

Positive Solutions for Fractional Differential Equations with Multi-Point Boundary Value Problems

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Abstract

In this paper, a fractional multi-point boundary value problem is considered. By using the fixed point index theory and Krein-Rutman theorem, some results on existence are obtained.

Keywords

Caputo Fractional Derivative, Fractional Integral, Boundary Value Problem, Fixed Point Index Theory

1. Introduction

Fractional differential equations have been of great interest recently. This is due to the intensive development of the theory of fractional calculus itself as well as its applications. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, dynamical processes in self similar and porous structures, electrical networks, visco-elasticity, chemical physics, and many other branches of science. For details, see [1]-[7].

It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations on terms of special functions. Recently, there are some papers dealing with the existence and multiplicity of solution to the nonlinear fractional differential equations boundary value problems, see [8]-[14].

Zhao [11] investigated the existence and uniqueness of positive solutions for a local boundary value problem of fractional differential equation.

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0 & 0 < t < 1 \\ u'(0) - \beta u(\xi) = 0, u'(1) + \gamma u(\eta) = 0 \end{cases}$$

where α is a real number with $1 < \alpha \leq 2, 0 \leq \beta, \gamma \leq 1, 0 \leq \xi \leq \eta \leq 1$, D_{0+}^{α} is the Caputo's derivative.

Inspired by above work, we will consider the fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0 & 0 < t < 1 \\ u'(0) - \beta u(\xi) = 0, u'(1) + \sum_{i=1}^{m-3} \gamma_i u(\eta_i) = 0 \end{cases} \quad (1)$$

where α is a real number with $1 < \alpha \leq 2, 0 \leq \beta \leq 1, 0 \leq \gamma_i \leq 1, i = 1, 2, \dots, m-3, 0 \leq \xi < \eta_1 < \eta_2 < \dots < \eta_{m-3} \leq 1$, D_{0+}^{α} is the Caputo's derivative.

$$\text{Let } \Delta = \beta(1 + \sum_{i=1}^{m-3} \gamma_i \eta_i) + \sum_{i=1}^{m-3} \gamma_i (1 - \beta \xi).$$

Now we list some conditions for convenience.

$$(H1) \quad (\alpha - 1)(1 - \beta \xi) > \Delta, 1 + \sum_{i=1}^{m-3} \gamma_i (\eta_i - 1) > 0$$

(H2) $f : [0, 1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfied Carathéodory condition, that is $f(\cdot, u)$ is measurable for each fixed $u \in \mathbb{R}^+$ and $f(t, \cdot)$ is continuous for a.e. $t \in [0, 1]$. For any $r > 0$, there existed $\Phi(t) \in L^1[0, 1]$ such that $f(t, u) \leq \Phi(t)$, where $u \in [0, r]$; a.e. $t \in [0, 1]$.

(H3) $\forall l > 0, \exists L > 0, |f(t, u)| < L$ where $u \in [0, l]$; a.e. $t \in [0, 1]$.

2. Preliminary

For the convenience of readers, we provide some background material in this section. **Definition 2.1**[7] The Riemann-Liouville fractional of order α for function y is defined as

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds$$

Definition 2.2 [7] The Caputo's derivative for function y is defined as

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds$$

Lemma 2.1 [14] Let $\alpha > 0$, then the fractional differential equation $D_{0+}^{\alpha} u(t) = 0$

has solutions $u(t) = c_1 + c_2 t + c_3 t^2 + \dots + c_n t^{n-1}, c_i \in \mathbb{R}, i = 1, 2, \dots, n, n = [\alpha] + 1$

Lemma 2.2 [14] Let $\alpha > 0$, then for some $c_i \in \mathbb{R}, i = 1, 2, \dots, n, n = [\alpha] + 1$

Lemma 2.3 (Krein-Rutman) [15] Let K be a reproducing cone in a real Banach space X and let $L : X \rightarrow X$ be a compact linear operator with $L(K) \subseteq K$. $r(L)$ is the spectral radius of L . If $r(L) > 0$, then there exists $\varphi_1 \in K \setminus \{0\}$ such that $L\varphi_1 = r(L)\varphi_1$.

Lemma 2.4 [16] Let X is a Banach space, P be a cone in X and $\Omega(P)$ be a bounded open subset in P .

Suppose that $A : \overline{\Omega(P)} \rightarrow P$ is a completely continuous operator. Then the following results hold:

(1) If there exists $u_0 \in P \setminus \{0\}$ such that $u \neq Au + \lambda u_0, \forall u \in \partial\Omega(P), \lambda > 0$ then the fixed point index $i(A, \Omega(P), P) = 0$.

(2) If $0 \in \Omega(P)$ and $Au \neq \lambda u, \forall u \in \partial\Omega(P), \lambda \geq 1$, then the fixed point index $i(A, \Omega(P), P) = 1$.

Take $X = C[0, 1]$ with norm $\|x(t)\| = \max_{t \in [0, 1]} |x(t)|$, $Y = L^1[0, 1]$ with norm $\|x(t)\|_1 = \int_0^1 |x(t)| dt$.

$K = \{u \in X \mid u(t) \geq 0, t \in [0, 1]\}$ Obviously K is a reproducing cone of X .

Lemma 2.5 If $y \in C[0, 1], 1 < \alpha \leq 2$, then the unique solution of

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0 & 0 < t < 1 \\ u'(0) - \beta u(\xi) = 0, u'(1) + \sum_{i=1}^{m-3} \gamma_i u(\eta_i) = 0 \end{cases}$$

is $u(t) = \int_0^1 G(t, s) y(s) ds$, where

$$G(t,s) = \begin{cases} -\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} + \frac{\beta[1 + \sum_{i=1}^{m-3} \gamma_i(\eta_i - t)]}{\Delta\Gamma(\alpha)}(\xi - s)^{\alpha-1} \\ + \frac{1 - \beta\xi + \beta t}{\Delta\Gamma(\alpha-1)}(1-s)^{\alpha-2} + \frac{(1 - \beta\xi + \beta t)}{\Delta\Gamma(\alpha)} \sum_{i=1}^{m-3} \gamma_i(\eta_i - s)^{\alpha-1} & s \leq \xi, s \leq t \\ \frac{\beta[1 + \sum_{i=1}^{m-3} \gamma_i(\eta_i - t)]}{\Delta\Gamma(\alpha)}(\xi - s)^{\alpha-1} + \frac{1 - \beta\xi + \beta t}{\Delta\Gamma(\alpha-1)}(1-s)^{\alpha-2} \\ + \frac{(1 - \beta\xi + \beta t)}{\Delta\Gamma(\alpha)} \sum_{i=1}^{m-3} \gamma_i(\eta_i - s)^{\alpha-1} & s \leq \xi, t \leq s \\ -\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} + \frac{1 - \beta\xi + \beta t}{\Delta\Gamma(\alpha-1)}(1-s)^{\alpha-2} \\ + \frac{(1 - \beta\xi + \beta t)}{\Delta\Gamma(\alpha)} \sum_{i=1}^{m-3} \gamma_i(\eta_i - s)^{\alpha-1} & \xi \leq s \leq \eta_1, s \leq t \\ \frac{1 - \beta\xi + \beta t}{\Delta\Gamma(\alpha-1)}(1-s)^{\alpha-2} + \frac{(1 - \beta\xi + \beta t)}{\Delta\Gamma(\alpha)} \sum_{i=1}^{m-3} \gamma_i(\eta_i - s)^{\alpha-1} & \xi \leq s \leq \eta_1, t \leq s \\ -\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} + \frac{1 - \beta\xi + \beta t}{\Delta\Gamma(\alpha-1)}(1-s)^{\alpha-2} \\ + \frac{(1 - \beta\xi + \beta t)}{\Delta\Gamma(\alpha)} \sum_{j=i}^{m-3} \gamma_j(\eta_j - s)^{\alpha-1} & \eta_{i-1} \leq s \leq \eta_i, s \leq t \\ \frac{1 - \beta\xi + \beta t}{\Delta\Gamma(\alpha-1)}(1-s)^{\alpha-2} + \frac{(1 - \beta\xi + \beta t)}{\Delta\Gamma(\alpha)} \sum_{j=i}^{m-3} \gamma_j(\eta_j - s)^{\alpha-1} & \eta_{i-1} \leq s \leq \eta_i, t \leq s \end{cases}$$

Proof: The equation $D_{0+}^\alpha u(t) + y(t) = 0$ has a unique solution

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + c_1 + c_2 t$$

where $c_1, c_2 \in R$. By $u'(0) - \beta u(\xi) = 0, u'(1) + \sum_{i=1}^{m-3} \gamma_i u(\eta_i) = 0$, we have

$$c_1 = \frac{1}{\Delta} [\beta(1 + \sum_{i=1}^{m-3} \gamma_i \eta_i) I_{0+}^\alpha y(\xi) + (1 - \beta\xi) I_{0+}^{\alpha-1} y(1) + (1 - \beta\xi) \sum_{i=1}^{m-3} \gamma_i I_{0+}^\alpha y(\eta_i)]$$

$$c_2 = \frac{1}{\Delta} [\beta I_{0+}^{\alpha-1} y(1) + \beta \sum_{i=1}^{m-3} \gamma_i I_{0+}^\alpha y(\eta_i) - \beta \sum_{i=1}^{m-3} \gamma_i I_{0+}^\alpha y(\xi)]$$

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{\beta[1 + \sum_{i=1}^{m-3} \gamma_i(\eta_i - t)]}{\Delta\Gamma(\alpha)} \int_0^t (\xi - s)^{\alpha-1} y(s) ds + \frac{1 - \beta\xi + \beta t}{\Delta\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) ds \\ + \frac{(1 - \beta\xi + \beta t)}{\Delta\Gamma(\alpha)} \sum_{i=1}^{m-3} \gamma_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} y(s) ds$$

The proof is complete.

Lemma 2.6 If (H1) hold, then there exist a constant M such that $0 \leq G(t,s) \leq M(1-s)^{\alpha-2}, t, s \in [0,1]$

Proof: Obviously $G(t,s) \geq 0$,

$$\begin{aligned} \max_{0 \leq t \leq 1} G(t, s) &\leq \frac{\beta[1 + \sum_{i=1}^{m-3} \gamma_i(\eta_i - t)]}{\Delta\Gamma(\alpha)} (\xi - s)^{\alpha-1} + \frac{1 - \beta\xi + \beta t}{\Delta\Gamma(\alpha - 1)} (1 - s)^{\alpha-2} + \frac{(1 - \beta\xi + \beta t)}{\Delta\Gamma(\alpha)} \sum_{i=1}^{m-3} \gamma_i(\eta_i - s)^{\alpha-1} \\ &\leq \frac{\beta[1 + \sum_{i=1}^{m-3} \gamma_i\eta_i]}{\Delta\Gamma(\alpha)} (1 - s)^{\alpha-2} + \frac{1 - \beta\xi + \beta}{\Delta\Gamma(\alpha - 1)} (1 - s)^{\alpha-2} + \frac{(1 - \beta\xi + \beta)}{\Delta\Gamma(\alpha)} \sum_{i=1}^{m-3} \gamma_i (1 - s)^{\alpha-2} \\ &= \frac{\beta(1 + \sum_{i=1}^{m-3} \gamma_i\eta_i) + (\alpha - 1)(1 - \beta\xi + \beta) + \sum_{i=1}^{m-3} \gamma_i(1 - \beta\xi + \beta)}{\Delta\Gamma(\alpha)} (1 - s)^{\alpha-2} \end{aligned}$$

Let $M = \frac{\beta(1 + \sum_{i=1}^{m-3} \gamma_i\eta_i) + (\alpha - 1)(1 - \beta\xi + \beta) + \sum_{i=1}^{m-3} \gamma_i(1 - \beta\xi + \beta)}{\Delta\Gamma(\alpha)}$. The proof is completed.

Define an operator $A : K \rightarrow K$ and a linear operator $T : X \rightarrow X$ as follows:

$$Au(t) = \int_0^1 G(t, s) f(s, u(s)) ds$$

$$Tu(t) = \int_0^1 G(t, s) u(s) ds$$

Then the fixed point $u \in K \setminus \{0\}$ of A is the positive solutions of (1)

3. Main Results

In order to obtain our main results, we firstly present and prove some lemmas.

Lemma 3.1 If (H1)-(H3) hold, then $A : K \rightarrow K$ and $T : X \rightarrow X$ are completely continuous.

Proof: According to the Lebesgue Dominated Convergence Theorem and **Lemma 2.6**, we have $A : K \rightarrow K$ is uniformly bounded and equicontinuous. It follows from Ascoli-Arzela theorem that $A : K \rightarrow K$ is completely continuous.

By the same method, we can get that $T : X \rightarrow X$ is completely continuous also.

Lemma 3.2 If (H1)-(H2) hold, then $r(T) > 0$ (r is the spectral radius of T)

Proof: Take $u(t) \equiv 1$

$$\begin{aligned} Tu(t) &= \int_0^1 G(t, s) u(s) ds \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} ds + \frac{\beta[1 + \sum_{i=1}^{m-3} \gamma_i(\eta_i - t)]}{\Delta\Gamma(\alpha)} \int_0^t (\xi - s)^{\alpha-1} ds + \frac{1 - \beta\xi + \beta t}{\Delta\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha-2} ds \\ &\quad + \frac{(1 - \beta\xi + \beta t)}{\Delta\Gamma(\alpha)} \sum_{i=1}^{m-3} \gamma_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} ds \\ &= -\frac{1}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha} + \frac{\beta[1 + \sum_{i=1}^{m-3} \gamma_i(\eta_i - t)]}{\Delta\Gamma(\alpha)} \frac{\xi^\alpha}{\alpha} + \frac{1 - \beta\xi + \beta t}{\Delta\Gamma(\alpha - 1)} \frac{1}{\alpha - 1} + \frac{(1 - \beta\xi + \beta t)}{\Delta\Gamma(\alpha)} \frac{\sum_{i=1}^{m-3} \gamma_i \eta_i^\alpha}{\alpha} \\ &\geq -\frac{1}{\Gamma(\alpha + 1)} + \frac{1 - \beta\xi}{\Delta\Gamma(\alpha)} + \frac{(1 - \beta\xi)}{\Delta\Gamma(\alpha + 1)} \sum_{i=1}^{m-3} \gamma_i \eta_i^\alpha := l \end{aligned}$$

$$T^2u(t) = T(Tu(t)) \geq T(l) \geq l^2, T^n(u(t)) \geq l^n$$

$$\|T^n\|^{\frac{1}{n}} \geq l, r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} \geq l > 0$$

The proof is completed.

By **Lemma 2.3**, we can get there exists $\varphi_0 \in K \setminus \{0\}$ such that $T\varphi_0 = r(T)\varphi_0$

Define $f_0 = \liminf_{u \rightarrow 0} \inf_{t \in [0,1] \setminus E} \frac{f(t,u)}{u}$, $f_\infty = \limsup_{u \rightarrow \infty} \sup_{t \in [0,1] \setminus E} \frac{f(t,u)}{u}$, where $E \subset [0,1]$, with $m(E) = 0$ ($m(E)$ is the lebesgue measure of E and the same as follows).

Set $K_\rho = \{u \in K \mid \|u\| < \rho\}$, $\mu = \frac{1}{r(T)}$.

Lemma 3.3 Suppose $\mu < f_0 \leq \infty$, then there exists $\rho_0 > 0$ such that for $\rho \in (0, \rho_0]$, if $u \neq Au, \forall u \in \partial K_\rho$, then $i(A, K_\rho, K) = 0$.

Proof: It follows from $\mu < f_0$ that there exists $\varepsilon > 0$ and $\rho_0 > 0$ such that for a.e. $t \in [0,1], 0 \leq u \leq \rho_0$

$$f(t, u) \geq (\mu + \varepsilon)u \quad (2)$$

For $0 < \rho \leq \rho_0$, assume $u \neq Au, \forall u \in \partial K_\rho$. By **Lemma 2.4**, we need only to prove that

$$u \neq Au + \lambda \varphi_0, \forall u \in \partial K_\rho, \lambda > 0$$

where $\varphi_0 \in K \setminus \{0\}, T\varphi_0 = r(T)\varphi_0$.

Otherwise, there exists $u_0 \in \partial K_\rho, \lambda_0 > 0$ such that

$$u_0 = Au_0 + \lambda_0 \varphi_0 \quad (3)$$

Then $u_0 \geq Au_0, u_0 \geq \lambda_0 \varphi_0$, by (2), we can get that

$$Au_0 = \int_0^1 G(t, s) f(s, u_0(s)) ds \geq (\mu + \varepsilon)Tu_0 \quad (4)$$

Considering $u_0 \geq \lambda_0 \varphi_0$, we get $Au_0 \geq (\mu + \varepsilon)\lambda_0 T\varphi_0 > \lambda_0 \varphi_0$.

This together with (3) means that $u_0 \geq 2\lambda_0 \varphi_0$, by (4) we get $Au_0 \geq 2\lambda_0 \varphi_0$. So $u_0 \geq 3\lambda_0 \varphi_0$.

Repeating this process, we get that $u_0 \geq n\lambda_0 \varphi_0$, so we have $\|u_0\| \geq n\lambda_0 \|\varphi_0\| \rightarrow \infty, n \rightarrow \infty$. This is a contradiction.

It follows from **Lemma 2.4** that $i(A, K_\rho, K) = 0, \rho \in (0, \rho_0]$. The proof is completed.

Lemma 3.4 Suppose $0 \leq f_\infty < \mu$, then there exists $r_0 > 0$ such that $i(A, K_r, K) = 1$, for each $r > r_0$

Proof: Let $\varepsilon > 0$ satisfy $f_\infty < \mu - \varepsilon$, then there exists $r_1 > 0$ such that

$$f(t, u) \leq (\mu - \varepsilon)u, \text{ for } u > r_1, \text{ a.e. } t \in [0, 1]$$

By (H3), there existed $\Phi(t) \in L^1[0,1]$ such that $f(t, u) \leq \Phi(t)$, where $u \in [0, r_1]; \text{ a.e. } t \in [0, 1]$.

Thus, for all $u \in R^+ \text{ a.e. } t \in [0, 1]$

$$f(t, u) \leq (\mu - \varepsilon)u + \Phi(t) \quad (5)$$

Since $\mu = \frac{1}{r(T)}$, $(\frac{I}{\mu - \varepsilon} - T)^{-1}$ exists. Let $C = \left\| \int_0^1 G(t, s) \Phi(s) ds \right\|, r_0 = \left\| (\frac{I}{\mu - \varepsilon} - T)^{-1} \frac{C}{\mu - \varepsilon} \right\|$

Take $r > r_0$, we will show $Au \neq \lambda u$, for each $u \in \partial K_r, \lambda \geq 1$.

Otherwise, there exist $u_0 \in \partial K_r, \lambda_0 \geq 1$, such that $Au_0 = \lambda_0 u_0$. This together with (5), implies

$u_0 \leq \lambda_0 u_0 = Au_0 \leq (\mu - \varepsilon)Tu_0 + C$ Then $(\frac{I}{\mu - \varepsilon} - T)u_0(t) \leq \frac{C}{\mu - \varepsilon}$. So we get $\frac{C}{\mu - \varepsilon} - (\frac{I}{\mu - \varepsilon} - T)u_0(t) \in K$. It

follows from $(\frac{I}{\mu - \varepsilon} - T)^{-1} = \sum_{n=0}^{\infty} (\mu - \varepsilon)^{n+1} T^n$ and $T(K) \subset K$, we get $u_0(t) \leq (\frac{I}{\mu - \varepsilon} - T)^{-1} \frac{C}{\mu - \varepsilon}$. Therefore,

we have $\|u_0\| \leq r_0 \leq r$. This is a contradiction.

By **Lemma 2.4** (2), we get $i(A, K_r, K) = 1$, for each $r > r_0$. The proof is completed.

Theorem 3.1 Suppose $\mu < f_0 \leq \infty$ and $0 \leq f_\infty < \mu$ then (1) has at least one positive solution.

Proof: It follows from $0 \leq f_\infty < \mu$ and **Lemma 3.4** there exists $r > 0$ such that $i(A, K_r, K) = 1$.

By $\mu < f_0 < \infty$ and **Lemma 3.3**, we can get there exists $0 < \rho < r$ such that either there exists $u \in \partial K$ with $u = Au$ or $i(A, K_\rho, K) = 0$. In the second case, A has a fixed point $u \in K$ with $\rho \leq \|u\| < r$ by the properties of index. The proof is completed.

4. Example

Let's consider the following boundary value problem

$$\begin{cases} D_{0+}^{\frac{3}{2}} u(t) + f(t, u(t)) = 0 & 0 < t < 1 \\ u'(0) - \frac{1}{5}u(\frac{1}{4}) = 0, u'(1) + \frac{1}{5}u(\frac{3}{4}) = 0 \end{cases} \quad (6)$$

where $f(t, u) = \begin{cases} \sqrt{u} + \sqrt{t}, & t \in [0, 1] \text{ is a irrational number} \\ 0, & t \in [0, 1] \text{ is a rational number} \end{cases}$.

Corresponding to the problem (1), we have that $\alpha = \frac{3}{2}, \xi = \frac{1}{4}, \eta = \frac{3}{4}, \beta = \frac{1}{5}, \gamma = \frac{1}{5}$. Let

$E = \{t \mid t \in [0, 1] \text{ is a rational number}\}$, then $m(E) = 0$. Obviously, (H1)-(H3) are satisfied. By simple calculation, we get $f_0 = \infty, f_\infty = 0$. By Theorem 3.1, we get that (6) has at least one solution. This problem can be not solved by the theorem in [11].

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