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The Modified Kadomtsev-Petviashvili Equation with Binary Bell Polynomials

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Abstract

Binary Bell Polynomials play an important role in the characterization of bilinear equation. The bilinear form, bilinear Bäcklund transformation and Lax pairs for the modified Kadomtsev-Petviashvili equation are derived from the Binary Bell Polynomials.

Keywords

Binary Bell Polynomials, Bilinear Bäcklund Transformation, Lax Pair

1. Introduction

There are some techniques that can be used to solve the nonlinear evolution equations, such as inverse scattering transformation, Hirota method, Darboux transformation and the tanh method [1]-[4]. Among this methods, the bilinear method and bilinear Bäcklund transformation have proved particularly powerful. Through the dependent variable transformations, some nonlinear evolution equations can be transformed into bilinear forms. Applying the bilinear method developed by Hirota, we can obtian the soliton solutions and quasiperiodic wave solutions [5]-[7]. The construction of the bilinear Bäcklund transformation [8] by using Hirota method relies on a particular skill in using appropriate exchange formulas which are connected with the linear presentation of the system. Yet, the construction of bilinear Bäcklund transformation is complicated. Recently, Lambert, Gilson *et al*. [9]-[11] proposed an alternative procedure based on the use of the Bell polynomials which enabled one to obtain parameter families of bilinear Bäcklund transformation and Lax pairs for the soliton equations in a lucid and systematic way. In Ref [12], Fan has constructed bilinear formalism, bilinear Bäcklund transformation, Lax pairs and infinite conservation laws for the nonisospectral and variable-coefficient KdV equation.

In this paper, we will extend the Binary Bell polynomials to deal with the modified Kadomtsev-Petviashvili (mKP) equation. First, we derive the bilinear form for the mKP equation by the binary Bell polynomials. Second,

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the bilinear Bäcklund transformation and Lax pairs are obtained in a quick and natural manner.

2. The Bilinear Form for the mKP Equation

The mail tool used here is a class of generalized multi-dimensional binary Bell polynomials. First, we give some notations on the Bell polynomials to easily understand our presentation.

Lambert *et al*. proposed a generalization of the Bell polynomial [9]-[11]. Let $n_k \geq 0, k = 1, \cdots, l$, denote arbitrary integers, $f = f(x_1, \cdots, x_l)$ be a C^∞ multi-variable function, the following polynomials

$$\mathcal{Y}_{n_1 x_1 \cdots n_l x_l}(f) = \exp(-f)\partial_{x_1}^{n_1} \cdots \partial_{x_l}^{n_l} \exp(f) \tag{1}$$

is called multi-dimensional Bell polynomial (generalized Bell polynomial or Y-polynomials). If all partial derivatives $f_{r_1 x_1 \cdots r_l x_l} = \partial_{x_1}^{r_1} \cdots \partial_{x_l}^{n_l} (r_k = 0, \cdots, n_k, k = 1, \cdots l)$ are taken as different variable elements, then the generalized Bell polynomial $\mathcal{Y}_{n_1 x_1 \cdots n_l x_l}(f)$ is the multivariable polynomial with respect to these variable elements $f_{r_1 x_1, \cdots, r_l x_l}$. The subscripts in the notation $Y_{n_1 x_1, \cdots, n_l x_l}(f)$ denote the highest order derivatives of f with respect to the variable $x_k, k = 1, \cdots, l$ respectively.

For the special case $f = f(x,t)$, the associated two-dimensional Bell polynomials defined by (1) read

$$\mathcal{Y}_x(f) = f_x,\ \mathcal{Y}_{2x}(f) = f_{2x} + f_x^2,\ \mathcal{Y}_{3x}(f) = f_{3x} + 3f_x f_{2x} + f_x^3. \tag{2}$$

$$\mathcal{Y}_{x,t}(f) = f_{x,t} + f_x f_t,\ \mathcal{Y}_{2x,t}(f) = f_{2x,t} + f_{2x} f_t + 2 f_{x,t} f_x + f_x^2 f_t. \tag{3}$$

Base on the use of above Bell polynomials (1), the multidimensional binary Bell polynomials ($\mathcal{Y}$-polynomials) can be defined as follows

$$\mathcal{Y}_{n_1 x_1, \cdots, n_l x_l}(v, w) = Y_{n_1 x_1, \cdots, n_l x_l}(f)\Big|_{f_{r_1 x_1, \cdots, r_l x_l}} = \begin{cases} v_{r_1 x_1, \cdots, r_l x_l}, & r_1 + \cdots + r_l \text{ is odd} \\ w_{r_1 x_1, \cdots, r_l x_l}, & r_1 + \cdots + r_l \text{ is even} \end{cases} \tag{4}$$

which is a multivariable polynomials with respect to all partial derivatives $v_{r_1 x_1, \cdots, r_l x_l}(r_1 + \cdots + r_l \text{ odd})$ and

$$w_{r_1 x_1, \cdots, r_l x_l}(r_1 + \cdots + r_l \text{ even}), r_k = 0, \cdots, n_k, k = 0, \cdots, l.$$

The binary Bell polynomials also inherits the easily recognizable partial structure of the Bell polynomials. The lowest order binary Bell polynomials are

$$\mathcal{Y}_x(v) = v_x,\ \mathcal{Y}_{2x}(v, w) = w_{2x} + v_x^2, \tag{5}$$

$$\mathcal{Y}_{x,t}(v, w) = w_{xt} + v_x v_t,\ \mathcal{Y}_{3x}(v, w) = v_{3x} + 3v_x w_{2x} + v_x^3. \tag{6}$$

The link between binary Bell polynomials $\mathcal{Y}_{n_1 x_1 \cdots n_l x_l}(v, w)$ and the standard Hirota bilinear equation $D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} F \cdot G$ can be given by an identity

$$\mathcal{Y}_{n_1 x_1 \cdots n_l x_l}(v = \ln F/G, w = \ln FG) = (FG)^{-1} D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} F \cdot G \tag{7}$$

in which $n_1 + n_2 + \cdots + n_l \geq 1$, and operators $D_{x_1}, \cdots, D_{x_l}$ are classical Hirota bilinear operators defined by

$$D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} F \cdot G = \left(\partial_{x_1} - \partial_{x_1'}\right)^{n_1} \cdots \left(\partial_{x_l} - \partial_{x_l'}\right)^{n_l} F(x_1, \cdots, x_l) \times G(x_1', \cdots, x_l')\Big|_{x_1' = x_1, \cdots, x_l' = x_l}. \tag{8}$$

In the particular case when $F = G$, the formula (7) becomes

$$\begin{aligned} & G^{-2} D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} G \cdot G \\ & = \mathcal{Y}_{n_1 x_1, \cdots, n_l x_l}(0, q = 2\ln G) \\ & = \begin{cases} 0, & n_1 + \cdots + n_l \text{ is odd}, \\ P_{n_1 x_1, \cdots, n_l x_l}(q), & n_1 + \cdots + n_l \text{ is even} \end{cases} \end{aligned} \tag{9}$$

in which the P-polynomials can be characterized by an equally recognizable even part partitional structure

$$P_{2x}(q)=q_{2x},\ P_{x,t}(q)=q_{xt},\ P_{4x}=q_{4x}+3q_{2x}^2,\ P_{6x}(q)=q_{6x}+15q_{2x}q_{4x}+15q_{2x}^2. \tag{10}$$

The formulaes (7),(9) and (10) will prove particular useful in connecting nonlinear equations with their corresponding bilinear equations. This means that once a nonlinear equation is expressible as a linear combination of the P-polynomials, then it can be transformed into a linear equation.

The binary Bell polynomials $\mathcal{Y}_{n_1x_1,\cdots,n_lx_l}(v,w)$ can be separated into P-polynomials and Y-polynomials

$$\begin{aligned}&(FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G\\&=\mathcal{Y}_{n_1x_1,\cdots,n_lx_l}(v,w)\big|_{v=\ln F/G,w=\ln FG}\\&=\mathcal{Y}_{n_1x_1,\cdots,n_lx_l}(v,v+q)\big|_{v=\ln F/G,q=2\ln G}\\&=\sum_{n_1+\cdots+n_l=\text{even}}\sum_{r_1=0}^{n_1}\cdots\sum_{r_l=0}^{n_l}\prod_{i=1}^{l}\binom{n_i}{r_i}P_{r_1x_1,\cdots,r_lx_l}(q)Y_{(n_1-r_1)x_1\cdots(n_l-r_l)x_l}(v).\end{aligned} \tag{11}$$

The key property of the multi-dimensional Bell polynomials

$$Y_{n_1x_1,\cdots,n_lx_l}(v)\big|_{v=\ln\psi}=\psi_{n_1x_1,\cdots,n_lx_l}/\psi. \tag{12}$$

implies that the binary Bell polynomials $\mathcal{Y}_{n_1x_1,\cdots,n_lx_l}(v,w)$ can still be linearized by means of the Hopf-Cole transformation $v=\ln\psi$, that is, $\psi=F/G$.

$$\begin{aligned}&(FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G\big|_{G=\exp(q/2),F/G=\psi}\\&=\psi^{-1}\sum_{r_1=0}^{n_1}\cdots\sum_{r_l=0}^{n_l}\prod_{i=1}^{l}\binom{n_i}{r_i}P_{r_1x_1,\cdots,r_lx_l}(q)\psi_{(n_1-r_1)x_1\cdots(n_l-r_l)x_l}(v)\end{aligned} \tag{13}$$

The formulaes (11) and (13) will then provide the shortest way to the associated Lax system of nonlinear equations.

In this paper we consider the mKP equation

$$u_t+u_{3x}+3\partial^{-1}u_{2y}-6u^2u_x+6\left(\partial^{-1}u_y\right)u_x=0. \tag{14}$$

Let a potential field q be

$$u=q_x. \tag{15}$$

Substituting (15) into (14), we have

$$E(q)=q_{xt}+q_{4x}+3q_{2y}-6q_x^2q_{2x}+6q_yq_{2x}=0. \tag{16}$$

Introducing two new variables

$$q=\ln g/f,\quad w=\ln fg, \tag{17}$$

using the binary Bell polynomials (5) and (6), Equation (16) can be written into

$$E(q)=\partial_x\left[\mathcal{Y}_t(q,w)+\mathcal{Y}_{3x}(q,w)\right]+3q_{2y}-3q_{2x}w_{2x}-3q_xw_{3x}-9q_x^2q_{2x}+6q_yq_{2x}=0. \tag{18}$$

A possible choice of such constraint maybe

$$q_y=w_{2x}+q_x^2, \tag{19}$$

then

$$q_{2y}=\partial_x\left[\mathcal{Y}_{xy}(q,w)\right]-q_{2x}q_y+q_xq_{xy}=\partial_x\left[\mathcal{Y}_{xy}(q,w)\right]-q_{2x}w_{2x}+q_xw_{3x}+3q_x^2q_{2x}. \tag{20}$$

Substituting (20) into (18) and using the relation (19), we have

$$\partial_x\left[\mathcal{Y}_t(q,w)+\mathcal{Y}_{3x}(q,w)+3\mathcal{Y}_{xy}(q,w)\right]=0. \tag{21}$$

Therefore, from (19) and (21), we deduce a couple system of $\mathcal{Y}$-polynomials for the mKP equation

$$\mathcal{Y}_y(q,w)=\mathcal{Y}_{2x}(q,w), \tag{22}$$

$$\mathcal{Y}_t(q,w)+\mathcal{Y}_{3x}(q,w)+3\mathcal{Y}_{xy}(q,w)=0. \tag{23}$$

By application of the identity (7) and the transformation (17), Equations (22) and (23) lead to the bilinear form for the mKP equation

$$D_y g\cdot f=D_x^2 g\cdot f, \tag{24}$$

$$D_t g\cdot f+D_x^3 g\cdot f+3D_xD_y g\cdot f=0. \tag{25}$$

Using Hirota's bilinear method, it is easy to solve the multisoliton solutions for the mKP equation. For example, the one-soliton solution reads

$$u=\left(\ln\frac{1+qe^{\xi+\eta}}{1-ke^{\xi+\eta}}\right)_x,\ \xi=kx-k^2y-4k^3t+\xi^{(0)},\ \eta=qx+q^2y-4q^3t+\eta^{(0)} \tag{26}$$

where $\xi^{(0)}$ and $\eta^{(0)}$ are variable constant.

3. The Bilinear Bäcklund Transformation and Lax Pairs for the mKP Equation

In this section, we consider the bilinear Bäcklund transformation and Lax pair for the mKP equation.

Set

$$q=\ln\frac{g}{f},\ \tilde{q}=\ln\frac{\tilde{g}}{\tilde{f}},\ w=\ln fg,\ \tilde{w}=\ln\tilde{f}\tilde{g}, \tag{27}$$

be two different solutions of (16), respectively. We associate the two-field condition

$$\begin{aligned}E(\tilde{q})-E(q)&=(\tilde{q}-q)_{xt}+(\tilde{q}-q)_{4x}+3(\tilde{q}-q)_{2y}-6\tilde{q}_x^2\tilde{q}_{2x}+6\tilde{q}_y\tilde{q}_{2x}+6q_x^2q_{2x}-6q_yq_{2x}\\&=(\tilde{q}-q)_{xt}+(\tilde{q}-q)_{4x}+3(\tilde{q}-q)_{2y}+6\tilde{q}_{2x}(\tilde{q}_y-\tilde{q}_x^2)-6q_{2x}(q_y-q_x^2)\end{aligned} \tag{28}$$

By the relation

$$q_y=w_{2x}+q_x^2, \tag{29}$$

$$\tilde{q}_y=\tilde{w}_{2x}+\tilde{q}_x^2. \tag{30}$$

Equation (28) can be transformed into

$$\begin{aligned}E(\tilde{q})-E(q)&=(\tilde{q}-q)_{xt}+(\tilde{q}-q)_{4x}+3(\tilde{q}-q)_{2y}+6\tilde{q}_{2x}\tilde{w}_{2x}-6q_{2x}w_{2x}\\&=(\tilde{q}-q)_{xt}+(\tilde{q}-q)_{4x}+3(\tilde{q}-q)_{2y}+3(\tilde{q}-q)_{2x}(\tilde{w}+w)_{2x}+3(\tilde{q}+q)_{2x}(\tilde{w}-w)_{2x}.\end{aligned} \tag{31}$$

Let

$$\tilde{q}-q=\ln\frac{\tilde{g}}{g}-\ln\frac{\tilde{f}}{f}=v_1-v_2, \tag{32}$$

$$\tilde{q}+q=\ln\tilde{g}g-\ln\tilde{f}f=w_1-w_2, \tag{33}$$

$$\tilde{w}-w=\ln\frac{\tilde{g}}{g}+\ln\frac{\tilde{f}}{f}=v_1+v_2, \tag{34}$$

$$\tilde{w}+w=\ln\tilde{g}g+\ln\tilde{f}f=w_1+w_2, \tag{35}$$

so Equation (31) becomes

$$\begin{aligned}E(\tilde{q})-E(q)&=(v_1-v_2)_{xt}+(v_1-v_2)_{4x}+3(v_1-v_2)_{2y}+3(v_1-v_2)_{2x}(w_1+w_2)_{2x}+3(w_1-w_2)_{2x}(v_1+v_2)_{2x}\\&=v_{1,xt}+v_{1,4x}+3v_{1,2y}+6v_{1,2x}w_{1,2x}-(v_{2,xt}+v_{2,4x}+3v_{2,2y}+6v_{2,2x}w_{2,2x})=0.\end{aligned} \tag{36}$$

Similar to the (21), by the relation

$$v_{1,y}=w_{1,2x}+v_{1,x}^2, \tag{37}$$

$$v_{2,y}=w_{2,2x}+v_{2,x}^2. \tag{38}$$

Equation (36) can be transformed into

$$\partial_x\left[\mathcal{Y}_t(v_1,w_1)+3\mathcal{Y}_{xy}(v_1,w_1)+\mathcal{Y}_{3x}(v_1,w_1)\right]-\partial_x\left[\mathcal{Y}_t(v_2,w_2)+3\mathcal{Y}_{xy}(v_2,w_2)+\mathcal{Y}_{3x}(v_2,w_2)\right]=0. \tag{39}$$

Then from (27) to (39), we get the system of $\mathcal{Y}$-polynimials

$$\mathcal{Y}_y(v,w)=\mathcal{Y}_{2x}(v,w), \tag{40}$$

$$\mathcal{Y}_y(\tilde{v},\tilde{w})=\mathcal{Y}_{2x}(\tilde{v},\tilde{w}), \tag{41}$$

$$\mathcal{Y}_y(v_1,w_1)=\mathcal{Y}_{2x}(v_1,w_1), \tag{42}$$

$$\mathcal{Y}_y(v_2,w_2)=\mathcal{Y}_{2x}(v_2,w_2), \tag{43}$$

$$\mathcal{Y}_t(v_1,w_1)+3\mathcal{Y}_{xy}(v_1,w_1)+\mathcal{Y}_{3x}(v_1,w_1)=0, \tag{44}$$

$$\mathcal{Y}_t(v_2,w_2)+3\mathcal{Y}_{xy}(v_2,w_2)+\mathcal{Y}_{3x}(v_2,w_2)=0. \tag{45}$$

Using the link between Bell Polynomials and Hirota bilinear bilinear Bäcklund transformation (7), the bilinear Bäcklund transformation can be written as

$$D_y g\cdot f=D_x^2 g\cdot f, \tag{46}$$

$$D_y \tilde{g}\cdot \tilde{f}=D_x^2 \tilde{g}\cdot \tilde{f}, \tag{47}$$

$$D_y g\cdot \tilde{g}=-D_x^2 g\cdot \tilde{g}, \tag{48}$$

$$D_y \tilde{f}\cdot f=D_x^2 \tilde{f}\cdot f, \tag{49}$$

$$\left(D_t-3D_xD_y+D_x^3\right)g\cdot\tilde{g}=0, \tag{50}$$

$$\left(D_t+3D_xD_y+D_x^3\right)\tilde{f}\cdot f=0. \tag{51}$$

Through the bilinear Bäcklund transformation, we can get the soliton solutions for the mKP equation.

In the following, we will give the Lax pair for the mKP equation. By transformations

$$\theta=\frac{g}{\tilde{g}},q=2\ln\tilde{q}, \tag{52}$$

and the relation (13), the formulaes (48) and (50) become

$$\theta_y=-\theta_{2x}-P_{2x}(q)\theta, \tag{53}$$

$$\theta_t-3\theta_{xy}-3P_{xy}(q)\theta+\theta_{3x}+3P_{2x}(q)\theta_x=0. \tag{54}$$

Set

$$\psi=\frac{g}{\tilde{f}}, \tag{55}$$

then $\theta=\psi\frac{\tilde{f}}{\tilde{g}}$, by the relation $u=\ln\frac{\tilde{g}}{\tilde{f}}$, (53) and (54) grow

$$\psi_y=-\psi_{2x}+2u\psi_x, \tag{56}$$

$$\psi_t+4\psi_{xxx}-12u\psi_{xx}-6u_x\psi_x+6u^2\psi_x+6\partial^{-1}u_y\psi_x=0, \tag{57}$$

which is the Lax pair of the mKP equation.

Similar to the (56) and (57). Let $\vartheta=\frac{\tilde{f}}{f}, q=2\ln f$, (49) and (51) make

$$\vartheta_y=\vartheta_{2x}+P_{2x}(q)\vartheta, \tag{58}$$

$$\vartheta_t+3\vartheta_{xy}+3P_{xy}(q)\vartheta+\vartheta_{3x}+3P_{2x}(q)\vartheta_x=0. \tag{59}$$

Let $\phi=\frac{\tilde{f}}{g}$, then $\vartheta=\phi\frac{g}{f}$, by the relation $u=\ln\frac{g}{f}$, (58) and (59) develop into

$$\phi_y=\phi_{xx}+2u\phi_x, \tag{60}$$

$$\phi_t+4\varphi_{xxx}+12u\phi_{xx}+6u_x\phi_x+6u^2\phi_2+6\partial^{-1}u_y\phi_x=0, \tag{61}$$

which is the Lax pair for the mKP equation.

4. Conclusion

Binary Bell Polynomials play an important role in the characterization of bilinear equation. By the Binary Bell Polynomials, we give the bilinear form, bilinear Bäcklund transformation and Lax pairs for the modified Kadomtsev-Petviashvili equation. This method is a lucid and systematic way. This method can be extended to the other soliton equations.

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