1 + bY_2 + cY_6 \quad$ Cannot get explicit reduction $A$   |
| $Y_4^1 = Y_4 + Y_6 \quad$ Cannot get explicit reduction $V = (y^2)^{\frac{i}{2}y^1} W(z), \quad z = \frac{y^1}{(y^2)^2}$ |
| $Y_5^1 = Y_3 + Y_1 \quad W_{zz} - \frac{1}{2}zW + r W ^2 W = 0 \quad B$  |
| $Y_6^1 = Y_6 + Y_2 \quad iW_z + \frac{1}{4}W - r W ^2 W = 0 \quad V = (y^2)^{\frac{i}{2}y^1} W(z), \quad z = y^1$        |

**Table 5.** The reductions for the subalgebras of Equation (15).

| Generator of optimal system the second reductions similarity variable                                 |     |
|---|-----|
| $Y_1^6 = Z_1 \quad 2izW_z + iW - 2rz W ^2 W = 0$  | $A$ |
| $Y_2^6 = Z_2 \quad 2izW_z - \frac{2iz}{a^2 - z^2}W - 2r W ^2 W = 0$                                   | $B$ |
| $Y_3^6 = Z_3 + aZ_4 \quad W_{zz} + \left(ia^2z^2 - \frac{a}{2}\right)W + \frac{1}{4}r W ^2 W = 0$     | $C$ |
| $Y_4^6 = Z_4 + aZ_1 \quad 2iz^2W_z + \left[\frac{1}{2a^2}(a^2 + z^2) + iz\right]W - 2rz^2 W ^2 W = 0$ | $D$ |

$$A: 4W_z + 4zW_{zz} - \frac{1}{z}b^2W + \frac{1}{4c}(a^2 + b^2 - 2cq + c^2z)W + r|W|^2 W = 0;$$

$$B: 4W_z + 4zW_{zz} + 2ibzW_z + \frac{1}{4}[2i(2b + iq) - acz]W + \frac{1}{4}r|W|^2 W = 0;$$

$$C: -2iz^2W_z - \left[\frac{1}{2a^2}(a^2 + z^2) + iz\right]W + 2rz^2|W|^2 W = 0;$$

$$D: V = e^{\frac{i}{2}y^1} W(z); \quad E: V = W(z);$$

$$F: V = e^{\frac{i\sqrt{a}\left(b^2y^1 - 2aby^2 - ay^1(y^2)^2\right) + i(b^2 + 2ac)\left(a + (y^1)^2\right) \arctan \frac{y^1}{\sqrt{a}}}{4a^{\frac{3}{2}}\left(a + (y^1)^2\right)} W(z);$$

$$G: V = (y^2)^{\frac{i}{2}a} W(z); \quad H: V = e^{\frac{i}{6}y^1\left((y^1)^2 - 3y^2\right)} W(z); \quad I: V = e^{\frac{i}{2}y^2} W(z); \quad J: V = e^{-iay^1} W(z);$$

$$K: V = e^{-iay^2} W(z); \quad L: V = e^{-ai \arctan \left(\frac{y^1}{\sqrt{(y^2)^2}}\right)} W(z); \quad M: \text{Cannot get similarity variable};$$

**Table 6.** The rest reductions of the 2D-CNLS (1) by optimal system (6).

| First step opts         | second step opts     | reduced equation   | similarity variable   |
|-------------------------|----------------------|--|---|
| $X^1$ or $X^2$ or $X^9$ | $Y_1^1 = Y_1 + Y_6$  | $W_{zz} + \frac{1}{2}aW + r W ^2 W = 0$                        | $D$   |
|                         |                      | $Y_2^2 = Y_2 - iW_z + r W ^2 W = 0$                            | $E$   |
|                         |                      | $Y_3^3 = Y_3 + aY_1 + bY_2 + cY_6$                             | Cannot be obtained explicit reduction                         |
|                         |                      | $Y_4^4 = Y_4 + aY_6$   | Cannot be obtained explicit reduction                         |
|                         |                      | $Y_5^5 = Y_5 + Y_1$  | $W_{zz} - \frac{1}{2}zW + r W ^2 W = 0$                       |
|                         |                      | $Y_6^6 = Y_6 + Y_2 - iW_z - \frac{1}{4}W + r W ^2 W = 0$       | $I$   |
| $X^3$ or $X^8$          | $Y_1^3 = W_1 + aW_5$ | Cannot be obtained explicit reduction                          | $J$   |
|                         |                      | $Y_2^3 = W_2 + aW_5$   | Cannot be obtained explicit reduction                         |
|                         |                      | $Y_3^3 = W_3 + aW_5$   | $A L$   |
|                         |                      | $Y_4^3 = W_4 + aW_5 + bW_5$                                    | Cannot be obtained explicit reduction                         |
|                         |                      | $Y_5^3 = W_5$  | Cannot be obtained explicit reduction                         |
| $X^4$ or $X^5$          | $Y_1^4 = V_1$        | $B O$  |   |
|                         |                      | $Y_2^4 = V_2$  | Cannot be obtained explicit reduction                         |
| $X^6$ or $X^7$          | $Y_1^6 = Z_1$        | $-2izW_z - iW + 2rz W ^2 W = 0$                                | $Q$   |
|                         |                      | $Y_2^6 = Z_2 - 2iW_z + \frac{2iz}{a^2 - z^2}W + 2r W ^2 W = 0$ | $R$   |
|                         |                      | $Y_3^6 = Z_3 + aZ_4$   | $W_{zz} + (ia^2z^2 - \frac{a}{2})W + \frac{1}{4}r W ^2 W = 0$ |
|                         |                      | $Y_4^6 = Z_4 + aZ_1$   | $C T$   |

$N$ : Cannot get similarity variable;  $O$ :  $V = W(z)$ ;  $P$ : Cannot get similarity variable;

$$Q: V = W(z); \quad R: V = e^{-\frac{iy^1(y^2)^2}{4((y^1)^2 - a^2)}} W(z);$$

$$S: V = e^{\frac{1}{4} \left[ \frac{iy^1((y^1)^2 - a^2)(y^2)^2 + 2i \arctan\left(\frac{a}{y^1}\right)}{a^2 + (y^1)^2} \right]} \left( a^2 + (y^1)^2 \right)^{-\frac{1}{2}} W(z); \quad T: V = e^{-\frac{i}{2a}y^2} W(z).$$

### 6. Some Exact Invariant Solutions of PDE (1)

In Table 4, we have

$$iW_z - r|W|^2 W = 0 \tag{16}$$



Transform the (16) to real case by transformation  $W \rightarrow \bar{W} + i\tilde{W}$ , where  $W$  and  $\tilde{W}$  are real functions, we find the simplified

$$\tilde{W}_z = r\bar{W}(\bar{W}^2 + \tilde{W}^2) \quad (17)$$

$$\bar{W}_z = -r\tilde{W}(\bar{W}^2 + \tilde{W}^2) \quad (18)$$

Dividing Equation (17) by Equation (18), we obtain

$$\tilde{W}\tilde{W}_z + \bar{W}\bar{W}_z = 0$$

which yields

$$\tilde{W} = \varepsilon\sqrt{c - \bar{W}^2} \quad (19)$$

where  $\varepsilon = \pm 1$ ,  $c$  is arbitrary constant. Substituting (19) to (18), it has general solutions

$$\bar{W} = \varepsilon\sqrt{c} \sin(rc\varepsilon z - c_1)$$

$$\tilde{W} = \varepsilon\sqrt{c} \cos(rc\varepsilon z - c_1)$$

Therefore

$$W = \varepsilon\sqrt{c}e^{-(rc\varepsilon z - c_1)i} \quad (20)$$

In the same manner, solving the equation

$$r|W|^2 W - iW_z - \frac{1}{4}W = 0$$

in **Table 4**, we have solution

$$W = \varepsilon\sqrt{c}e^{-\left[\left(\frac{rc+1}{4}\right)\varepsilon z - c_1\right]i}$$

In the same manner, solving the equation in **Table 5**

$$-2iz^2W_z - \left[\frac{1}{2a^2}(a^2 + z^2) + iz\right]W + 2rz^2|W|^2W = 0$$

one has solution

$$W = \varepsilon\sqrt{c}e^{-\left[\left(\frac{rc-1}{4a^2}\right)\varepsilon z + \frac{\varepsilon}{4z} - \frac{i\varepsilon}{2}\ln|z-c_1\right]i}$$

These solutions  $W$  yield exact solutions of (1) through connection with  $V = W(z)$ , (20) and (7).

## 7. Conclusion

The 1-dimensional optimal system of the Lie algebra of 2D-CNLS equation has been constructed by matrix form method and twice reductions of the equation by the optimal system are given. Consequently, some exact invariant solutions of the equation are formed by symmetry method.

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