Journal of Applied Mathematics and Physics, **2014, 2, 60-70**
Published Online February 2014 (http://www.scirp.org/journal/jamp)
http://dx.doi.org/10.4236/jamp.2014.23007

Application and Generalization of Eigenvalues Perturbation Bounds for Hermitian Block Tridiagonal Matrices*

Jicheng Li#, Jing Wu, Xu Kong
School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, China
Email: #jcli@mail.xjtu.edu.cn

Received December 15, 2013; revised January 15, 2014; accepted January 21, 2014

ABSTRACT

The paper contains two parts. First, by applying the results about the eigenvalue perturbation bounds for Hermitian block tridiagonal matrices in paper [1], we obtain a new efficient method to estimate the perturbation bounds for singular values of block tridiagonal matrix. Second, we consider the perturbation bounds for eigenvalues of Hermitian matrix with block tridiagonal structure when its two adjacent blocks are perturbed simultaneously. In this case, when the eigenvalues of the perturbed matrix are well-separated from the spectrum of the diagonal blocks, our eigenvalues perturbation bounds are very sharp. The numerical examples illustrate the efficiency of our methods.

KEYWORDS

Singular Value; Eigenvalue Perturbation; Hermitian Matrix; Block Tridiagonal Matrix; Eigenvector

1. Introduction

There are many known results about eigenvalue perturbation bounds of Hermitian matrices. See example [2-5]. Among them, one well-known theory is the following result.

Theorem 2.1 [2]. Let A and $A+E$ be n-by-n Hermitian matrices. Let λ_i and $\hat{\lambda}_i$ denote the ith smallest eigenvalues of A and $A+E$, respectively. Then for $i=1,2,\cdots,n$, we have

$$\lambda_i + \lambda_{\min}(E) \le \hat{\lambda}_i \le \lambda_i + \lambda_{\max}(E), \tag{1.1}$$

where all the eigenvalues of A and $A+E$ are indexed in ascending order.

This is the Weyl's theorem, which is one of the most classic eigenvalue perturbation theories. When the perturbation matrix E is an arbitrary Hermitian matrix, the bounds obtained by Weyl's theorem can be very small. However, for Hermitian matrices with special sparse structures such as block tridiagonal Hermitian matrix, the Weyl's theorem may not be the best choice. For this reason, [1] considered the difference between eigenvalues of the block tridiagonal Hermitian matrices A and $A+E_s$, where

$$A=\begin{pmatrix} A_1 & B_1^H & & \\ B_1 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1}^H \\ & & B_{n-1} & A_n \end{pmatrix} \text{ and } E_s=\begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 0 & 0 & & \\ & 0 & \Delta A_s & \Delta B_s^H & \\ & & \Delta B_s & 0 & 0 \\ & & & 0 & \ddots \end{pmatrix}, \tag{1.2}$$

*The work was supported by the Fundamental Research Funds for the Central Universities (xjj20100114) and the National Natural Science Foundation of China (11171270).
#Corresponding author.

in which $A_j \in C^{n_j \times n_j}$, $j = 1, 2, \cdots, n,$ are Hermitian matrices and $B_k \in C^{n_{j+1} \times n_j}$, $j = 1, 2, \cdots, n-1,$ are arbitrary matrices, the perturbation matrices ΔA_s and ΔB_s have the same order with the matrices A_s and B_s, respectively. Let λ_i and $\hat{\lambda}_i$ denote the ith smallest eigenvalues of matrices A and $A + E_s$, respectively. Let $\lambda(X)$ denote the set of all the eigenvalues of the Hermitian matrix X. By defining $gap_j = \min \left| \lambda_i - \lambda(A_j) \right|$, and assuming that there exists an integer $\ell > 0$ such that $gap_j > \|B_j\|_2 + \|B_{j-1}\|_2 + \|E_s\|_2$, $j = 1, 2, \cdots, s + \ell$, the paper [1] obtained the shaper eigenvalue perturbation bounds

$$\left| \lambda_i - \hat{\lambda}_i \right| \le \|\Delta A_s\| \left(\prod_{j=0}^{l} \delta_j \right)^2 + 2 \|\Delta B_s\|_2 \, \delta_0 \left(\prod_{j=1}^{l} \delta_j \right)^2, \tag{1.3}$$

in which

$$\delta_0 = \frac{\|B_s\|_2 + \|\Delta B_s\|_2}{gap_s - \|E_s\|_2 - \|\Delta A_s\|_2 - \|B_{s-1}\|_2}, \quad \delta_1 = \frac{\|B_{s+1}\|_2}{gap_{s+1} - \|E_s\|_2 - \|B_s\|_2 - \|\Delta B_s\|_2},$$

and

$$\delta_j = \frac{\|B_{s+j}\|_2}{gap_{s+j} - \|E_s\|_2 - \|B_{s+j-1}\|_2}, \quad j = 2, \cdots, \ell.$$

The natural questions are that whether the above results can be used to estimate the perturbation bounds for singular values of a block tridiagonal matrix, and how to get the eigenvalues perturbation bounds when two adjacent blocks of the matrix A in the formula (1.2) are perturbed simultaneously. If we apply the results above repeatedly, we can obtain a weaker upper bounds. Inspired by these questions, in this paper, we expect to obtain the perturbation bounds for singular value of a block tridiagonal matrix. Further, we give a new idea to obtain the eigenvalues perturbation bounds by directly using the bounds of eigenvector elements rather than applying the results in [1] repeatedly.

The structure of this paper is organized as follows. In Section 2, we provide preliminaries to outline our basic idea of deriving eigenvalue perturbation bounds via bounding eigenvector components [1]. In Section 3, we present a new approach to estimate the perturbation bounds for the singular values of the block tridiagonal matrix via applying the ideas in paper [1]. In Section 4, we consider the case which the sth block and $(s+1)$th block of the matrix A are perturbed simultaneously and present a new perturbation bound of the i smallest eigenvalue λ_i. Further, we discuss the eigenvalue perturbation bounds when the first s blocks of A are perturbed simultaneously and provide an algorithm to estimate the bounds. In Section 5, we present a numerical example to show the efficiency of our approach.

Notations. Let $\|\cdot\|_2$ denote the matrix spectrum norm.

2. Preliminaries

For simplicity, the eigenvalues that we mention in this paper are all simple eigenvalues. We need the following conclusion about the partial derivative of simple eigenvalue of $A + tE$ for further discussion, where $t \in [0,1]$.

Lemma 2.1 [1]. Let A and E be n-by-n Hermitian matrices. Denote by $\lambda_i(t)$ the ith eigenvalue of $A + tE$, and define the vector-valued function $x(t)$ such that $(A + tE)x(t) = \lambda_i(t)x(t)$ where $\|x(t)\|_2 = 1$ for some $t \in [0,1]$. If $\lambda_i(t)$ is simple, then

$$\frac{\partial \lambda_i(t)}{\partial t} = x(t)^H Ex(t). \tag{2.1}$$

Especially, the perturbation matrix E has the special structure. For example, the perturbation matrix E has the form as the matrix E_s whose block elements are zero except for the sth block. Moreover if $x(t)$ has small components in the positions corresponding to the nonzero elements of E, then $\frac{\partial \lambda_i(t)}{\partial t}$ is small. Hence if we know a bound for the components of $x(t)$ that are in the position corresponding to the nonzero elements of E, then we can obtain a bound for $\left| \lambda_i - \hat{\lambda}_i \right| = \left| \lambda_i(0) - \lambda_i(1) \right|$ via integrating the Equation (2.1) over $0 \le t \le 1$.

Yuji Nakatsukasa [1] has derived the eigenvalues perturbation bounds for the case (1.2) with this idea. In the following, we shall describe in detail how this idea can be exploited to derive perturbation bounds of singular values for block tridiagonal matrix, and how this idea is expanded to derive eigenvalue perturbation bounds for our cases.

Note that the Lemma 2.1 holds under the condition that $\lambda_i(t)$ is a simple eigenvalue of $A+tE$. Similarly, we also assume that $\lambda_i(t)$ is simple for all $t\in[0,1]$. For multiple eigenvalues, we can discuss this case by referring to the method of the paper [1,6,7].

3. Singular Value Perturbation Bounds

In this section, we use the results in paper [1] to study the perturbation bounds of singular values for the block tridiagonal matrices. For the sake of convenience, we define the sequence of nonzero singular values of a complex $p\times q$ matrix A by

$$\sigma(A)=(\sigma_1(A),\cdots,\sigma_r(A)),$$

where $r=rank(A^H A)$ and $\sigma_1(A)\le\cdots\le\sigma_r(A)$. Similarly, for the perturbation matrix E, we denote the rank of $A+E$ by $\tilde{r}$. Note that the nonzero eigenvalues of AA^H and $A^H A$ are the same. Generally, the nonzero singular values of A have important applications in many filed, so it's necessary to study singular value perturbation bounds. Just as the discussion of the [1,8] we only consider the simple singular values perturbation bounds.

3.1. 2 × 2 Case

Firstly, for the 2×2 case, we have the following results concerning the nonzero singular values perturbation bounds.

Theorem 3.1. *Let*

$$A_{2\times2}=\begin{array}{cc} & \begin{array}{cc} k & \ell \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{pmatrix} A_1 & B_1 \\ C_1 & A_2 \end{pmatrix} \end{array} \text{ and } E_{2\times2}=\begin{array}{cc} & \begin{array}{cc} k & \ell \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix} \end{array}$$

be two complex matrices, $0<\sigma_1\le\sigma_2\le\cdots\le\sigma_r$ and $0<\tilde{\sigma}_1\le\tilde{\sigma}_2\le\cdots\le\tilde{\sigma}_{\tilde{r}}$ be the nonzero singular values of $A_{2\times2}$ and $A_{2\times2}+E_{2\times2}$, respectively. Define $\gamma=\max\{\|B_1\|_2,\|C_1\|_2\}$, $\varepsilon=\max\{\|E_2\|_2,\|E_3\|_2\}$ and

$\tau_i=\dfrac{\gamma+\varepsilon}{\min|\sigma_i-\sigma(A_2)|-2\|E\|_2}$. For $i=1,2,\cdots,\min\{r,\tilde{r}\}$, if $\tau_i>0$ and the ith singular value $\sigma_i\notin\sigma(A_2)$, then we have $|\sigma_i-\tilde{\sigma}_i|\le\|E_1\|_2+\|E_2\|_2\tau_i^2+2\varepsilon\tau_i$.

Proof. Let

$$\tilde{A}_{4\times4}=\begin{pmatrix} 0 & A_{2\times2} \\ A_{2\times2}^H & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & A_1 & B_1 \\ 0 & 0 & C_1 & A_2 \\ A_1^H & C_1^H & 0 & 0 \\ B_1^H & A_2^H & 0 & 0 \end{pmatrix} \tag{3.1}$$

and

$$\tilde{E}_{4\times4}=\begin{pmatrix} 0 & E_{2\times2} \\ E_{2\times2}^H & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & E_1 & E_2 \\ 0 & 0 & E_3 & E_4 \\ E_1^H & E_3^H & 0 & 0 \\ E_2^H & E_4^H & 0 & 0 \end{pmatrix}. \tag{3.2}$$

By Jordan-Wielandt theorem[2-Theorem I.4.2], we know that the eigenvalues of the matrix $\tilde{A}_{4\times4}$ are $\pm\sigma_i$, where $1\le i\le m+n+k+\ell$. The same statement holds for $\tilde{E}_{4\times4}$. Permuting the rows and columns of the matrix $\tilde{A}_{4\times4}$ appropriately, we can get that the matrix $\tilde{A}_{4\times4}$ is similar to

$$\hat{A}_{4\times 4}=\begin{pmatrix} 0 & A_1 & 0 & B_1 \\ A_1^H & 0 & C_1^H & 0 \\ 0 & C_1 & 0 & A_2 \\ B_1^H & 0 & A_2^H & 0 \end{pmatrix}$$

and the matrix $\tilde{E}_{4\times 4}$ is similar to

$$\hat{E}_{4\times 4}=\begin{pmatrix} 0 & E_1 & 0 & E_2 \\ E_1^H & 0 & E_3^H & 0 \\ 0 & E_3 & 0 & E_4 \\ E_2^H & 0 & E_4^H & 0 \end{pmatrix}.$$

Let

$$A_{11}=\begin{pmatrix} 0 & A_1 \\ A_1^H & 0 \end{pmatrix},\ A_{21}^H=\begin{pmatrix} 0 & B_1 \\ C_1^H & 0 \end{pmatrix},\ A_{22}^H=\begin{pmatrix} 0 & A_2 \\ A_2^H & 0 \end{pmatrix} \text{ and } E_{21}^H=\begin{pmatrix} 0 & E_2 \\ E_3^H & 0 \end{pmatrix}$$

Obviously, the matrix $\hat{A}_{4\times 4}=\begin{pmatrix} A_{11} & A_{21}^H \\ A_{21} & A_{22} \end{pmatrix}$ is a 2×2 block Hermitian matrix, so is $\hat{E}_{4\times 4}$. Note that the $\|A_{21}\|_2=\max\left\{\|B_1\|_2, \|C_1\|_2\right\}$, the eigenvalue set of A_{22} is $\sigma(A_2)$, and $\left\|E_{21}^H\right\|_{21}=\max\left\{\|E_2\|_2, \|E_3\|_2\right\}$. So it is natural that we can apply the result of [1-Theorem 3.2] to get the conclusion. □

3.2. 3 × 3 Case

Secondly, we study the perturbation bounds for singular values of 3×3 case. Let

$$A_{3\times 3}=\begin{pmatrix} A_1 & B_1 & 0 \\ C_1 & A_2 & B_2 \\ 0 & C_2 & A_3 \end{pmatrix} \quad\text{and}\quad E_{3\times 3}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta A_2 & \Delta B_2 \\ 0 & \Delta C_2 & 0 \end{pmatrix}$$

be two complex matrices, where $A_i\in\mathbb{C}^{m_i\times k_i}\ (i=1,2,3)$ and $\Delta A_2\in\mathbb{C}^{m_2\times k_2}$, $0<\sigma_1\le\sigma_2\le\cdots\le\sigma_r$ and $0<\tilde{\sigma}_1\le\tilde{\sigma}_2\le\cdots\le\tilde{\sigma}_{\tilde{r}}$ be the singular values of $A_{3\times 3}$ and $A_{3\times 3}+E_{3\times 3}$, respectively. Similar to the discussion above, by permuting the rows and columns of $\begin{pmatrix} 0 & A_{3\times 3} \\ A_{3\times 3}^H & 0 \end{pmatrix}$ appropriately, we can get that the matrix $\begin{pmatrix} 0 & A_{3\times 3} \\ A_{3\times 3}^H & 0 \end{pmatrix}$ is similar to

$$\hat{A}_{6\times 6}=\begin{pmatrix} 0 & A_1 & B_1 & 0 & 0 & 0 \\ A_1^H & 0 & 0 & C_1^H & 0 & 0 \\ B_1^H & 0 & 0 & A_2^H & C_2^H & 0 \\ 0 & C_1 & A_2 & 0 & 0 & B_2 \\ 0 & 0 & C_2 & 0 & 0 & A_3 \\ 0 & 0 & 0 & B_2^H & A_3^H & 0 \end{pmatrix}$$

and the matrix $\begin{pmatrix} 0 & E_{3\times 3} \\ E_{3\times 3}^H & 0 \end{pmatrix}$ is similar to

$$\hat{E}_{6\times 6}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta A_2^H & \Delta C_2^H & 0 \\ 0 & 0 & \Delta A_2 & 0 & 0 & \Delta B_2 \\ 0 & 0 & \Delta C_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta B_2^H & 0 & 0 \end{pmatrix}.$$

Obviously, both $\hat{A}_{6\times6}$ and $\hat{E}_{6\times6}$ are block tridiagonal Hermitian matrices. Applying [1-Theorem 4.2], we can get the following theorem.

Theorem 3.2. *Let* σ_i *and* $\tilde{\sigma}_i$ *be the ith smallest nonzero singular values of* $A_{3\times3}$ *and* $A_{3\times3}+E_{3\times3}$, *respectively. Define* $\gamma=\max\{\|B_2\|_2,\|C_2\|_2\}$, $\varepsilon=\max\{\|\Delta B_2\|_2,\|\Delta C_2\|_2\}$, $\delta=\max\{\|B_1\|_2,\|C_1\|_2\}$ *and* $\tau_i=\dfrac{\gamma+\varepsilon}{\min|\sigma_i-\sigma(A_2)|-\|\Delta A_2\|_2-\delta}$. *For* $i=1,2,\cdots,\min\{r,\tilde{r}\}$, *if* $\sigma_i\notin\sigma(A_1)\cup\sigma(A_2)$, *then we have* $|\sigma_i-\tilde{\sigma}_i|\le\|\Delta A_2\|_2\tau_i^2+2\|\Delta B_2\|_2\tau_i$.

3.3. n × n Case

Further, we gradually consider the general $n\times n$ case and extend above statements to the $n\times n$ block tridiagonal matrices. Let

$$A_{n\times n}=\begin{pmatrix}A_1 & B_1 & & \\ C_1 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1}\\ & & C_{n-1} & A_n\end{pmatrix}\quad\text{and}\quad E_{n\times n}=\begin{pmatrix}\ddots & \ddots & & & \\ \ddots & 0 & 0 & & \\ & 0 & \Delta A_s & \Delta B_s & \\ & & \Delta C_s & 0 & 0\\ & & & 0 & \ddots\end{pmatrix},\tag{3.4}$$

where $A_i\in\mathbb{C}^{m_i\times k_i}\ (i=1,2,\cdots,n)$ and $\Delta A_s\in\mathbb{C}^{m_s\times k_s}$, $0<\sigma_1\le\sigma_2\le\cdots\le\sigma_r$ and $0<\tilde{\sigma}_1\le\tilde{\sigma}_2\le\cdots\le\tilde{\sigma}_{\tilde{r}}$ be the nonzero singular values of $A_{n\times n}$ and $A_{n\times n}+E_{n\times n}$, respectively. The following conclusion can be demonstrated.

Theorem 3.3. *Let* σ_i *and* $\tilde{\sigma}_i$ *be the ith smallest nonzero singular values of* $A_{n\times n}$ *and* $A_{n\times n}+E_{n\times n}$, *respectively. Define* $gap_j=\min|\sigma_i-\sigma(A_j)|$, $\gamma_j=\max\{\|B_j\|_2,\|C_j\|_2\}$ *and* $\varepsilon_j=\max\{\|\Delta B_j\|_2,\|\Delta C_j\|_2\}$. *If there exists a positive integer* ℓ *such that* $gap_j>\gamma_j+\gamma_{j-1}+\varepsilon_j+\varepsilon_{j-1}+\|E\|_2+\|\Delta A_j\|_2$, *where* $j=1,\cdots,s+\ell$, *and*

$$\delta_0=\frac{\gamma_s+\varepsilon_s}{gap_s-\gamma_{s-1}-\|E\|_2-\|\Delta A_s\|_2},\quad \delta_1=\frac{\gamma_{s+1}}{gap_{s+1}-\gamma_s-\varepsilon_s-\|E\|_2},$$

and

$$\delta_k=\frac{\gamma_{s+k}}{gap_{s+k}-\gamma_{s+k}-\|E\|_2},\quad \text{for } k=1,\cdots,\ell,$$

then we have

$$|\sigma_i-\tilde{\sigma}_i|\le\|\Delta A_s\|_2\left(\prod_{j=0}^{\ell}\delta_j\right)^2+2\varepsilon_s\delta_0\left(\prod_{j=1}^{\ell}\delta_j\right)^2.$$

In what follows, we give an example to illustrate the singular values perturbation bounds obtained by our results.

Example 3.1. Consider the 4×4 matrices A and E represented by

$$A=\begin{pmatrix}A_1 & 0\\ 0 & A_2\end{pmatrix}\quad\text{and}\quad E=\begin{pmatrix}0 & E_3^*\\ E_3 & 0\end{pmatrix}$$

where

$$A_1=\begin{pmatrix}6 & 0\\ 0 & 5\end{pmatrix},\quad A_2=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad E_3=\begin{pmatrix}3\times10^{-4} & -2\times10^{-2}\\ 2\times10^{-4} & 10^{-2}\end{pmatrix}.$$

Obviously, the last two singular values of A are $\sigma_3=5,\sigma_4=6$. By computing, we can get that the two singular values of $A+E$ are

$$\tilde{\sigma}_3=5.000116663424745\quad\text{and}\quad\tilde{\sigma}_4=6.000000023715121.$$

Therefore, we can get

$$\left|\tilde{\sigma}_3-\sigma_3\right|=0.000116663424745 \quad \text{and} \quad \left|\tilde{\sigma}_4-\sigma_4\right|=0.000000023715121. \tag{3.5}$$

Through the Theorem 3.1 we know that

$$\left|\tilde{\sigma}_3-\sigma_3\right|\le 2.535e-4 \quad \text{and} \quad \left|\tilde{\sigma}_4-\sigma_4\right|\le 2.022e-4.$$

By comparing the differences in the equation (3.5) with the bounds obtained by the Theorem 3.1, we can find that the singular values perturbation bounds obtained by the Theorem 3.1 are sharp and this estimating method is efficient.

4. Eigenvalue Perturbation Bounds

On the basis of conclusions of the paper [1], in this section we study eigenvalue perturbation bounds of block tridiagonal matrix for the cases where two adjacent blocks of A are perturbed and the first s blocks of A are perturbed by the perturbation matrix $E_{1,\cdots,s}$.

4.1. Two Adjacent Blocks of A Being Perturbed

In this subsection, we discuss eigenvalue perturbation bounds when two adjacent blocks of A are perturbed. In other words, we consider the matrices in the following form

$$A=\begin{pmatrix} A_1 & B_1^H & & \\ B_1 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1}^H \\ & & B_{n-1} & A_n \end{pmatrix},\ E_{s,s+1}=\begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & 0 & 0 & & & & \\ & 0 & \Delta A_s & \Delta B_s^H & & & \\ & & \Delta B_s & \Delta A_{s+1} & \Delta B_{s+1}^H & & \\ & & & \Delta B_{s+1} & 0 & 0 & \\ & & & & 0 & \ddots & \end{pmatrix}. \tag{4.1}$$

Similar to discussion of the paper [1], we need the following assumption.

Assumption 1. There exists an integer $\ell>0$ such that $\lambda_i\notin\left[\lambda_{\min}\left(A_j\right)-\eta_j,\lambda_{\max}\left(A_j\right)+\eta_j\right]$, where $\eta_j=\left\|B_j\right\|_2+\left\|B_{j-1}\right\|_2+\left\|E_{s,s+1}\right\|_2+\left\|\Delta A_j\right\|_2+\left\|\Delta B_j\right\|_2+\left\|\Delta B_{j-1}\right\|_2$, $j=1,\cdots,s+\ell.$

Roughly, the assumption demands that λ_i is far away from the eigenvalues of $A_1,\cdots,A_{s+l}$, respectively, and the norms of $E_{s,s+1}$ and $B_1,\cdots,B_{s+l}$ are not too large.

Now, on the basis of the Assumption 1, we first discuss upper bounds for the eigenvector components of the matrix $A+tE_{s,s+1}$.

Lemma 4.1. Let A and $E_{s,s+1}$ be Hermitian block-tridiagonal matrices in (4.1), λ_i be the ith smallest eigenvalue of A. For $t\in[0,1]$, let $\left(A+tE_{s,s+1}\right)x(t)=\lambda_i(t)x(t)$, where $\left\|x(t)\right\|_2=1$ and $x(t)^H=\left[x_1(t)^H,x_2(t)^H,\cdots,x_n(t)^H\right]^H$ satisfying that $x_j(t)$ and A_j have the same number of rows. De*fine*

$$\delta_0=\frac{\left\|B_s\right\|_2+\left\|\Delta B_s\right\|_2}{gap_s-\left\|E_{s,s+1}\right\|_2-\left\|\Delta A_s\right\|_2-\left\|B_{s-1}\right\|_2},$$

$$\delta_1=\frac{\left\|B_{s+1}\right\|_2+\left\|\Delta B_{s+1}\right\|_2}{gap_{s+1}-\left\|E_{s,s+1}\right\|_2-\left\|B_s\right\|_2-\left\|\Delta B_s\right\|_2-\left\|\Delta A_{s+1}\right\|_2},$$

$$\delta_2=\frac{\left\|B_{s+2}\right\|_2}{gap_{s+2}-\left\|E_{s,s+1}\right\|_2-\left\|B_{s+1}\right\|_2-\left\|\Delta B_{s+1}\right\|_2},$$

and for $j=3,\cdots,\ell$,

$$\delta_j=\frac{\left\|B_{s+j}\right\|_2}{gap_{s+j}-\left\|E_{s,s+1}\right\|_2-\left\|B_{s+j-1}\right\|_2}. \tag{4.2}$$

If λ_i satisfies Assumption 1, then, for all $t\in[0,1]$ we have

$$\left\|x_s(t)\right\|_2 \le \prod_{j=0}^{\ell}\delta_j, \quad \left\|x_{s+1}(t)\right\|_2 \le \prod_{j=1}^{\ell}\delta_j \quad \text{and} \quad \left\|x_{s+2}(t)\right\|_2 \le \prod_{j=2}^{\ell}\delta_j. \tag{4.3}$$

Proof. The first block component of $\left(A + tE_{s,s+1}\right)x(t) = \lambda_i(t)x(t)$ is

$$A_1x_1(t) + B_1^H x_2(t) = \lambda_i(t)x_1(t).$$

Since $gap_j > \left\|B_j\right\|_2 + \left\|B_{j-1}\right\|_2 + \left\|E_{s,s+1}\right\|_2 + \left\|\Delta A_j\right\|_2 + \left\|\Delta B_j\right\|_2 + \left\|\Delta B_{j-1}\right\|_2$ for $j = 1,\cdots,s+\ell$, by Weyl's theorem, we have $\lambda_i(t) \notin \lambda\left(A_j\right)$. Therefore, we have

$$x_1(t) = \left(\lambda_i(t)I - A_1\right)^{-1} B_1^H x_2(t).$$

Further, by applying the Theorem 2.1[2], we know $\lambda_i(t) \in \left[\lambda_i - \left\|E_{s,s+1}\right\|_2, \lambda_i + \left\|E_{s,s+1}\right\|_2\right]$ and $\left\|\left(\lambda_i(t)\mathrm{I} - A_1\right)^{-1}\right\|_2 \le \dfrac{1}{gap_1 - \left\|E_{s,s+1}\right\|_2}$. So we can bound $\left\|x_1(t)\right\|_2 / \left\|x_2(t)\right\|_2$ by

$$\frac{\left\|x_1(t)\right\|_2}{\left\|x_2(t)\right\|_2} \le \frac{\left\|B\right\|_2}{gap_2 - \left\|E_{s,s+1}\right\|_2 - \left\|B_1\right\|_2} < 1, \tag{4.4}$$

where the right inequality follows from Assumption 1. Continuously, the second block component of $\left(A + tE_{s,s+1}\right)x(t) = \lambda_i(t)x(t)$ is

$$B_1x_1(t) + A_2x_2(t) + B_2^H x_3(t) = \lambda_i(t)x_2(t).$$

So,

$$x_2(t) = \left(\lambda_i(t)I - A_2\right)^{-1}\left(B_1x_1(t) + B_2^H x_3(t)\right).$$

Similarly, by Weyl's theorem, we have $\left\|\left(\lambda_i(t)I - A_2\right)^{-1}\right\|_2 \le \dfrac{1}{gap_2 - \left\|E_{s,s+1}\right\|_2}$. Combining this inequality with (4.4), we can get

$$\frac{\left\|x_2(t)\right\|_2}{\left\|x_3(t)\right\|_2} \le \frac{\left\|B_2\right\|_2}{gap_2 - \left\|E_{s,s+1}\right\|_2 - \left\|B_1\right\|_2} < 1.$$

Hence, $\left\|x_2(t)\right\|_2 < \left\|x_3(t)\right\|_2$.

By the same argument, we can prove $\left\|x_1(t)\right\|_2 < \left\|x_2(t)\right\|_2 < \cdots < \left\|x_{s+\ell}(t)\right\|_2$ for all $t \in [0,1]$.

To consider the *s*th block component of $\left(A + tE_{s,s+1}\right)x(t) = \lambda_i(t)x(t)$, we have

$$B_{s-1}x_{s-1}(t) + \left(A_s + t\Delta A_s\right)x_s(t) + \left(B_s^H + t\Delta B_s^H\right)x_{s+1}(t) = \lambda_i(t)x_s(t),$$

thus,

$$x_s(t) = \left(\lambda_i(t)I - A_s - t\Delta A_s\right)^{-1}\left(B_{s-1}x_{s-1}(t) + \left(B_s^H + t\Delta B_s^H\right)x_{s+1}(t)\right).$$

By using the results of the Assumption 1 and Theorem 2.1[2], we know that $\lambda_i(t)I - A_s - t\Delta A_s$ is invertible and $\left\|\left(\lambda_i I - A_s - t\Delta A_s\right)^{-1}\right\|_2 \le \dfrac{1}{gap_s - \left\|E_{s,s+1}\right\|_2 - \left\|\Delta A_s\right\|_2}$. Since $\left\|x_{s-1}(t)\right\|_2 < \left\|x_s(t)\right\|_2$, we can get

$$\left\|x_s(t)\right\|_2 \le \frac{\left\|B_{s-1}\right\|_2\left\|x_s(t)\right\|_2 + \left\|B_s + t\Delta B_s\right\|_2\left\|x_{s+1}(t)\right\|_2}{gap_s - \left\|E_{s,s+1}\right\|_2 - \left\|\Delta A_s\right\|_2}.$$

Therefore, for all $t \in [0,1]$ we can obtain the following result

$$\frac{\|x_s(t)\|_2}{\|x_{s+1}(t)\|_2} \le \frac{\|B_s\|_2 + \|\Delta B_s\|_2}{gap_s - \|E_{s,s+1}\|_2 - \|\Delta A_s\|_2 - \|B_{s-1}\|_2} = \delta_0.$$

Continuously, considering the $s+1^{\text{th}}$ block of $(A + tE_{s,s+1})x(t) = \lambda_i(t)x(t)$,

$$(B_s + t\Delta B_s)x_s(t) + (A_{s+1} + t\Delta A_{s+1})x_{s+1}(t) + (B_{s+1} + t\Delta B_{s+1}^H)x_{s+2}(t) = \lambda_i(t)x_{s+1}(t),$$

we have

$$x_{s+1}(t) = (\lambda_i(t)I - A_{s+1} + t\Delta A_{s+1})^{-1}\left((B_s + t\Delta B_s)x_s(t) + (B_{s+1} + t\Delta B_{s+1}^H)x_{s+2}(t)\right).$$

Similarly, by using the results of the Assumption 1 and Theorem 2.1[2], we know that $\lambda_i(t)I - A_{s+1} + t\Delta A_{s+1}$ is invertible and $\left\|(\lambda_i(t)I - A_{s+1} + t\Delta A_{s+1})^{-1}\right\|_2 \le \frac{1}{gap_{s+1} - \|E_{s,s+1}\|_2 - \|\Delta A_{s+1}\|_2}$. Since $\|x_s(t)\|_2 < \|x_{s+1}(t)\|_2$, for all $t \in [0,1]$, we can get

$$\|x_{s+1}(t)\|_2 \le \frac{(\|B_s\|_2 + \|\Delta B_s\|_2)\|x_{s+1}(t)\|_2 + \left((\|B_{s+1}\|_2 + \|\Delta B_{s+1}\|_2)\|x_{s+2}(t)\|_2\right)}{gap_{s+1} - \|E_{s,s+1}\|_2 - \|\Delta A_{s+1}\|_2}.$$

Hence,

$$\frac{\|x_{s+1}(t)\|_2}{\|x_{s+2}(t)\|_2} \le \frac{\|B_{s+1}\|_2 + \|\Delta B_{s+1}\|_2}{gap_{s+1} - \|E_{s,s+1}\|_2 - \|\Delta A_{s+1}\|_2 - \|B_s\|_2 - \|\Delta B_s\|_2} = \delta_1.$$

Similar to the discussion above, we also have

$$(B_{s+1} + \Delta B_{s+1})x_{s+1}(t) + A_{s+2}x_{s+2}(t) + B_{s+2}^H x_{s+3}(t) = \lambda_i(t)x_{s+2}(t),$$

$$x_{s+2}(t) = (\lambda_i(t)I - A_{s+2})^{-1}\left((B_{s+1} + \Delta B_{s+1})x_{s+1}(t) + B_{s+2}^H x_{s+3}(t)\right).$$

By $\left\|(\lambda_i(t)I - A_{s+2})^{-1}\right\|_2 \le \frac{1}{gap_{s+2} - \|E_{s,s+1}\|_2}$ and $\|x_{s+1}(t)\|_2 < \|x_{s+2}(t)\|_2$, we have

$$\|x_{s+2}(t)\|_2 \le \frac{(\|B_{s+1}\|_2 + \|\Delta B_{s+1}\|_2)\|x_{s+2}(t)\|_2 + \|B_{s+2}\|_2\|x_{s+3}(t)\|_2}{gap_{s+2} - \|E_{s,s+1}\|_2}.$$

Consequently, for all $t \in [0,1]$,

$$\frac{\|x_{s+2}(t)\|_2}{\|x_{s+3}(t)\|_2} \le \frac{\|B_{s+2}\|_2}{gap_{s+2} - \|E_{s,s+1}\|_2 - \|B_{s+1}\|_2 - \|\Delta B_{s+1}\|_2} = \delta_2.$$

Similar to the discussion above, we can prove $\frac{\|x_{s+j}(t)\|_2}{\|x_{s+j+1}(t)\|_2} \le \delta_j$, $j = 1, \cdots, \ell$. In addition, $\|x_{s+\ell+1}(t)\|_2 \le \|x(t)\|_2 \le 1$.

Based on the discussion above, we conclude that for all $t \in [0,1]$,

$$\|x_s(t)\|_2 \le \prod_{j=0}^{\ell}\delta_j, \quad \|x_{s+1}(t)\|_2 \le \prod_{j=1}^{\ell}\delta_j \quad \text{and} \quad \|x_{s+2}(t)\|_2 \le \prod_{j=2}^{\ell}\delta_j.$$

The following Theorem 4.1 is aiming to present perturbation bounds for λ_i.

Theorem 4.1. Let λ_i and $\hat{\lambda}_i$ be the *ith* smallest eigenvalues of matrix A and $A + E_{s,s+1}$, respectively, and δ_i be defined as in (4.2). If λ_i satisfies the Assumption 1, we have

$$\left|\lambda_i - \hat{\lambda}_i\right| \le \|\Delta A_s\|_2\left(\prod_{j=0}^{\ell}\delta_j\right)^2 + \left(2\|\Delta B_s\|_2\,\delta_0 + \|\Delta A_{s+1}\|_2\right)\left(\prod_{j=1}^{\ell}\delta_j\right)^2 + 2\|B_{s+1}\|_2\,\delta_1\left(\prod_{j=2}^{\ell}\delta_j\right)^2.$$

Proof. Integrating (2.1) over $0 \le t \le 1$ we get

$$
\begin{aligned}
\left|\lambda_i-\hat{\lambda}_i\right| &= \left|\int_0^1 x^H(t)E_{s,s+1}x(t)\,\mathrm{d}t\right| \\
&\le \left|\int_0^1 x_s^H(t)\Delta A_s x_s(t)\,\mathrm{d}t\right| + 2\left|\int_0^1 x_{s+1}^H(t)\Delta B_s x_s(t)\,\mathrm{d}t\right| \\
&\quad + \left|\int_0^1 x_{s+1}^H(t)\Delta A_s x_{s+1}(t)\,\mathrm{d}t\right| + 2\left|\int_0^1 x_{s+1}^H(t)\Delta B_{s+1} x_{s+2}(t)\,\mathrm{d}t\right| \\
&\le \left\|\Delta A_s\right\|_2 \left|\int_0^1 \left\|x_s(t)\right\|_2^2\,\mathrm{d}t\right| + 2\left\|\Delta B_s\right\|_2 \left|\int_0^1 \left\|x_s(t)\right\|_2 \left\|x_{s+1}(t)\right\|_2\,\mathrm{d}t\right| \\
&\quad + \left\|\Delta A_{s+1}\right\|_2 \left|\int_0^1 \left\|x_{s+1}(t)\right\|_2^2\,\mathrm{d}t\right| + 2\left\|\Delta B_{s+1}\right\|_2 \left|\int_0^1 \left\|x_{s+1}(t)\right\|_2 \left\|x_{s+2}(t)\right\|_2\,\mathrm{d}t\right|
\end{aligned}
$$

Together with (4.3), it follows that

$$
\begin{aligned}
\left|\lambda_i-\hat{\lambda}_i\right| &\le \left\|\Delta A_s\right\|_2 \left|\int_0^1 \left(\prod_{j=0}^{\ell}\delta_j\right)^2 \mathrm{d}t\right| + 2\left\|\Delta B_s\right\|_2 \left|\int_0^1 \left(\prod_{j=0}^{\ell}\delta_j\right)\left(\prod_{j=1}^{\ell}\delta_j\right)\mathrm{d}t\right| \\
&\quad + \left\|\Delta A_{s+1}\right\|_2 \left|\int_0^1 \left(\prod_{j=1}^{\ell}\delta_j\right)^2 \mathrm{d}t\right| + 2\left\|\Delta B_{s+1}\right\|_2 \left|\int_0^1 \left(\prod_{j=1}^{\ell}\delta_j\right)\left(\prod_{j=2}^{\ell}\delta_j\right)\mathrm{d}t\right| \\
&\le \left\|\Delta A_s\right\|_2 \left(\prod_{j=0}^{\ell}\delta_j\right)^2 + \left(2\left\|\Delta B_s\right\|_2 \delta_0 + \left\|\Delta A_{s+1}\right\|_2\right)\left(\prod_{j=1}^{\ell}\delta_j\right)^2 \\
&\quad + 2\left\|B_{s+1}\right\|_2 \delta_1 \left(\prod_{j=2}^{\ell}\delta_j\right)^2. \qquad \square
\end{aligned}
$$

4.2. The First s Blocks of A Being Perturbed

In this subsection, we gradually consider the bounds of eigenvalues of the matrix A, whose the first s blocks are perturbed simultaneously. In other words, we consider the perturbation matrix

$$
E_{1,2,\cdots,s} = \begin{pmatrix}
\Delta A_1 & \Delta B_1^H & & & & \\
\Delta B_1 & \ddots & \ddots & & & \\
 & \ddots & \Delta A_s & \Delta B_s^H & & \\
 & & \Delta B_s & 0 & 0 & \\
 & & & 0 & \ddots &
\end{pmatrix}, \tag{4.5}
$$

where s is a positive integer.Let $\left(\lambda_i(t), x(t)\right)$ denote the ith eigenpair of $A+tE_{1,2,\cdots,s}$ satisfying $\left(A+tE_{1,2,\cdots,s}\right)x(t)=\lambda_i x(t)$, and the partition of $x(t)^H = \left[x_1(t)^H, x_2(t)^H, \cdots, x_n(t)^H\right]^H$ satisfies that $x_j(t)$ and A_j have the same number of rows, where $t\in[0,1]$.

If λ_i satisfies the Assumption 1, through the similar discussion as above, we can derive a similar conclusion for calculating the eigenvalue perturbation bounds. For simplicity, we don't repeat the proof here. The **Algorithm 1** below shows the calculation in detail, where $B_0=0$, $B_n=0$ and $\Delta B_0=0$.

5. Numerical Example

In this section, we use the following example to illustrate the validity of our method and to show the advantage of the our method over the method proposed in [1].

Example 5.1 [1]. Let $A+E$ be the 1000×1000 tridiagonal matrix

$$
A+E = tridiag\left\{\begin{matrix} & 1 & & 1 & \cdot & 1 & & 1 & \\ 1000 & & 999 & & \cdot & & \cdot & 2 & 1 \\ & 1 & & 1 & \cdot & 1 & & 1 & \end{matrix}\right\} \tag{5.1}
$$

Algorithm 1. Eigenvalue perturbation bound algorithm for the first s blocks of A being perturbed.

Input:	A, $A+E_{1,2\cdots s}$ and s;
step 1:	Compute the ith eigenvalues λ_i and $\hat{\lambda}_i$ of A and $A+E_{1,2\ldots,s}$, respectively;
	Choose an integer $\ell>0$ such that λ satisfies the Assumption 1;
step 2:	for $j=1,2,\cdots,s+l_i$ do
	Compute the $gap_j=\min\left\|\lambda_i-\lambda\left(A_j\right)\right\|$;
	$\mu_j=\left\|B_j\right\|_2+\left\|\Delta B_j\right\|_2$; $\nu_j=gap_j-\left\|E_{1,2\ldots,s}\right\|_2-\left\|B_{j-1}\right\|_2-\left\|\Delta A_j\right\|_2-\left\|\Delta B_{j-1}\right\|_2$;
	$\delta_j=\mu_j/\nu_j$;
	end for
step 3:	for $k=1$ to $s+\ell-1$ do
	$\omega_k=\delta_k$;
	for $j=k$ to $s+\ell-1$ do
	$\omega_k=\omega_k\delta_{j+1}$;
	end for
	end for
step 4:	$\Delta\lambda_1=\omega_1^2\left\|\Delta A_1\right\|_2+2\left\|\Delta B_1\right\|\omega_1\omega_2$; $\omega_{s+\ell+1}=0$;
step 5:	for $k=2,3\cdots,s+\ell$ do
	$\Delta\lambda_k=\Delta\lambda_{k-1}+\omega_k^2\left\|\Delta A_k\right\|_2+2\left\|\Delta B_k\right\|\omega_k\omega_{k+1}$;
	end for
Output:	the eigenvalue perturbation bounds $\Delta\lambda_k$;

where all the elements of E are zero except for the 900th and 901th off diagonal, which are 1 $(i.e., s=900)$. Note that none of the off-diagonals is negligibly small. We focus on λ_i (the ith smallest eigenvalue of A) for $i=1,\cdots,10$, which are smaller than 10. For such λ_i we have $\ell=87$, and give bounds for $i=1,\cdots,10$ with our method. The results are outlined in **Table 1**.

Meanwhile, we use the method in the paper [1] to give the perturbation bounds for $i=1,\cdots,10$. The results are outlined in **Table 2**.

Further, we partition the matrix $A+E$ as in the (5.1) again so that the block size is one except for the 900th block, which is 2-by-2 matrix $\begin{pmatrix}901 & 1\\ 1 & 900\end{pmatrix}$. In other words, we set $A_{900}=\begin{pmatrix}901 & 0\\ 0 & 900\end{pmatrix}$, $B_{900}=\begin{pmatrix}0 & 0\end{pmatrix}$, and set $\Delta A_{900}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\Delta B_{900}=\begin{pmatrix}0 & 1\end{pmatrix}$ (*i.e.*, s = 900). Using the method in the paper [1] we have the following perturbation bounds for $i=1,\cdots,10$, which are outlined in **Table 3**.

Obviously, comparing the **Table 1** with the **Table 2**, we can see that our method saves CPU times and improves the perturbation bounds. In addition, comparing the **Table 1** with the **Table 3**, although our CPU time is close to the CPU time in **Table 3**, we see that the perturbation bounds are also improved . So we can say that our method is efficient and improved.

6. Conclusion

We have obtained a new efficient method to estimate the perturbation bounds for singular values of block tridiagonal matrix. Further, under the bases of the paper [1], we present a new conclusion for estimating the perturbation bound when the sth block and $(s+1)$th block of the matrix A are perturbed simultaneously and

Table 1. The eigenvalue perturbation bounds and CUP times.

i	1	2	3	4	5
bound	8.5e – 292	6.8e – 289	1.5e – 286	2.1e – 284	3.3e – 282
time (s)	160.81	164.94	165.11	168.75	171.84
i	6	7	8	9	10
bound	6.7e – 280	1.8e – 277	7.2e – 275	4.5e – 272	5.4e – 269
time (s)	154.41	158.65	156.44	171.61	161.98

Table 2. The eigenvalue perturbation bounds and CUP times.

i	1	2	3	4	5
bound	1.05e – 290	9.7e – 288	2.4e – 285	3.96e – 283	7.2e – 281
time (s)	330.93	372.81	316.51	383.42	378.66
i	6	7	8	9	10
bound	1.7e – 278	5.7e – 276	2.9e – 273	2.6e – 270	5.1e – 267
time (s)	381.42	374.66	378.98	378.85	381.69

Table 3. The eigenvalue perturbation bounds and CUP times.

i	1	2	3	4	5
bound	1.1e – 289	1.1e – 286	3.1e – 284	5.7e – 282	1.2e – 279
time (s)	157.95	154.78	156.89	156.57	157.30
i	6	7	8	9	10
bound	3.2e – 277	1.2e – 274	7.9e – 272	9.3e – 269	2.9e – 265
time (s)	154.95	154.93	154.02	155.58	157.10

provide an algorithm for the general case when the first s blocks of A are perturbed simultaneously. Number examples are presented to show the effectiveness of our methods.

REFERENCES

[1] Y. Nakatsukasa, "Eigenvalue Perturbation Bounds for Hermitian Block Tridiagonal Matrices," Applied Numerical Mathematics, Vol. 62, No. 1, 2012, pp. 67-78.

[2] G. W. Stewart and J.-G. Sun, "Matrix Perturbation Theory," Academic Press, Boston, 1990.

[3] J. Demmel, "Applied Numerical Linear Algebra," SIAM, Philadelphia, 1997.

[4] G. H. Golub and C. F. Van Loan, "Matrix Computations," Johns Hopkins University Press, Baltimore, 1996.

[5] E.-X. Jiang, "Perturbation in Eigenvalues of a Symmetric Tridiagonal Matrix," *Linear Algebra and its Applications*, Vol. 399, 2005, pp. 91-107.

[6] J. Barlow and J. Demmel, "Computing Accurate Eigensystems of Scaled Diagonally Dominant Matrices," *SIAM Journal on Numerical Analysis*, Vol. 27, No. 3, 1990, pp. 762-791.

[7] J. Barlow and I. Slapnicar, "Optimal Perturbation Bounds for the Hermitian Eigenvalue Problem," Linear Algebra and its Applications, Vol. 309, No. 1-3, 2000, pp. 19-43.

[8] C.-K. Li and R.-C. Li, "A Note on Eigenvalues of Perturbed Hermitian Matrices," *Linear Algebra and its Applications*, Vol. 395, 2005, pp. 183-190. http://dx.doi.org/10.1016/j.laa.2004.08.026