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Application of Trial Equation Method for Solving the Benjamin Ono Equation

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ABSTRACT

In the article, the nonlinear equation is reduced to an ordinary differential equation under the travelling wave transformation. Using trial equation method, the ODE is reduced to the elementary integral form. In the end, complete discrimination system for polynomial is used to solve the corresponding integrals and obtain the classification of all single travelling wave solutions to the equation.

KEYWORDS

The Nonlinear Partial Differential Equation; Complete Discrimination System for Polynomial; Trial Equation Method; Traveling Wave Transform; The Benjamin Ono Equation

1. Introduction

Nonlinear phenomena are general problems in every field of engineering technology, science research, natural world and human society activities. So the investigation of exact solutions of nonlinear equations plays a important role not only in theoretic research but in application. To obtain the travelling wave solutions, many methods were attempted, such as the inverse scattering method [1], Hirotas bilinear transformation [2,3], the tanh method [4], sine-cosine method [5], homogeneous balance method [6,7], exp-function method [8], and so on. These methods derived many solutions to most nonlinear evolution equations. Recently, Professor Liu proposed a powerful method named trial equation method for finding exact solutions to nonlinear differential equations [9-11]. By using his method, the nonlinear differential equation is reduced to an ordinary differential equation under the travelling wave transformation. Using the trial equation method, the ODE is reduced to the elementary integral form. In the end, the complete discrimination system for polynomial is used to solve the corresponding integrals. We can obtain the classification of all single travelling wave solutions [12-16] to the equation. This idea is so good that many types of nonlinear differential equations can be solved by it. Using the trial equation method and complete discrimination system for polynomial, we have obtained a lot of new solutions to many nonlinear differential equations. As an application, some new solutions to the Benjamin Ono equation are given.

2. Application of the Trial Equation Method

The Benjamin Ono equation reads as

$$u_{tt} + \beta \left(u^2 \right)_{xx} + \gamma u_{xxxx} = 0. \tag{1}$$

where β, γ are parameters. Taking the traveling wave transformation $u = u(\xi_1)$ and $\xi_1 = kx + \omega t$, we can obtain the corresponding reduced ODE.

$$\left(\omega^2\right)u'' + 2k^2\beta uu'' + 2k^2\beta\left(u'\right)^2 + \gamma k^2 u'''' = 0. \tag{2}$$

we take the trial equation as follows: $u'' = a_0 + a_1 u + \cdots + a_m u^m$.

According to the trial equation method of rank homogeneous equation, balancing u'''' with uu'' (or $\left(u'\right)^2$) gets $m = 2$. Equation (4) has the following specific form

$$u'' = a_0 + a_1 u + a_2 u^2. \tag{3}$$

Integrating the Equation (3)once with respect to ξ_1, we get

$$\left(u'\right)^2 = \frac{2}{3}a_2 u^3 + a_1 u^2 + 2a_0 u + d. \tag{4}$$

By Equation (3) and Equation(4), we derive the following equation

$$u'''' = \frac{10}{3}a_2^2 u^3 + 5a_1 a_2 u^2 + \left(6a_0 a_2 + a_1^2\right)u + 2a_2 d + a_0 a_1. \tag{5}$$

Substituting Equations (3)-(5) into Equation (2), we have

$$r_3 u^3 + r_2 u^2 + r_1 u + r_0 = 0. \tag{6}$$

where

$$r_0 = a_0\omega^2 + 2k^2\beta d + \left(2a_2 d + a_0 a_1\right)k^4\gamma. \tag{7}$$

$$r_1 = a_1\omega^2 + 6k^2\beta a_0 + \left(6a_0 a_2 + a_1^2\right)k^4\gamma. \tag{8}$$

$$r_2 = a_2\omega^2 + 4k^2\beta a_1 + 5a_1 a_2 k^4\gamma. \tag{9}$$

$$r_3 = \frac{10}{3}k^2\beta a_2 + \frac{10}{3}k^4\gamma a_2^2. \tag{10}$$

Let the coefficient $r_i = 0\,(i = 0,1,2,3)$ be zero,we will yield nonlinear algebraic equations.Solving the equations, we will determine the values of a_0, a_1, a_2, d.

We get $a_1 = -\dfrac{\omega^2}{k^4\gamma}, a_2 = -\dfrac{\beta}{k^2\gamma}, a_0$ and d are two arbitrary constants. When the above conditions are

satisfied, we use the complete discrimination system for the third order polynomial and have the following solving process.

Let

$$v = \left(\frac{2}{3}a_2\right)^{\frac{1}{3}}u, \xi = \left(\frac{2}{3}a_2\right)^{\frac{1}{3}}\xi_1, d_2 = a_1\left(\frac{2}{3}a_2\right)^{-\frac{2}{3}}, d_1 = 2a_0\left(\frac{2}{3}a_2\right)^{-\frac{1}{3}}, d_0 = d. \tag{11}$$

Then Equation(4) becomes

$$\left(v'\right)^2 = v^3 + d_2 v^2 + d_1 v + d_0. \tag{12}$$

where v is a function of ξ.The integral form of Equation(12) is

$$\pm\left(\xi - \xi_0\right) = \int\frac{dv}{\sqrt{v^3 + d_2 v^2 + d_1 v + d_0}}. \tag{13}$$

Denote

$$F\left(v\right) = v^3 + d_2 v^2 + d_1 v + d_0. \tag{14}$$

$$\Delta = -27\left(\frac{2d_2^3}{27} + d_0 - \frac{d_1 d_3}{3}\right)^2 - 4\left(d_1 - d_2^2\right)^3, D_1 = d_1 - \frac{d_2^2}{3}. \tag{15}$$

According to the complete discrimination system, we give the corresponding single traveling wave solutions

to Equation(1).

Case 1. $\Delta = 0, D_1 < 0 \cdot F(v) = 0$ has a double real root and a simple real root. Then we have

$$F(v) = (v - \lambda_1)^2 (v - \lambda_2), \ \lambda_1 \neq \lambda_2. \tag{16}$$

when $v > \lambda_2$, the corresponding solutions are

$$u_1 = \left(\frac{2}{3} a_2\right)^{-\frac{1}{3}} \left\{(\lambda_1 - \lambda_2)\tanh^2\left[\frac{\sqrt{\lambda_1 - \lambda_2}}{2}\left(\frac{2}{3} a_2\right)^{\frac{1}{3}}(\xi_1 - \xi_0)\right] + \lambda_2\right\}, (\lambda_1 > \lambda_2); \tag{17}$$

$$u_2 = \left(\frac{2}{3} a_2\right)^{-\frac{1}{3}} \left\{(\lambda_1 - \lambda_2)\coth^2\left[\frac{\sqrt{\lambda_1 - \lambda_2}}{2}\left(\frac{2}{3} a_2\right)^{\frac{1}{3}}(\xi_1 - \xi_0)\right] + \lambda_2\right\}, (\lambda_1 > \lambda_2); \tag{18}$$

$$u_3 = \left(\frac{2}{3} a_2\right)^{-\frac{1}{3}} \left\{(-\lambda_1 + \lambda_2)\sec^2\left[\frac{\sqrt{-\lambda_1 + \lambda_2}}{2}\left(\frac{2}{3} a_2\right)^{\frac{1}{3}}(\xi_1 - \xi_0)\right] + \lambda_1\right\}, (\lambda_1 < \lambda_2). \tag{19}$$

Case 2. $\Delta = 0, D_1 = 0 \cdot F(v) = 0$ has a triple root. Then we have

$$F(v) = (v - \lambda)^3. \tag{20}$$

The corresponding solution is

$$u_4 = 4\left(\frac{2}{3} a_2\right)^{-\frac{2}{3}} (\xi_1 - \xi_0)^{-2} + \lambda. \tag{21}$$

Case 3. $\Delta > 0, D_1 < 0 \cdot F(v) = 0$ has three different real roots. Then we have

$$F(v) = (v - \lambda_1)(v - \lambda_2)(v - \lambda_3), \ \lambda_1 < \lambda_2 < \lambda_3. \tag{22}$$

when $\lambda_1 < v < \lambda_2$, we take the transformation as follows

$$v = \lambda_1 + (\lambda_2 - \lambda_1)\sin^2 \varphi. \tag{23}$$

According to the Equation(12), we have

$$\pm(\xi - \xi_0) = \int \frac{dv}{\sqrt{F(v)}} = \frac{2}{\sqrt{\lambda_3 - \lambda_1}} \int \frac{d\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}}. \tag{24}$$

where $m^2 = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}$. On the basis of the Equation(24) and the definition of the Jacobi elliptic sine function, we have

$$v = \lambda_1 + (\lambda_2 - \lambda_1) sn^2 \left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2}(\xi_1 - \xi_0), m\right). \tag{25}$$

The corresponding solutions is

$$u_5 = \left(\frac{2}{3} a_2\right)^{-\frac{1}{3}} \left[\lambda_1 + (\lambda_2 - \lambda_1) sn^2 \left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2}(\xi_1 - \xi_0), m\right)\right]. \tag{26}$$

when $v > \lambda_3$, we take the transformation as follows

$$v = \frac{-\lambda_2 \sin^2 \varphi + \lambda_3}{\cos^2 \varphi}. \tag{27}$$

The corresponding solutions is

$$u_6 = \left(\frac{2}{3}a_2\right)^{-\frac{1}{3}} \left[\frac{\lambda_3 - \lambda_2 sn^2\left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2}\left(\frac{2}{3}a_2\right)^{\frac{1}{3}}(\xi_1 - \xi_0), m\right)}{cn^2\left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2}\left(\frac{2}{3}a_2\right)^{\frac{1}{3}}(\xi_1 - \xi_0), m\right)} \right]. \tag{28}$$

where $m^2 = \dfrac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}$.

Case 4. $\Delta < 0 \cdot F(v) = 0$ has only a real root. Then we have

$$F(v) = (v - \lambda)(v^2 + pv + q), \quad p^2 - 4q < 0. \tag{29}$$

when $v > \lambda_1$, we take the transformation as follows

$$v = \lambda + \sqrt{\lambda^2 + p\lambda + q} \tan^2 \frac{\varphi}{2}. \tag{30}$$

According to the Equation(13), we have

$$\xi - \xi_0 = \int \frac{dv}{\sqrt{(v-\lambda)(v^2 + pv + q)}} = \frac{1}{(\lambda^2 + p\lambda + q)^{\frac{1}{4}}} \int \frac{d\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}}. \tag{31}$$

where $m^2 = \dfrac{1}{2}\left(1 - \dfrac{\lambda + \dfrac{p}{2}}{\sqrt{\lambda^2 + p\lambda + q}}\right)$. On the basis of the Equation(31) and the definition of the Jacobi elliptic

cosine function, we have

$$v = \lambda + \frac{2\sqrt{\lambda^2 + p\lambda + q}}{1 + cn\left((\lambda^2 + p\lambda + q)^{\frac{1}{4}}(\xi_1 - \xi_0), m\right)} - \sqrt{\lambda^2 + p\lambda + q}. \tag{32}$$

The corresponding solutions is

$$u_7 = \left(\frac{2}{3}a_2\right)^{-\frac{1}{3}} \left[\lambda + \frac{2\sqrt{\lambda^2 + p\lambda + q}}{1 + cn\left((\lambda^2 + p\lambda + q)^{\frac{1}{4}}\left(\frac{2}{3}a_2\right)^{-\frac{1}{3}}(\xi_1 - \xi_0), m\right)} - \sqrt{\lambda^2 + p\lambda + q} \right]. \tag{33}$$

In Equations (17), (18), (19), (21), (26), (28) and (33), the integration constant ξ_0 has been rewritten, but we still use it. The solutions $u_i (i = 1, \cdots, 7)$ are all possible exact traveling wave solutions to Equation (1). It is easy to write the corresponding solutions to the Benjamin Ono equation. For brevity, we omitted.

3. Conclusion

Trial equation method is a systematic method to solve nonlinear differential equations. The advantage of this method is that we can deal with nonlinear equations with linear methods. This method has the characteristics of simple steps and clear effectivity. Based on the idea of the trial equation method and the aid of the computerized symbolic computation, some exact traveling wave solutions to the Benjamin Ono equation have been obtained. With the same method, some of other equations can be dealt with.

Acknowledgements

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