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Steffensen-Type Method of Super Third-Order Convergence for Solving Nonlinear Equations

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Abstract

In this paper, a one-step Steffensen-type method with super-cubic convergence for solving nonlinear equations is suggested. The convergence order 3.383 is proved theoretically and demonstrated numerically. This super-cubic convergence is obtained by self-accelerating second-order Steffensen's method twice with memory, but without any new function evaluations. The proposed method is very efficient and convenient, since it is still a derivative-free two-point method. Its theoretical results and high computational efficiency is confirmed by Numerical examples.

Keywords

Newton's Method, Steffensen's Method, Derivative Free, Super-Cubic Convergence, Nonlinear Equation

1. Introduction

Finding the root of a nonlinear equation

$$f(x) = 0 \tag{1}$$

is a classical problem. It is well-known in scientific computation that Newton's method (NM, see [1]):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \cdots, \tag{2}$$

is widely used for root-finding, where x_0 is an initial guess of the root. However, when the derivative f' is

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unavailable or is expensive to be obtained, the derivative-free method is necessary. If the derivative $f'(x_n)$ is replaced by the divided difference in (2), Steffensen's method (SM, see [1]) is obtained as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x, x_n + f(x_n)]}, \quad n = 0,1,2,\cdots, \tag{3}$$

NM/SM converges quadratically and requires two function evaluations per iteration. The efficiency index of them is $\sqrt{2} = 1.414$.

Besides H.T. Kung and J.F. Traub conjectured that an iterative method based on m evaluations per iteration without memory would arrive at the optimal convergence of order 2^{m-1} (see [2]), Traub proposed a self-accelerating two-point method of order 2.414 with memory (see [3]):

$$\begin{cases} x_{n+1} = x_n - \dfrac{f(x_n)}{f[x_n + \beta_n f(x_n)]}, \\ \beta_n = -\dfrac{1}{f[x_n, z_{n-1}]}, \end{cases} \tag{4}$$

where $z_{n-1} = x_{n-1} + \beta_{n-1} f(x_{n-1})$, and $\beta_0 = -\mathrm{sign}(f'(x_0))$ or $-1/f[x_0, x_0 + f(x_0)]$, etc.

A lot of self-accelerating Steffensen-type methods were derived in the literature (see [1]-[7]). Steffensen-type methods and their applications in the solution of nonlinear systems and nonlinear differential equations were discussed in [1] [4] [5] [8]. Recently, by a new self-accelerating technique based on the second-order Newtonian interpolatory polynomial $N_2(x) = f(x_n) + f[x_n, z_{n-1}](x - x_n) + f[x_n, z_{n-1}, x_{n-1}](x - x_n)(x - z_{n-1})$, J. Džunića and M.S. Petkovića proposed a cubically convergent Steffensen-like method (see [7]):

$$\begin{cases} x_{n+1} = x_n - \dfrac{f(x_n)}{f[x_n, x_n + \beta_n f(x_n)]}, \\ \beta_n = -\dfrac{1}{f[x_n, z_{n-1}] + f[x_n, x_{n-1}] - f[x_{n-1}, z_{n-1}]}, \end{cases} \tag{5}$$

In this study, a one-step Steffensen-type method is proposed by doubly-self-accelerating in Section 2, its super-cubic convergence is proved in Section 3, and numerical examples are demonstrated in Section 4.

2. The Method of Steffensen-Type

By the first-order Newtonian interpolatory polynomial $N_1(x) = f(x_n) + f[x_n, z_n](x - x_n)$ and

$$z_n = x_n + \beta_n f(x_n),$$

we have $f(x) = N_1(x) + R_1(x)$,

where

$$R_1(x) = f(x) - N_1(x) = f[x_n, z_n, x](x - x_n)(x - z_n).$$

So, with some $\mu_n \approx f[x_n, z_n, x]$,

$$\tilde{N}_2(x) = f(x_n) + f[x_n, z_n](x - x_n) + \mu_n(x - x_n)(x - z_n) \tag{6}$$

should be better than $N_1(x)$ to approximate $f(x)$.

Therefore, we suggest $x_{n+1} = x_n - \dfrac{\tilde{N}_2(x_n)}{N_2'(x_n)}$, *i.e.*, a two-parameter Steffensen's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, z_n] + \mu(x_n - z_n)}, \quad n = 0,1,2,\cdots, \tag{7}$$

where $z_n = x_n + \beta_n f(x_n)$, $\{\beta_n\}$ and $\{\mu_n\}$ are bounded constant sequences. The error equation of (7) is $e_{n+1} = \left[\left(1+\beta_n f'(a)\right)\frac{f''(a)}{2f'(a)} - \mu_n\beta_n\right]e_n^2 + O(e_n^3)$. By defining $\mu_0 = 0$ and $\mu_n = \frac{1+\beta_n f[x_n, z_n]}{\beta_n f[x_n, z_n]} f[z_{n-1}, x_n, z_n]$ $(n > 0)$ recursively as the iteration proceeds without any new evaluation to vanish the asymptotic convergence constant, we establish a self-accelerating Steffensen's method with super quadratic convergence as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, z_n] - \left(1 + \frac{1}{\beta_n f[x_n, z_n]}\right)\left(f[z_{n-1}, z_n] - f[x_n, z_{n-1}]\right)}, \quad n = 0, 1, 2, \cdots, \tag{8}$$

Furthermore, we propose a one-step Steffensen-type method with super cubic convergence by doubly-self-accelerating as follows:

$$\begin{cases} x_{n+1} = x_n - \dfrac{f(x_n)}{f[x_n, z_n] - \left(1 + 1/\left(\beta_n f[x_n, z_n]\right)\right)\left(f[z_{n-1}, z_n] - f[x_n, z_{n-1}]\right)} \\ \beta_n = \dfrac{1}{f[x_{n-1}, z_{n-1}] - f[x_n, z_{n-1}] - f[x_{n-1}, x_n]} \end{cases} \tag{9}$$

3. Its Super Third-Order Convergence

Lemma 3.1 $\beta_n \sim -\frac{1}{f'(a)} f\left(1 + c_3 e_{n-1} e_{n-1}^z\right)$, where $c_k = \frac{f^{(k)}(a)}{k!\, f'(a)}$, $e_n = x_n - a$ and $e_n^z = z_n - a$.

Proof. By Taylor formula, we have

$$\begin{aligned}
& f[x_n, z_{n-1}] + f[x_{n-1}, x_n] - f[x_{n-1}, z_{n-1}] \\
&= \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} + \frac{f(x_n) - f(z_{n-1})}{x_n - z_{n-1}} - \frac{f(z_{n-1}) - f(x_{n-1})}{z_{n-1} - x_{n-1}} \\
&= \frac{f(x_n) - f(x_{n-1})}{e_n - e_{n-1}} + \frac{f(x_n) - f(z_{n-1})}{e_n - e_{n-1}^z} - \frac{f(z_{n-1}) - f(x_{n-1})}{e_{n-1}^z - e_{n-1}} \\
&= f'(a)\left[\frac{e_n - e_{n-1} + c_2\left(e_n^2 - e_{n-1}^2\right) + c_3\left(e_n^3 - e_{n-1}^3\right) + \cdots}{e_n - e_{n-1}}\right. \\
&\quad + \frac{e_n - e_{n-1}^z + c_2\left(e_n^2 - \left(e_{n-1}^z\right)^2\right) + c_3\left(e_n^3 - \left(e_{n-1}^z\right)^3\right) + \cdots}{e_n - e_{n-1}^z} \\
&\quad \left. - \frac{e_{n-1}^z - e_{n-1} + c_2\left(\left(e_{n-1}^z\right)^2 - e_{n-1}^2\right) + c_3\left(\left(e_{n-1}^z\right)^3 - e_{n-1}^3\right) + \cdots}{e_{n-1}^z - e_{n-1}}\right] \\
&\sim f'(a)\left(1 - c_3 e_{n-1} e_{n-1}^z\right).
\end{aligned}$$

So,

$$\beta_n = \frac{1}{f[x_{n-1}, z_{n-1}] - f[x_n, z_{n-1}] - f[x_n, x_{n-1}]} \sim -\frac{1}{f'(a)} f\left(1 + c_3 e_{n-1} e_{n-1}^z\right).$$

Then, the proof can be completed.

Theorem 3.2 *Let* $f: D \to R$ *be a sufficiently differentiable function with simple root* $a \in D$, $D \subset R$ *be an open set,* x_0 *be close enough to* a, *then* (9) *achieve the convergence of order* 3.383.

Proof. If z_n converges to a with order $p > 1$ as:

$$e_n^z = C_n e_n^p + o\left(e_n^p\right),$$

and if x_n converges to a with order $r > 2$ as:

$$e_{n+1} = D_n e_n^r + o\left(e_n^r\right),$$

Then

$$e_n^z = C_n\left(D_{n-1}e_{n-1}^r\right)^p + o\left(e_{n-1}^{rp}\right) = C_n D_{n-1}^p e_{n-1}^{rp} + o\left(e_{n-1}^{rp}\right),$$

$$e_{n+1} = D_n\left(D_{n-1}e_{n-1}^r\right)^r + o\left(e_{n-1}^{r^2}\right) = D_n D_{n-1}^r e_{n-1}^{r^2} + o\left(e_{n-1}^{r^2}\right).$$

By Taylor formula and Lemma 3.1, we also have

$$\begin{aligned} e_n^z &= \left(1+\beta_n f\left[x_n,a\right]\right)e_n = -c_3 e_{n-1}C_{n-1}e_{n-1}^p D_{n-1}e_{n-1}^r + o\left(e_{n-1}^{r+p+1}\right) \\ &= --c_3 C_{n-1}D_{n-1}e_{n-1}^{r+p+1} + o\left(e_{n-1}^{r+p+1}\right). \end{aligned}$$

$$\begin{aligned} e_{n+1} &= e_n - \frac{f\left[x_n,a\right]e_n}{f\left[x_n,z_n\right]-\left(1+1/\left(\beta_n f\left[x_n,z_n\right]\right)\right)\left(f\left[z_{n-1},z_n\right]-f\left[z_{n-1},x_n\right]\right)} \\ &= e_n\frac{f\left[x_n,z_n\right]+\left(1+1/\left(\beta_n f\left[x_n,z_n\right]\right)\right)\left(f\left[z_{n-1},x_n\right]-f\left[z_{n-1},z_n\right]\right)-f\left[x_n,a\right]}{f\left[x_n,z_n\right]-\left(1+1/\left(\beta_n f\left[x_n,z_n\right]\right)\right)\left(f\left[z_{n-1},z_n\right]-f\left[z_{n-1},x_n\right]\right)} \\ &= e_n\frac{f\left[x_n,z_n,a\right]e_n^z+\left(1+1/\left(\beta_n f\left[x_n,z_n\right]\right)\right)f\left[z_{n-1},x_n,z_n\right]\left(-\beta_n f\left[x_n,a\right]\right)}{f\left[x_n,z_n\right]-\left(1+1/\left(\beta_n f\left[x_n,z_n\right]\right)\right)\left(f\left[z_{n-1},z_n\right]-f\left[z_{n-1},x_n\right]\right)} \\ &= e_n\frac{f\left[x_n,z_n,a\right]\left(1+\beta_n f\left[x_n,a\right]\right)e_n-\left(1+\beta_n f\left[x_n,z_n\right]\right)-\dfrac{f\left[x_n,a\right]}{f\left[x_n,z_n\right]}f\left[z_{n-1},x_n,z_n\right]e_n}{f\left[x_n,z_n\right]-\left(1+1/\left(\beta_n f\left[x_n,z_n\right]\right)\right)\left(f\left[z_{n-1},z_n\right]-f\left[z_{n-1},x_n\right]\right)} \\ &= e_n^2\left(1+\beta_n f\left[x_n,a\right]\right)\frac{f\left[x_n,z_n,a\right]f\left[x_n,z_n\right]-f\left[x_n,a\right]f\left[z_{n-1},x_n,z_n\right]}{f^2\left[x_n,z_n\right]-\left(1+1/\left(\beta_n f\left[x_n,z_n\right]\right)\right)f\left[x_n,z_n\right]\left(f\left[z_{n-1},z_n\right]-f\left[z_{n-1},x_n\right]\right)} \\ &= e_n^2\left(1+\beta_n f\left[x_n,a\right]\right)\frac{f^2\left[x_n,z_n,a\right]e_n^z-f\left[x_n,a\right]f\left[z_{n-1},x_n,z_n\right]e_{n-1}^z}{f^2\left[x_n,z_n\right]-\left(1+1/\left(\beta_n f\left[x_n,z_n\right]\right)\right)f\left[x_n,z_n\right]\left(f\left[z_{n-1},z_n\right]-f\left[z_{n-1},x_n\right]\right)} \\ &= e_n^2\left(-c_3 e_{n-1}e_{n-1}^z+\cdots\right)\frac{-f'(a)\dfrac{f'''(a)}{3!}e_{n-1}^z+\cdots}{f'^2(a)+\cdots} = c_3^2 C_{n-1}^2 D_{n-1}^2 e_{n-1}^{2r+2p+1} + o\left(e_{n-1}^{2r+2p+1}\right) \end{aligned}$$

.

So, comparing the exponents of e_{n-1} in expressions of e_n^z and e_{n+1} for (9), we obtain the same system of two equations:

$$\begin{cases} rp = r + p + 1, \\ r^2 = 2r + 2p + 1. \end{cases}$$

From its non-trivial solution $r \approx 3.383$ and $p \approx 1.839$, we prove that the convergence of (9) is of order 3.383.

As the efficiency index is $p^{1/w}$, without any additional function evaluations, the efficiency indices of (4), (5) and (9) are $\sqrt{1+\sqrt{2}} = 1.554$, $\sqrt{3} = 1.732$ and $\sqrt{3.383} = 1.839$, respectively.

4. Numerical Examples

Related one-step methods only using two function evaluations per iteration are showed in the following numerical examples. The proposed method is a derivative-free two-point method with high computational efficiency.

Example 1. The numerical results of NM, SM, (4), (5) and (9) in **Table 1** agree with the theoretical analysis. The computational order of convergence is defined by

$$\text{COC} = \frac{\log\left(|e_n|/|e_{n-1}|\right)}{\log\left(|e_{n-1}|/|e_{n-2}|\right)}.$$

Example 2. The numerical results of NM, SM, (4), (5) and (9) are in **Table 2** for the following nonlinear functions:

$$f_1(x) = 0.5\left(e^{x-2} - 1\right), \quad a = 2, \; x_0 = 2.5,$$

$$f_2(x) = e^{x^2} + \sin x - 1, \quad a = 0, \; x_0 = 0.25,$$

$$f_3(x) = e^{-x^2+x+2} - 1, \quad a = -1, \; x_0 = -0.85,$$

$$f_4(x) = e^{-x} - \arctan x - 1, \quad a = 0, \; x_0 = -0.2.$$

Table 1. $f(x) = x^2 - e^{-x} - 3x + 1, a = 0, x_0 = 0.2$.

Methods	n	1	2	3	4	5	6
NM	$\|x_n - a\|$	0.53279e−2	0.35561e−5	0.15808e−11	0.31235e−24	0.12195e−49	0.15890e−100
	COC	2.25256	2.01691	0.15808e−11	2.00000	2.00000	2.00000
SM	$\|x_n - a\|$	0.28174e−1	0.51325e−3	0.16476e−6	0.16966e−13	0.17989e−27	0.20226e−55
	COC	1.21776	2.04376	2.00830	2.00009	2.00000	2.00000
(4)	$\|x_n - a\|$	0.28174e−1	0.15996e−4	0.13132e−12	0.43283e−32	0.38442e−79	0.99936−193
	COC	1.21776	3.81335	2.49109	2.40945	2.41512	2.41406
(5)	$\|x_n - a\|$	0.28174e−1	0.16560e−6	0.11521e−21	0.39821e−67	0.16444e−203	0.11580e−612
	COC	1.21776	6.14536	2.89776	2.99925	3.00000	3.00000
(9)	$\|x_n - a\|$	0.28174e−1	0.43010e−7	0.21604e−27	0.23153e−94	0.20021e−321	0.69689e−1090
	COC	1.21776	6.83322	3.49004	3.29917	3.39052	3.38434

Table 2. Numerical results for solving $f_i(x), i = 1,2,3,4.$

Methods	NM	SM	(4)	(5)	(9)
$f_1 : \|e_6\|$	0.19785e−40	0.88156e−29	0.50439e−84	0.19314e−313	0.75162e−578
COC	2.0000	2.0000	2.4141	3.0000	3.3831
$f_2 : \|e_6\|$	0.32328e−44	0.42920e−26	0.19843e−85	0.57587e−282	0.13494e−706
COC	2.0000	2.0000	2.4141	3.0000	3.3825
$f_3 : \|e_6\|$	0.18813e−51	0.15758e−18	0.12013e−86	0.34524e−286	0.27679e−677
COC	2.0000	2.0000	2.4140	3.0000	3.3796
$f_4 : \|e_6\|$	0.35988e−79	0.96290e−84	0.16834e−248	0.21536e−597	0.25291e−1154
COC	2.0000	2.0000	2.4161	3.0000	3.3831

5. Conclusion

By theoretical analysis and numerical experiments, we confirm that the proposed method which is a derivative-free two-point method has high computational efficiency. Its convergence order is 3.383 and its efficiency index is 1.839. We can see that the suggested method is suitable to solve nonlinear equations and can also be used for solving boundary-value problems of nonlinear ordinary differential equations.

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