**Journal of Applied Mathematics and Physics, 2014, 2, 194-203**
Published Online April 2014 in SciRes. http://www.scirp.org/journal/jamp
http://dx.doi.org/10.4236/jamp.2014.25024

# Remarks on the Harnak Inequality for Local-Minima of Scalar Integral Functionals with General Growth Conditions

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Received September 2013

## Abstract

**In this paper we proof a Harnack inequality and a regularity theorem for local-minima of scalar intagral functionals with general growth conditions.**

## Keywords

**Harnack Inequality, Regularity, Hölder Continuity**

## 1. Introduction

In this paper we proof a Harnack inequality for local-minima of scalar intagral functionals of the calculus of variation of that type

$$J[u,\Omega]=\int_{\Omega} f\left(x,u(x),\nabla u(x)\right)dx \tag{1.1}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $\Phi:[0,+\infty)\to[0,+\infty)$ is a N-function and $\Phi$ globally satisfies the $\Delta'$-condition in $[0,+\infty)$, $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function and there exist $L_1, L_2\in(0,+\infty)$ $L_2$ and

$$\Phi(|z|)\le f(x,s,z)\le L_2\ \Phi(|z|)$$

for a. e. $x\in\Omega$ and for every $(s,z)\in\mathbb{R}\times\mathbb{R}^N$. The risearch of regularity results for elliptic and parabolic equations start from the basic and most important results of E. De Giorgi [5] and J. Nash [27]. In 1990s, beginning from the papers of G. Astarita and G. Marrucci [3] and J. P. Gosez [13] has been developed a remarkable production of regularity results for functionals with general growths. In [7], [8] and [25], M. Fuchs, G. Mingione, G. Seregin and F. Siepe have studied functionals of the type

$$J[u,\Omega]=\int\Omega\left|\nabla u(x)\right| ln\left(1+|\nabla u(x)|\right)dx \tag{1.2}$$

showing results of partial and global regularity for the minimizer of such functional in the scalar and vectorial case. Moreover in [8] M. Fuchs and G. Mingione, have already studied functionals of this type

$$J[u,\Omega]=\int_{\Omega}\Phi(|\nabla u|)dx. \tag{1.3}$$

**How to cite this paper:** Granucci, T. (2014) Remarks on the Harnak Inequality for Local-Minima of Scalar Integral Functionals with General Growth Conditions. *Journal of Applied Mathematics and Physics*, **2**, 194-203.
http://dx.doi.org/10.4236/jamp.2014.25024

In papers [7,8,25] the regularity of the minimizer of the functionals (1.2) and (1.3) has been obtained starting from the weak Eulero-Lagrange equations using the hypothesis: $\Phi \in C^2$. We remember that in [7,8,25] there are important estimations on the L^{∞} norm of the gradient of the minima both in the scalar case and in the vectorial one. In [24] E. Mascolo and G. Papi have determined an inequality of Harnack for the minimizer of the functional (1.3) under the condition $\Phi \in \Delta_2 \cap \nabla_2$. We observe that $\Phi \in \Delta_2 \cap \nabla_2$ implies

$$t^p - c_2 < \Phi(t) < c_3 \; t^m + c_4 \; for \; t > 0 \tag{1.4}$$

with real positive constants $c_1, c_2, c_3, c_4$ and $1 < p \le m$. Therefore the functional (1.3) satisfies non-standard growth conditions. Classical regularity theorem for functionals with standard growth conditions ($p = m$) has been proved in [9] and [10] (for a didactic explanation refer to [2,11,12]). In [26], G. Moscariello and L. Nania has obtain a results of hölder continuity for the local-minima of functional of the type (1.1) under the hypothesis that (1.4) holds with $1 < p \le m < ((Np)/(N-p))$. In [17], G. M. Lieberman proved an Harnack inequality for the local-minima of the functional (1.1) with $\Phi \in C^2$ suth that verifies the following relation

$$c_5 \le t\Phi''(t)/\Phi'(t) \le c_6 \; for \; t > 0$$

with $0 < c_5 < c_6$. We are interested in functionals with quasi-linear growths and we will proof a regularity result which extend the ones obtained in [17,24,26] to a wider N-functional class. In particular we get that the local-minima of the following functionals:

$$J[u,\Omega] = \int_\Omega |\nabla u|^p \ln(1 + |\nabla u|) dx \; with \; p > 1 \tag{1.5}$$

are holder continuous functions. In [14] and [15] we start to study the regularity of the local-minima introducing a maximal $L^\Phi$-$L^\infty$ inequality and estimating the measure of the level set A(k,R). Moreover in [15] and [16] we have shown that the following hypothesis can be used in order to give a new estimation of the measure of the livel set A(k,R):

H-1) $\Phi$ globally satisfies the $\Delta'$-condition in $[0,+\infty)$;

H-2) there exists a constant $c_{H_2} > 0$

$$\Phi(t)\Phi(1/t) \le c_{H_2} \; for \; every \; t \in (0,1); \tag{1.6}$$

H-3) there exists a constant $c_{H_3} > 0$

$$\Phi^{-1}(t) \le c_{H_3} t^{1/m} \; for \; every \; t \in (0,1). \tag{1.7}$$

Under these hypotheses we can show the following result.

Theorem 1: If $u \in W^1L^\Phi(\Omega)$ is a quasi-minima of the functional (1.1) and if $\Phi$ confirm the hypotheses H-1, H-2 and H-3; then u is locally hölder continuous.

In these pages we show the hypotheses H-2 and H-3 are purely technical and they can be eliminated. We can subsequently weaken besides H-1.

We will suppose that the following hypothesis hold.

G-1) Let $\varpi$: $\mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that

$$\Phi(\varepsilon t) \le c_G \varepsilon \varpi(\varepsilon) \Phi(t) \tag{1.8}$$

for every $t \in \mathbb{R}^+$ and for every $\varepsilon \in (0,1)$, where $c_G > 0$ is a real constant. Moreover we suppose that

$$\lim_{x \to 0^+} \varpi(s) = 0.$$

We say that $\Phi \in G$ if (1.8) holds. The hypothesis G-1 implicates a type of quasi-sub-homogeneity condition on the N-function $\Phi$.

Remark 1: We observe that if $\Phi \in \Delta_2 \cap \nabla_2$ then by Lemma 3 (i) we have

$$\Phi(\varepsilon t) = \varepsilon^r (1/\varepsilon^r) \Phi(\varepsilon t) \le \varepsilon^r \Phi(t).$$

Then the functions $\Phi \in \Delta_2 \cap \nabla_2$ verify the hypothesis G-1.

Remark 2: We observe that if $\Phi \in \Delta'$ on $(0,+\infty)$ then $\Phi$ verify the hypothesis G-1; in fact

$$\Phi(\varepsilon t) \le c\Phi(\varepsilon)\Phi(t).$$

Our principal results will be, a weak inequality of Harnack [Theorem 5] and the corollary of regularity that it follows of it [Corollary 2]. The proof of the Harnack inequality uses the techniques introduced in [6,17] and [24]. The only present novelty in the demonstrative technique is the use of an ε-Young inequality. This simple trick allows to recover the results introduced in [15-17,24,26] in a simple way and without using the properties of the functions $\Delta_2 \cap \nabla_2$ (see Lemma of [15,24] and [26]). We finally observe that the hypotheses $\Delta_2 \cap \nabla_2$ it is not, in general, equivalent to H-1; therefore the hypothesis G-1 seems to be slightly more general of those introduced in [15-17,24,26].

Definition 1: Let p be a real valued function defined on $[0,+\infty)$ and having the following properties: $p(0) = 0$, $p(t) > 0$ if $t > 0$, p is nondecreasing and right continuous on $(0,+\infty)$. Then the real valued function Φ defined on $[0,+\infty)$ by

$$\Phi(t) = \int_{[0,t]} p(s)\,ds \tag{1.9}$$

is called an N-function.

The function $\Phi: [0,+\infty)\rightarrow[0,+\infty)$ defined by (1.9) satisfies the following properties:

$$\Phi(0) = 0 \textit{ and } \Phi(t) > 0 \textit{ if } t > 0;$$

$$\Phi \textit{ is continuous on } [0,+\infty);$$

$$\Phi \textit{ is strictly increasing on } [0,+\infty);$$

$$\Phi \textit{ is convex on } [0,+\infty);$$

$$\lim_{x\to 0} \Phi(t)/t = 0 \textit{ and } \lim_{x\to\infty} \Phi(t)/t = +\infty;$$

$$\textit{if } s > t > 0, \textit{ then } \Phi(s)/s > \Phi(t)/t.$$

Definition 2: Let p be a real valued function defined on $[0,+\infty)$ and having the following properties: $p(0) = 0$, $p(t) > 0$ if $t > 0$, p is nondecreasing and right continuous on $(0,+\infty)$. We define

$$q(s) = sup_{p(t)\le s}(t)$$

and

$$\Psi(t) = \int_{[0,t]} q(s)\,ds. \tag{1.10}$$

The N-functions Φ and Ψ given by (1.9) and (1.10) are said to be complementary.

Particularly for us it will be important the following Lemma.

Lemma 1: Let Φ be an N-function, let Ψ be the complemantary N-function of Φ then we have

$$st \le \Phi(s) + \Psi(t) \tag{1.11}$$

$\forall s, t \in \mathbb{R}^+$. Moreover for every $\varepsilon > 0$ we get

$$st \le (1/\varepsilon)\Phi(\varepsilon s) + (1/\varepsilon)\Psi(t) \quad \forall s, t \in \mathbb{R}^+. \tag{1.12}$$

Definition 3: A N-function Φ is of class $\Delta_2$ globally in $(0,+\infty)$ if exists $k > 1$ such that

$$\Phi(2t) \le k\Phi(t) \quad \forall t \in (0,+\infty). \tag{1.13}$$

Definition 4: A N-function Φ is of class $\Delta_2$^{m} globally in $(0,+\infty)$, with $m>1$, if for every $\lambda>1$

$$\Phi(\lambda t) \le \lambda^m \Phi(t) \quad \forall t \in (0,+\infty). \tag{1.14}$$

The N-functions $\Phi \in \Delta_2^m$ are characterized by the following result

Lemma 2: Let Φ be a N-function and let $\Phi'_-$ be its left derivative. For $m > 1$ the following properties are equivalent:

1) $\Phi(\lambda t) \le \lambda^m\Phi(t)$, for every $t \ge 0$, for every $\lambda>1$;

2) $t\Phi'_-(t) \le m\Phi(t)$, for every $t \ge 0$;

3) the function $\Phi(t)/t^r$ is non-increasing on $(0,+\infty)$.

The N-functions $\Phi \in \nabla_2^r$ are characterized by the following result

Lemma 3: Let Φ be a N-function and let $\Phi'_-$ be its left derivative. For r>1 the following properties are equivalent:

1) $\Phi(\lambda t) \geq \lambda^r \Phi(t)$, for every $t \geq 0$, for every $\lambda > 1$;

2) $t\Phi'_-(t) \geq r\Phi(t)$, for every $t \geq 0$;

3) the function $\Phi(t)/\lambda^r$ is non-decreasing on $(0,+\infty)$.

Definition 5: We say that a N-function Φ belongs to the class $\Phi \in \nabla_2^r$ if any of the three condition (i)', (ii)' or (iii)' is satisfied.

Definition 6: We say that the N-function Φ satisfies the Δ′-condition if there exist positive constants—c and $t_0$—such that

$$\Phi(ts) \leq c_4 \ \Phi(t)\Phi(s) \tag{1.15}$$

for every $t,s \geq t_0$.

Definition 7: We say that the N-function Φ globally satisfies the Δ′-condition in $[0,+\infty)$ if (1.12) holds for every $t, s \geq 0$.

We remember that if $\Phi \in C^2$ then $\Phi \in \Delta'$ if $t\Phi''(t)/\Phi'(t)$ is a non-increasing function, for further details refer to Theorems 5.1 and 5.2 and to the Lemma 5.2 of [19].

Lemma 4: If the N-function Φ satisfies the Δ′-condition then it also satisfies the $\Delta_2$-condition

The N-functions

$$\Phi_1(t) = t^p with\ p > 1;$$

$$\Phi_2(t) = t^p \left(|ln(t)| + 1\right) \ with\ p > 1;$$

$$\Phi_3(t) = (1+t)ln(1+t) - t;$$

$$\Phi_4(t) = (t^2)/\left(1 + ln(1+t)\right).$$

satisfy the Δ′-condition. Moreover $\Phi_1$ and $\Phi_2$ satisfy the Δ′-condition globally in $[0,+\infty)$ and belong to the class $\nabla_2$ globally in $[0,+\infty)$. The function $\Phi_3$ does not satisfy Δ′-condition for all $t,s \geq 0$ and $\Phi_3 \notin \nabla_2$. Osserviamo inoltre che la funzione $\Phi_4 \in \nabla_2 \cap \Delta_2$ but $\Phi_4$ does not satisfy the Δ′-condition. For further details refer to [1,19,28]. Now we can introduce Orlicz spaces and Orlicz Sobolev Spaces, $L^\Phi$ and $W^1L^\Phi$; in these definitions and throughout the article we assume that Φ is a N-function of class $\Delta_2^m$ for some $m > 1$ and that $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary.

Definition 8: If u is a $L^N$-measurable function on Ω and: $\int_\Omega \Phi(|u|)dx < +\infty$ then $u \in L^\Phi(\Omega)$. Moreover

$$W^1L^\Phi(\Omega) = \left\{u \in L^\Phi(\Omega) : \partial_i u \in L^\Phi(\Omega)\ for\ i = 1,\dots,N\right\} \tag{1.16}$$

where $\partial_i u$, for I = 1,...,N, are the weak derivatives of u.

Theorem 2: $L^\Phi(\Omega)$ e $W^1L^\Phi(\Omega)$ are Banach spaces with the following norms

$$\|u\|_{\Phi,\Omega} = inf\left(k > 0 : \int_\Omega \Phi\left((|u|)/k\right)dx \leq 1\right) \tag{1.17}$$

and

$$\|u\|_{1,\Phi,\Omega} = \|u\|_{\Phi,\Omega} + \sum_{i=1,\dots,N} \|\partial_i u\|_{\Phi,\Omega}. \tag{1.18}$$

For greater details we refer to [1,19,28]. If $u \in W_{loc}^1L^\Phi(\Omega)$, k is a real number and $Q_R \Subset \Omega$, we set

$$A(k,R) = \left\{x \in Q_R : u(x) > k\right\} = \left\{u > k\right\} \cap Q_R,$$

$$B(k,R) = \left\{x \in Q_R : u(x) < k\right\} = \left\{u < k\right\} \cap Q_R.$$

Remark 3: For almost each $k \in \mathbb{R}$ we get $|A(k,R)| = |Q_R| - |B(k,R)|$.

Definition 9: If $u \in W_{loc}^1L^\Phi(\Omega)$, we say that $u \in ODG_\Phi^+(\Omega,H,\bar{R})$ if for every couple of concentric balls $Q_0 \subset Q_R \Subset Q_{R_0} \Subset \Omega$, with $R < R_0$, and for every $k \in \mathbb{R}$ we have

$$\int_{A(k,\bar{r})} \Phi(|\nabla u|)\ dx \leq H \int_{A(k,R)} \Phi\left((u-k)/(R-\bar{r})\right) dx \tag{1.19}$$

Definition 10: If $u\in W_{loc}^{1}L^{\Phi}(\Omega)$, we say that $u\in ODG_{\Phi}^{-}(\Omega,H,R_0)$ if for every couple of concentric balls $Q_{\varrho}\subset Q_R\Subset Q_{R_0}\Subset\Omega$, with $R < R_0$, and for every $k\in\mathbb{R}$ we have

$$\int_{B(k,\varrho)}\Phi(|\nabla u|)\,dx \le H\int_{B(k,R)}\Phi\big((k-u)/(R-\varrho)\big)dx \tag{1.20}$$

Definition 11: If $u\in W_{loc}^{1}L^{\Phi}(\Omega)$, we say that $u\in ODG\_\{\Phi\}(\Omega,H,R_0)$ if $u\in ODG\_\{\Phi\}\^\{\pm\}(\Omega,H,R_0)$, that is

$$ODG_{\Phi}(\Omega,H,R_0)=ODG_{\Phi}^{+}(\Omega,H,R_0)\cap ODG_{\Phi}^{-}(\Omega,H,R_0).$$

Theorem 3: If $u\in ODG_{\Phi}^{+}(\Omega,H,R_0)$ then u is locally bounded above on $\Omega$. Furthermore, for each $x_0\in\Omega$ and $R\le\min(R_0,d(x_0,\partial\Omega),1)$ there exists an universal constant $c_5 = c_5(N,m,H)$ such that

$$\Phi\big(ess-sup_{QR/2}(u_+(x))\big)\le(c_7/|Q_R|)\int_{QR}\Phi(u_+)dx$$

Proof: The proof follows using the demonstration methods presented in [24].

Corollary 1: If $u\in ODG_{\Phi}(\Omega,H,R_0)$ then u is locally bounded on $\Omega$. Furthermore, for each $x_0\in\Omega$ and $R\le\min(R_0,d(x_0,\partial\Omega),1)$ there exists an universal constant $c_6 = c_6(N,m,H)$ such that

$$\Phi\big(ess-sup_{QR/2}(|u(x)|)\big)\le(c_7/|Q_R|)\int_{QR}\Phi(|u|)dx.$$

Proof: The proof comes after Theorem 3 remembering that if $u\in DG_{\Phi}^{-}(\Omega,H,R_0)$ then $-u\in DG_{\Phi}^{+}(\Omega,H,R_0)$.

Moreover the following lemma is valid:

Lemma 5: If $u\in DG_{\Phi}^{+}(\Omega,H,R_0)$ then u is locally bounded above on $\Omega$. Furthermore, for each $x_0\in\Omega$, $R\le\min(R_0,d(x_0,\partial\Omega),1)$ and for every $p > 1$ there exists an universal constant $c_7 = c_7(p,N,m,H)$ such that

$$\Phi^{1/p}\big(ess-sup_{Q\varrho}|u|\big)\le\big(c_7/(R-\varrho)^N\big)\int_{QR}\Phi^{1/p}(|u|)dx \tag{1.21}$$

for each $Q_{\varrho}\Subset Q_R$ and $0<\varrho<R$.

Proof: The proof comes after Theorem 3 using the demonstration methods presented in [24].

Definition 12: Let $u\in W_{loc}^{1}L^{\Phi}(\Omega)$ then it is a local minima of (1.1) if for every $\phi\in W_0^{1}L^{\Phi}(\Omega)$ we have

$$J\big(u,supp(\phi)\big)\le J\big(u+\phi,supp(\phi)\big).$$

Moreover we get:

Theorem 4 (Caccioppoli inequalities): If $\Phi\in\Delta_2$ and $u\in W_{loc}^{1}L^{\Phi}(\Omega)$ is a local minima of (1.1) then $u\in ODG_{\Phi}(\Omega,H,R_0)$.

Using the previous results we obtain the following theorems:

Theorem 5 (Weak Harnack inequality): Let $\Phi$ be a N-function. Let u be a positive function satisfying (1,17). If $\Phi\in G$; then there exists $p > 1$ and a constant $c > 0$ such that

$$\Phi^{1/p}\big(ess-inf_{QR/2}(u)\big)\ge c\,(1/R^N)\int_{QR}\Phi^{1/p}(u)dx. \tag{1.22}$$

Theorem 6 (Main Theorem-Harnack inequality): Let $\Phi$ be a N-function. Let u be a positive local minimizer of (1.1). If $\Phi\in G$; then there exists a constant $c > 0$ such that, for $\sigma\in(0,1)$ we have

$$ess-sup_{Q\sigma R}(u)\le c\,ess-inf_{Q\sigma R}(u). \tag{1.23}$$

Proof (Proof of the Main Theorem): Using the (1.21) and (1.22) we have (1.23).

Corollary 2: Let $\Phi$ be a N-function. If $\Phi\in G$ and $u\in W^{1}L^{\Phi}(\Omega)$ is a local minimizer of the functional (1.1); then u is locally hölder continuous.

Proof: Using (1.20) and the technique introduced in [6,11,12] we get the proof.

We finish observing that with small changes our demonstrative technique can also be applied to the quasi-minima of the functional (1.1). Besides we can also apply this demonstration using equivalent N-functions. Unfortunately, $\Phi_2(t)=t\ln(1+t)$ does not verify H1; for this $\Phi_2\in\Delta'$ on $[t_0,+\infty)$ with $t_0>0$ but $\Phi_2\notin\Delta'$ globally on $[0,+\infty)$. We should think to solve this problem using the concept of equivalent N-function; the demonstrative technique allows it, but we do not know if it exists a N-function $\Phi_3$ equivalent to $\Phi_2$ which globally verifies $\Delta'$ globally on $[0,+\infty)$. It is still an unsolved problem. I thank the colleague Dott. Elisa Albano who translated the article into English supporting and encouraging me so much.

## 2. Proof of the Weak Harnack Inequality

### 2.1. Lemmata

Let define

$$v_R(y) = \left(\left(u(Ry)\right)/R\right),\ y \in Q_1$$

then we have the following Caccioppoli inequalities

$$\int_{A(k,\sigma,vR)} \Phi(|\nabla v_R|)\,dx \le H \int_{A(k,\tau,vR)} \Phi\left(\left(\left(v_R - k\right)/\left(\tau - \sigma\right)\right)\right)dx \tag{2.1}$$

and

$$\int_{B(k,\sigma,vR)} \Phi(|\nabla v_R|)\,dx \le H \int_{B(k,\tau,vR)} \Phi\left(\left(\left(k - v_R\right)/\left(\tau - \sigma\right)\right)\right)dx \tag{2.2}$$

where $0 < \sigma < \tau < 1$ and $k \in \mathbb{R}$.

Let us start remembering the following lemma:

Lemma 6: Let g(t), h(t) be a non-negative and increasing functions on $[0,+\infty)$ then $g(t)h(s) \le g(t)h(t) + g(s)h(s)$ for every $s, t \in [0,+\infty)$.

Proof: If $s \le t$ then $g(t)h(s) \le g(t)h(t) \le g(t)h(t) + g(s)h(s)$. If $t \le s$ then $g(t)h(s) \le g(s)h(s) \le g(t)h(t) + g(s)h(s)$.

Let us remember for the sake of completeness the following lemma:

Lemma 7: Let $\Phi \in \Delta_2$ and $u \in W^1L^{\Phi}(\Omega)$. Suppose that u is positive in $Q_R$ and satisfies (2.2) then there exists a positive constants $\delta_0$ such that if for some $\theta > 0$ we have $|B(\theta,u,R)| \le \delta_0|Q_R|$, then

$$inf_{QR/2}\{u\} \ge \left(\theta/2\right). \tag{2.3}$$

Proof: The proof follows using the demonstration methods presented in [24]. Refer to Lemma 4.1 of [24].

Our demonstration of the weak inequality of Harnack founds him on the following Lemma 8. We have shown the Lemma 8 using an opportune ε-Young inequality.

Lemma 8: Let be Φ a N-function and $\Phi \in G$. Let $u \in W^1L^{\Phi}(\Omega)$. Suppose that u satisfies (2.2). For every $\delta \in (0,1)$ and $T > (1/2)$, there exists a positive constant $\mu(\delta,T)$ such that if u is positive on $Q_{2TR}$ and there exists $\theta > 0$ such that $|B(\theta,u,R)| \le \delta|Q_R|$, we have

$$inf_{QTR}\{u\} \ge \mu\left(\delta,T\right)\theta. \tag{2.4}$$

Proof: Let $\delta \in (0,1)$. We first prove that if u is positive in $Q_R$ and there exists $\theta > 0$ such that $|B(\theta,u,R)| < \delta|Q_R|$, there exists a constant $\lambda(\theta)$ such that

$$inf_{QR/2}\{u\} \ge \lambda\left(\theta\right)\theta \tag{2.5}$$

We consider the function $w_R$ define by $w_R(y) = 0$ if $v_R(y) \ge k$, $w_R(y) = k - v_R$ if $k > v_R(y) > h$, $w_R(y) = k - h$ if $v_R(y) \le h$

where $v_R(y) = ((u(Ry))/R)$, $y \in Q_1$. Let us consider $k_i = (\theta/(2^iR))$ with $I \le \nu$, since $w_R = 0$ in $Q_1 \setminus B\left(k_i, v_R, 1\right)$ and

$$\left|Q_1 \setminus B\left(k_i, v_R, 1\right)\right| > \left(1 - \delta\right)\left|Q_R\right|$$

by Sobolev inequality we have

$$\Phi\left(k_i - k_{i-1}\right)\left|B\left(k_i, v_R, 1\right)\right| \le \left|B\left(k_i, v_R, 1\right)\right|^{1/N}\left[\int_{Q_1}\left(\Phi\left(w_R\right)\right)^{N/(N-1)}dx\right]^{(N-1)/N}$$

and

$$\Phi\left(k_i - k_{i-1}\right)\left|B\left(k_i, v_R, 1\right)\right| \le C_{SN}\left|B\left(k_i, v_R, 1\right)\right|^{1/N}\int_{\Delta i}\Phi'\left(w_R\right)\left|\nabla w_R\right|dx \tag{2.6}$$

where $\Delta_i = B(k_i,v_R,1) \setminus B(k_{i-1},v_R,1)$. Using the Young inequality $ab \le \Psi(a) + \Phi(b)$, where Φ is the complementary function of Φ, we have

$$\int_{\Delta i}\Phi\left(w_R\right)\left|\nabla w_R\right|dx = \left(m/\varepsilon\right)\int_{\Delta i}\left(\Phi'\left(w_R/m\right)\varepsilon\left|\nabla w_R\right|dx\right.$$

and

$$(m/\varepsilon)\int_{\Delta i}(\Phi'(w_R/m)\varepsilon|\nabla w_R|dx \le (m/\varepsilon)\int_{\Delta i}\Psi(\Phi'(w_R)/m)+\Phi(\varepsilon|\nabla w_R|)dx$$

then

$$\int_{\Delta i}\Phi(w_R)|\nabla w_R|dx \le (m/\varepsilon)\int_{\Delta i}\Psi(\Phi'(w_R)/m)+\Phi(\varepsilon|\nabla w_R|)dx. \tag{2.7}$$

Since

$$\Psi(\Phi'(w_R)/m) \le \Psi(w_R\Phi'(w_R)/(mw_{R)}) \le \Psi(\Phi(w_R)/w_R)$$

from the inequality

$$\Psi(\Phi(t)/t)) < \Phi(t)$$

(see inequality (6), page 230 of [1]) we have

$$\int_{\Delta i}\Phi(w_R)|\nabla w_R|dx \le (m/\varepsilon)\int_{\Delta i}\Phi(w_R)+\Phi(\varepsilon|\nabla w_R|)dx,$$

then

$$\Phi(k_i-k_{i-1})|B(k_i,v_R,1)| \le C_{SN}|B(k_i,v_R,1)|^{1/N}(m/\varepsilon)\int_{\Delta i}\Phi(w_R)+\Phi(\varepsilon|\nabla w_R|)dx.$$

Moreover, since $\Phi$ globally satisfies the $\Delta'$-condition in $[0,+\infty)$, it follows

$$\Phi(k_i-k_{i-1})|B(k_i,v_R,1)| \le C_{SN}|B(k_i,v_R,1)|^{1/N}\left[(m/\varepsilon)\Phi(k_i-k_{i-1})|\Delta_i|+(mc_1\ c_G\varpi(\varepsilon))\int_{\Delta i}\Phi(|\nabla w_R|)dx\right]$$

since

$$\int_{\Delta i}\Phi(|\nabla w_R|)dx = \int_{\Delta i}\Phi(|\nabla v_R|)dx$$

using Caccioppoli's inequality (2.2) we have

$$|B(k_i,v_R,1)||^{1-1/N} \le C_{SN}(m/\varepsilon)|\Delta_{i|}+C_{SN}mc_2\ \ \varpi(\varepsilon)|Q_2|.$$

Summing the last inequality on i from 0 to ν we have

$$(1+\nu)|B(k_i,v_R,1)|^{1-1/N} \le C_{SN}(m/\varepsilon)|Q_1|+C_{SN}mc_2\ \ \varpi(\varepsilon)|Q_2|(1+\nu)$$

and

$$|B(k_i,v_R,1)|^{1-1/N} \le C_{SN}(m/(\varepsilon(1+\nu)))|Q_1|+C_{SN}mc_2\ \ \varpi(\varepsilon)|Q_2|.$$

Fix $\varepsilon = (1/(1+\nu)^{1/2})$, then

$$|B(k_i,v_R,1)|^{1-1/N} \le C_{SN}\left(m/(1+\nu)^{1/2}\right)|Q_1|+C_{SN}mc_2\ \ \varpi\left(1/(1+\nu)^{1/2}\right)|Q_1|$$

and

$$|B(k_i,v_R,1)|^{1-1/N} \le C_{SN}m\left(1/(1+\nu)^{1/2}+c_2\ \ \varpi\left(1/(1+\nu)^{1/2}\right)\right)|Q_1|^{1-1/N} \tag{2.9}$$

From (2.9) we have

$$|B(k_i,v_R,1)| \le (C_{SN}m)^{N/(N-1)}(1/(1+\nu)^{1/2}+c_2\ \ \varpi\left(1/(1+\nu)^{1/2}\right)^{N/(N-1)}|Q_1|.$$

Since $\varpi(s)\downarrow 0$ for $s\downarrow 0$ then we can choose ν such that

$$(C_{SN}m)^{N/(N-1)}(1/(1+\nu)^{1/2}+c_2\ \ \varpi\left(1/(1+\nu)^{1/2}\right)^{N/(N-1)} \le (1/2)(\delta_0)^{(N-1)/N}$$

where $\delta_0$ is the constant in Lemma 7, then there exists $\lambda(\delta_0)$ such that

$$inf_{QR/2}\{u\} \ge \lambda(\delta)\theta.$$

Let now $T > (1/2)$ and assume $|B(\theta,u,R)| \le \delta|Q_R|$ and u positive in $Q_{2R}$. Since

$$\left|A(\theta,u,2TR)\right| \ge (1-\gamma)\left|Q_R\right| = \left((1-\delta)/(2T)^n\right)\left|Q_{2TR}\right|$$

we have

$$\left|B(\theta,u,2TR)\right| \le \left(1-(1-\delta)/(2T)^n\right)\left|Q_{2TR}\right|$$

then there exists a constant depending on $\delta$ and T such that (2.4) holds.

Using the technique introduces in [11] we get the following lemma.

Lemma 9: Let $u \in DG_\Phi^-$ with $k_0 = 0$ and let u be positive in $Q_2$. Let $\delta \in (0,1)$ and $t > 0$. If

$$\left|\{x \in Q_1 : u(x) > t\}\right| \ge 2^{-s}\left|Q_1\right|$$

then

$$inf_{Q1/2}\{u\} > c^s t$$

where $c = c(\delta)$ being as in Lemma 8 with $\delta = 2^{-N-1}$.

Proof: For s = 0 the claim is true by Lemma 8. Now we use the inductive process. We assume the claim true for some s and we prove it for s + 1. Let us define $A_i = \{x \in Q_1 : u(x) > c^i t\}$; by hipothesis, if $A_0 = \{x \in Q_1 : u(x) > t\}$ then

$$\left|A_0\right| > \left(1/2^{s+1}\right)\left|Q_1\right|.$$

We have two alternative.

1) We assume $|A_0| > 2^{-s}|Q_1|$, the by inductive hypothesis

$$inf_{Q1/2}\{u\} > c^s t > c^{s+1} t$$

2) Otherwise $2^{-s-1}|Q_1| < |A_0| < 2^{-s}|Q_1|$. Let us assume $g = \chi\_\{A_0\}$ and apply the Calderon-Zygmund argument to g in $Q_1$ with parameter (1/2) then we find a sequence of dyadic cubes $\{Q_j\}$ such that

$$(1/2) < \left(1/|Q_j|\right)\int Q_j g\,dx < 2^{N-1};$$

$$g < (1/2)\ in\ Q_1 \setminus \bigcup_j Q_j;$$

if $Q_j$ is one of the $2^N$ subcubes of $P_i$ arising during the Calderon-Zygmund process, then

$$\left(1/|P_i|\right)\int_{Pi} g\,dx \le (1/2).$$

From (2) and (3) we get

$$\left|A_0\right| = \left|A_0 \cap \bigcup_i P_i\right| = \sum_i \left|A_0 \cap P_i\right| \le (1/2)\sum_i \left|P_i\right|;$$

moreover

$$\left|A_0 \cap P_i\right| \ge \left|A_0 \cap Q_{j}\right| \ge (1/2)\left|Q_{j}\right| \ge \left(1/2^{N+1}\right)\left|P_i\right|.$$

We apply Lemma 8 and we obtain

$$inf_{Pi}\{u\} \ge ct$$

Let us consider

$$A_1 = \{x \in Q_1 : u(x) > ct\}$$

then $P_i \subset A_1$ and

$$\left|Q_1\right| 2^{-s-1} < \left|A_0\right| < (1/2)\left|A_1\right|$$

by inductive hypotesis

$$inf_{Q1/2}\{u\} > c^{s+1} t$$

## 2.2. Proof of the Weak Harnack Inequality

Now we can proof the inequality (1.19) using the technique introduced by Di Benedetto-Trudinger in [6].

Proof (Proof of Theorem 5); Given any t > 0 choose an integer s such that

$$\lambda_t = \left|\left\{x \in Q_R : u(x) > t\right\}\right| \geq 2^{-s}\left|Q_R\right|$$

*i.e.*

$$s \geq ln\left(\lambda_t / Q_R|\right) / ln\left(1/2\right);$$

then by Lemma 9 we get

$$ess - inf_{QR/2}\left\{u\right\} > c^s t$$

therefore

$$u(x) \geq t\left(\lambda_t / |Q_R|\right)^{ln(c)/ln(1/2)}.$$

Let us define

$$\xi = ess - inf_{QR/2}\left\{u\right\}$$

then

$$\lambda_t \leq \left(\xi^\alpha / t^\alpha\right)\left|Q_R\right|$$

where $\alpha = \ln((1/2)/\ln(c)$. Since $\Phi'(t)t \leq m\Phi(t)$ for $p > \max\{1,(m/\alpha)\}$ we have

$$\int_{QR} \Phi^{1/p}(u)dx = (1/p)\int_{[0,+\infty]} \Phi^{1/p-1}(t)\Phi(t)\lambda_t dt \leq (1/p)\Phi^{1/p}(\xi)\left|Q_R\right| + (m/p)\left|Q_R\right|\xi^\alpha \int_{[\xi,+\infty]} \Phi^{1/p}(t)/t^{\alpha+1}dt$$

Integrating by parts, we have

$$\int_{[\xi,+\infty]} \Phi^{1/p}(t)/t^{\alpha+1}dt \leq \left(1/\left(\alpha\left[1-\left(m/(p\alpha)\right)\right]\right)\right)\Phi^{1/p}(\xi)\xi^{-\alpha}$$

hence

$$\left(1/|Q_R|\right)\int_{QR} \Phi^{1/p}(u)dx \leq c\Phi^{1/p}\left(ess - inf_{QR/2}\{u\}\right).$$

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