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Similarity Reduction of Nonlinear Partial Differential Equations

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ABSTRACT

In this work, the HB method is extended to search for similarity reduction of nonlinear partial differential equations. This method is generalized and will apply for a (2 + 1)-dimensional higher order Broer-Kaup System. Some new exact solutions of Broer-Kaup System are found.

KEYWORDS

Similarity Reduction; Exact Solutions; Nonlinear Partial Differential Equations

1. Introduction

In the past few decades, there has been the noticeable progress in the construction of the exact solutions for nonlinear partial differential equations, which has long been a major concern for both mathematicians and physicists. The effort in finding exact solutions to nonlinear differential equation, when they exist, is very important for the understanding of most nonlinear physical phenomena. For instances, the nonlinear wave phenomena observed in fluid dynamics, plasma and optical fibers are often modelled by the bell shaped sech solutions and the kink shaped tanh solutions.

We consider the following a (2 + 1)-dimensional higher order Broer-Kaup system:

$$u_t + 4\left(u_{xx} + u^3 - 3uu_x + 3uw + 3p\right)_x = 0, \tag{1}$$

$$v_t + 4\left(v_{xx} + 3vu^2 + uv_x + 3vw\right)_x = 0, \tag{2}$$

$$w_y - v_x = 0, \tag{3}$$

$$p_y - (uv)_x = 0. \tag{4}$$

which is obtained from the Kadomtsev-Petviashvili (KP) equation by the symmetry constraint [1].

The systems (1)-(4) were given by Li *et al* [2] solving it via a transformation and tanh-function method to obtain many new exact solutions. Jain *et al.* [3] reduced a system to a simple (1 + 1)-dimensional nonlinear evolution equation through a simple transformation, and by using the new generally projective Riccati equation expansion method to explore many families of soliton-like and periodic solutions for it. Recently, Li *et al.* [4] have obtained some new types of multisoliton solutions for the systems (1)-(4) by using some simple transformations as $v = u_y, w = u_x + \alpha(x,t), p = uu_x + \beta$ and homogenous balance method.

The homogenous balance (HB) method is a powerful tool to find solitary wave solutions of nonlinear partial differential equations. Fan *et al*. [5] presented an improved HB method to obtain more other kinds of exact solutions and introduced a continuation of [5] in [6]. The traditional method for finding similarity reduction of nonlinear partial differential equations is to use classical Lie approach [7,8]. However, the method involves tedious algebraic calculations and still can not be used to find all similarity solutions. Recently,Clarkson and Kruskal devloped a direct and simple method to find more similarity solutions of nonlinear PDEs.

In this work, the HB method is extended to search for similarity reduction of nonlinear partial differential equations. So, more solutions can be obtained by the improved HB method. This method is generalized and can be applied to other nonlinear partial differential equations [9-15].

Similarity Reduction of Nonlinear Partial Differential Equations

We describe the main steps of our method. For a given PDE, say in three variables, say x, y, t

$$u_t = K\left(u, u_x, u_y, u_{xx}, u_{yy}, \cdots\right) \tag{5}$$

we seek its similarity reductions in the form

$$u\left(x, y, t\right) = \frac{\partial^{\alpha}}{\partial x^{\alpha}} f\left(s\right) + u_0, \tag{6}$$

where α is a constant to determine by balancing between the highest order derivative of the linear terms of u and the nonlinear terms of u, where s, u_0 are regarded as undetermined functions.

Substituting from Equation (6) into Equation (5) and collecting all terms of f with the same derivative and power. To make the associated equation be an ordinary equations of f and s, requiring ratios of their coefficients being functions of s, we obtain a set of determining equations for s, u_0 and other undermined functions, from which s and u_0 will be obtained.

To explain this method, we will apply for a (2 + 1)-dimensional higher order Broer-Kaup system (1)-(4), we suppose their similarity solutions are of the form

$$\begin{gathered} u\left(x, y, t\right) = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} f(s) + u_0,\ v\left(x, y, t\right) = \frac{\partial^{\alpha_2}}{\partial x^{\alpha_2}} g\left(s\right) + v_0, \\ w\left(x, y, t\right) = \frac{\partial^{\alpha_3}}{\partial x^{\alpha_3}} h\left(s\right) + w_0,\ p\left(x, y, t\right) = \frac{\partial^{\alpha_4}}{\partial x^{\alpha_4}} k\left(s\right) + p_0. \end{gathered} \tag{7}$$

where

$$s = s\left(x, y, t\right), u_0 = u_0\left(x, y, t\right), v_0 = v_0\left(x, y, t\right), w_0 = w_0\left(x, y, t\right), p_0 = p\left(x, y, t\right).$$

are determined functions. Balancing the highest order of linear term with the nonlinear terms in every equations (1)-(4) to determining $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, we obtain

$$\alpha_1 = 1, \alpha_2 = \alpha_3 = 2, \alpha_4 = 3 \tag{8}$$

the Equation (64) take the following form

$$u\left(x, y, t\right) = \frac{\partial}{\partial x} f\left(s\right) + u_0 = f' s_x + u_0, \tag{9}$$

$$v\left(x, y, t\right) = \frac{\partial^2}{\partial x^2} g\left(s\right) + v_0 = g'' s_x^2 + g' s_{xx} + v_0, \tag{10}$$

$$w\left(x, y, t\right) = \frac{\partial^2}{\partial x^2} h\left(s\right) + w_0 = h'' s_x^2 + h' s_{xx} + w_0, \tag{11}$$

$$p\left(x, y, t\right) = \frac{\partial^3}{\partial x^3} k\left(s\right) + p_0 = k''' s_x^3 + 3k'' s_x s_{xx} + k' s_{xxx} + p_0. \tag{12}$$

Substituting Equations (9)-(12) into the original system (1)-(4) and collecting all terms of f, g, h, k with the same derivative and power leads to

$$\begin{aligned}&\left(-12s_x s_{xxx}\right) f'^2+\left(12u_{0x}s_x^2+36u_0 s_x s_{xx}\right)h''+\left(36s_{xx}^2+48s_x s_{xx}\right)k''+12 f h' s_x s_{xxx}\\&+\left(24u_0u_{0x}s_x+12w_{0x}s_x+s_{xt}+12u_0^2 s_{xx}+12w_0 s_{xx}-24u_{0x}s_{xx}-12u_0 s_{xxx}+4s_{xxxx}\right) f'\\&+\left(12u_{0x}s_{xx}+12u_0 s_{xxx}\right)h'+12k' s_{xxxx}+\left(12 p_{0x}+u_{0t}+12u_0^2u_{0x}+12w_0u_{0x}-12u_{0x}^2-12u_0u_{0xx}+4u_{0xxx}\right)=0,\end{aligned}\tag{13}$$

$$\begin{aligned}&\left(24 f f'' g''+12 f'2g'''+4 f'' g'''+12h'' g'''+12g'' h''+4 f g''''+4g'''''\right)s_x^5\\&+\left(4u_0 s_x^4+40s_x^3 s_{xx}\right)g'''+\left(24u_0 s_x^4+28s_x^3 s_{xx}\right) f g''+\left(24u_0 s_x^4+12s_x^3 s_{xx}\right) f'' g''\\&+\left(24 f g' f''+60 f'^2 g''+72g'' h''+12h' g'''+12g' h'''\right)s_x^3 s_{xx}\\&+\left(s_x^2 s_t+12u_0^2 s_x^3+12w_0 s_x^3+4u_{0x}s_x^3+24u_0 s_x^2 s_{xx}+60s_x s_{xx}^2+40s_x^2 s_{xxx}\right)g'''\\&+\left(24u_{0x}s_x^3+96u_0 s_x^2 s_{xx}+24s_x s_{xx}^2+16s_x^2 s_{xxx}\right) f g''+\left(24s_x s_{xx}^2+12s_x^2 s_{xx}\right) f'^2 g'\\&+\left(24u_0 s_x^2 s_{xx}+4s_x^2 s_{xxx}\right) f'' g'+\left(36s_x s_{xx}^2+12s_x^2 s_{xxx}\right)h' g''\\&+\left(36s_x s_{xx}^2+12s_x^2 s_{xxx}\right)g' h''+\left(24 f f''+12h'''\right)v_0 s_x^3\left(24u_0 v_0 s_x^2+4v_{0x}s_x^2\right) f''\\&+\left(24u_0u_{0x}s_x^2+12w_{0x}s_x^2+2s_x s_{xt}+36u_0^2 s_x s_{xx}+36w_0 s_x s_{xx}+12u_{0x}s_x s_{xx}+12u_0 s_{xx}^2+16u_0 s_x s_{xxx}\right.\\&\left.+40s_{xx}s_{xxx}+20s_x s_{xxxx}\right)g''+\left(12v_{0x}s_x^2+36v_0 s_x s_{xx}\right)h''+\left(12v_{0x}s_x^2+24v_0 s_x s_{xx}\right) f'^2\\&+\left(24u_0 s_x s_{xx}+24u_0 s_{xx}^2+24u_0 s_x s_{xxx}+4s_{xx}s_{xxx}+4s_x s_{xxxx}\right) f g'+24s_{xx}s_{xxxx}g' h'\\&+\left(24v_0u_{0x}s_x+24u_0v_0 s_{xx}+4v_{0x}s_{xx}\right) f'\\&+\left(24u_0u_{0x}s_{xx}+12w_{0x}s_{xx}+s_{xxt}+12u_0^2 s_{xxx}+12w_0 s_{xxx}+4u_{0x}s_{xxx}+4u_0 s_{xxxx}+4s_{xxxxx}\right)g'\\&+\left(12v_{0x}s_{xx}+12v_0 s_{xxx}\right)h'\\&+\left(24u_0v_0u_{0x}+v_{0t}+12u_0^2v_{0x}+12w_0v_{0x}+4u_{0x}v_{0x}+12v_0w_{0x}+4u_0v_{0xx}+4v_{0xxx}\right)=0,\end{aligned}\tag{14}$$

$$-g''' s_x^3+h''' s_x^2 s_y+\left(s_{xx}s_y+2s_x s_{xy}\right)h''-3g' s_x s_{xx}+s_{xxy}h'-s_{xxx}g'+\left(w_{0y}-v_{0x}\right)=0,\tag{15}$$

$$\begin{aligned}&-\left(f'' g''+ f g'''\right)s_x^4-\left(u_{0x}s_x^2+3u_0 s_x s_{xx}\right)g''-\left(s_{xx}^2+s_x s_{xxx}\right) f g'\\&+\left(s_y s_{xxx}+3s_{xx}s_{xy}+3s_x s_{xxy}\right)k''-v_0 s_x^2 f''+\left(3s_x s_y s_{xx}+3s_x^2 s_{xy}+s_x^3 s_y\right)k'''\\&-\left(g' f''+4 f g''\right)s_x^2 s_{xx}-u_0 s_x^3 g'''-\left(v_{0x}s_x+v_0 s_{xx}\right) f'-\left(u_{0x}s_{xx}+u_0 s_{xxx}\right)g'\\&+s_{xxxy}k'+\left(p_{0y}-v_0u_{0x}-u_0v_{0x}\right)=0.\end{aligned}\tag{16}$$

To make Equations (13)-(16) be an ordinary differential equations of f, g, h and k only for s, the ratios of the coefficients of different derivative and power of f,g,h,k must be functions of s. That is to say, the following constrained conditions are satisfied

$$s_x^2 s_{xx}=s_x^4\Gamma_1\left(s\right)\tag{17}$$

$$-12u_0 s_x^3+24s_x^2 s_{xx}=s_x^4\Gamma_2\left(s\right)\tag{18}$$

$$24u_0 s_x^3-60s_x^2 s_{xx}=s_x^4\Gamma_3\left(s\right)\tag{19}$$

$$s_x s_t+12u_0 s_x^2+12w_0 s_x^2+12s_{xx}^2-24u_{0x}s_x^2-36u_0 s_x s_{xx}+16s_x s_{xxx}=s_x^4\Gamma_4\left(s\right)\tag{20}$$

$$12u_{0x}s_x^2+36u_0 s_x s_{xx}=s_x^4\Gamma_5\left(s\right)\tag{21}$$

$$36s_{xx}^2+48s_x s_{xx}=s_x^4\Gamma_6\left(s\right)\tag{22}$$

$$12u_{0x}s_x^2+24u_0 s_x s_{xx}-12s_{xx}^2-12s_x s_{xxx}=s_x^4\Gamma_7\left(s\right)\tag{23}$$

$$s_x s_{xxx}=s_x^4\Gamma_8\left(s\right)\tag{24}$$

$$12\left(w_0 s_x\right)_x + 12\left(u_0^2 s_x\right)_x + s_{xt} - 24u_{0x}s_{xx} - 12u_0 s_{xxx} + 4s_{xxxx} = s_x^4\Gamma_9(s) \tag{25}$$

$$12u_{0x}s_{xx} + 12u_0 s_{xxx} = s_x^4\Gamma_{10}(s) \tag{26}$$

$$s_{xxxx} = s_x^4\Gamma_{11}(s) \tag{27}$$

$$12p_{0x} + u_{0t} + 12u_0^2 u_{0x} + 12w_0 u_{0x} - 12u_{0x}^2 - 12u_0 u_{0xx} + 4u_{0xxx} = s_x^4\Gamma_{12}(s) \tag{28}$$

$$s_{xx}s_x^3 = s_x^4\Gamma_{13}(s) \tag{29}$$

$$4u_0 s_x^4 + 40s_x^3 s_{xx} = s_x^5\Gamma_{14}(s) \tag{30}$$

$$24u_0 s_x^4 + 28s_x^3 s_{xx} = s_x^5\Gamma_{15}(s) \tag{31}$$

$$24u_0 s_x^4 + 12s_x^3 s_{xx} = s_x^5\Gamma_{16}(s) \tag{32}$$

$$s_x^2 s_t + 12u_0^2 s_x^3 + 12w_0 s_x^3 + 4u_{0x}s_x^3 + 24u_0 s_x^2 s_{xx} + 60s_x s_{xx}^2 \tag{33}$$

$$s_x^2 s_{xx} = s_x^4\Gamma_1(s) \tag{34}$$

$$-12u_0 s_x^3 + 24s_x^2 s_{xx} = s_x^4\Gamma_2(s) \tag{35}$$

$$24u_0 s_x^3 - 60s_x^2 s_{xx} = s_x^4\Gamma_3(s) \tag{36}$$

$$s_x s_t + 12u_0 s_x^2 + 12w_0 s_x^2 + 12s_{xx}^2 - 24u_{0x}s_x^2 - 36u_0 s_x s_{xx} + 16s_x s_{xxx} = s_x^4\Gamma_4(s) \tag{37}$$

$$12u_{0x}s_x^2 + 36u_0 s_x s_{xx} = s_x^4\Gamma_5(s) \tag{38}$$

$$36s_{xx}^2 + 48s_x s_{xx} = s_x^4\Gamma_6(s) \tag{39}$$

$$12u_{0x}s_x^2 + 24u_0 s_x s_{xx} - 12s_{xx}^2 - 12s_x s_{xxx} = s_x^4\Gamma_7(s) \tag{40}$$

$$s_x s_{xxx} = s_x^4\Gamma_8(s) \tag{41}$$

$$12\left(w_0 s_x\right)_x + 12\left(u_0^2 s_x\right)_x + s_{xt} - 24u_{0x}s_{xx} - 12u_0 s_{xxx} + 4s_{xxxx} = s_x^4\Gamma_9(s) \tag{42}$$

$$12u_{0x}s_{xx} + 12u_0 s_{xxx} = s_x^4\Gamma_{10}(s) \tag{43}$$

$$s_{xxxx} = s_x^4\Gamma_{11}(s) \tag{44}$$

$$12p_{0x} + u_{0t} + 12u_0^2 u_{0x} + 12w_0 u_{0x} - 12u_{0x}^2 - 12u_0 u_{0xx} + 4u_{0xxx} = s_x^4\Gamma_{12}(s) \tag{45}$$

$$s_{xx}s_x^3 = s_x^4\Gamma_{13}(s) \tag{46}$$

$$4u_0 s_x^4 + 40s_x^3 s_{xx} = s_x^5\Gamma_{14}(s) \tag{47}$$

$$24u_0 s_x^4 + 28s_x^3 s_{xx} = s_x^5\Gamma_{15}(s) \tag{48}$$

$$24u_0 s_x^4 + 12s_x^3 s_{xx} = s_x^5\Gamma_{16}(s) \tag{49}$$

$$s_x^2 s_t + 12u_0^2 s_x^3 + 12w_0 s_x^3 + 4u_{0x}s_x^3 + 24u_0 s_x^2 s_{xx} + 60s_x s_{xx}^2 \tag{50}$$

$$+40s_x^2 s_{xxx} s_x^5 = \Gamma_{17}(s) \tag{51}$$

$$24u_{0x}s_x^3 + 96u_0 s_x^2 s_{xx} + 24s_x s_{xx}^2 + 16s_x^2 s_{xxx} = s_x^5\Gamma_{18}(s) \tag{52}$$

$$24s_x s_{xx}^2 + 12s_x^2 s_{xx} = s_x^5\Gamma_{19}(s) \tag{53}$$

$$24u_0 s_x^2 s_{xx} + 4s_x^2 s_{xxx} = s_x^5 \Gamma_{20}(s) \tag{54}$$

$$36 s_x s_{xx}^2 + 12 s_x^2 s_{xxx} = s_x^5 \Gamma_{21}(s) \tag{55}$$

$$36 s_x s_{xx}^2 + 12 s_x^2 s_{xxx} = s_x^5 \Gamma_{22}(s) \tag{56}$$

$$v_0 s_x^3 = s_x^5 \Gamma_{23}(s) \tag{57}$$

$$24u_0 v_0 s_x^2 + 4v_{0x} s_x^2 = s_x^5 \Gamma_{24}(s) \tag{58}$$

$$24u_0 u_{0x} s_x^2 + 12w_{0x} s_x^2 + 2s_x s_{xt} + 36u_0^2 s_x s_{xx} + 36w_0 s_x s_{xx} + 12u_{0x} s_x s_{xx} \tag{59}$$

$$+12u_0 s_{xx}^2 + 16u_0 s_x s_{xxx} + 40 s_{xx} s_{xxx} + 20 s_x s_{xxxx} = s_x^5 \Gamma_{25}(s) \tag{60}$$

$$12v_{0x} s_x^2 + 36v_0 s_x s_{xx} = s_x^5 \Gamma_{26}(s) \tag{61}$$

$$12v_{0x} s_x^2 + 24v_0 s_x s_{xx} = s_x^5 \Gamma_{27}(s) \tag{62}$$

$$24u_0 s_x s_{xx} + 24u_0 s_{xx}^2 + 24u_0 s_x s_{xxx} + 4s_{xx} s_{xxx} + 4s_x s_{xxxx} = s_x^5 \Gamma_{28}(s) \tag{63}$$

$$24 s_{xx} s_{xxxx} = s_x^5 \Gamma_{29}(s) \tag{64}$$

$$24v_0 u_{0x} s_x + 24u_0 v_0 s_{xx} + 4v_{0x} s_{xx} = s_x^5 \Gamma_{30}(s) \tag{65}$$

$$12\left(u_0^2 s_{xx}\right)_x + s_{xxt} + 12\left(w_0 s_{xx}\right)_x + \left(4u_0 s_{xxx}\right)_x + 4s_{xxxxx} s_x^5 = \Gamma_{31}(s) \tag{66}$$

$$12v_{0x} s_{xx} + 12v_0 s_{xxx} = s_x^5 \Gamma_{32}(s) \tag{67}$$

$$24u_0 v_0 u_{0x} + v_{0t} + 12u_0^2 v_{0x} + 12\left(w_0 v_0\right)_x + 4\left(u_0 v_{0x}\right)_x + 4v_{0xxx} = s_x^5 \Gamma_{33}(s) \tag{68}$$

$$-s_x^2 s_y = s_x^3 \Gamma_{34}(s) \tag{69}$$

$$-\left(s_{xx} s_y + 2s_x s_{xy}\right) = s_x^3 \Gamma_{35}(s) \tag{70}$$

$$3 s_x s_{xx} = s_x^3 \Gamma_{36}(s) \tag{71}$$

$$-s_{xxy} = s_x^3 \Gamma_{37}(s) \tag{72}$$

$$s_{xxx} = s_x^3 \Gamma_{38}(s) \tag{73}$$

$$v_{0x} - w_{0y} = s_x^3 \Gamma_{39}(s) \tag{74}$$

$$u_{0x} s_x^2 + 3u_0 s_x s_{xx} = s_x^4 \Gamma_{40}(s) \tag{75}$$

$$s_{xx}^2 + s_x s_{xxx} = s_x^4 \Gamma_{41}(s) \tag{76}$$

$$-\left(s_y s_{xxx} + 3s_{xx} s_{xy} + 3s_x s_{xxy}\right) = s_x^4 \Gamma_{42}(s) \tag{77}$$

$$v_0 s_x^2 = s_x^4 \Gamma_{43}(s) \tag{78}$$

$$-\left(3s_x s_y s_{xx} + 3s_x^2 s_{xy} + s_x^3 s_y\right) = s_x^4 \Gamma_{44}(s) \tag{79}$$

$$s_x^2 s_{xx} = s_x^4 \Gamma_{45}(s) \tag{80}$$

$$u_0 s_x^3 = s_x^4 \Gamma_{46}(s) \tag{81}$$

$$v_{0x} s_x + v_0 s_{xx} = s_x^4 \Gamma_{47}(s) \tag{82}$$

$$u_{0x} s_{xx} + u_0 s_{xxx} = s_x^4 \Gamma_{48}(s) \tag{83}$$

$$-s_{xxxy} = s_x^4 \Gamma_{49}(s) \tag{84}$$

$$-\left(p_{0y} - v_0 u_{0x} - u_0 v_{0x}\right) = s_x^4 \Gamma_{50}(s) \tag{85}$$

where $\Gamma_i\,(i = 1, \cdots, 50)$ are some arbitrary functions of s to be determined and take the following form

$$\Gamma_1(s) = \frac{12f'^2 f'' - 12f''^2 + 12f''h'' - 12f'f''' + 12f'h''' + 4f'''' + 12k''''}{12\left(f'^3 + h'f'' + 4f'h''\right)}$$

$$\Gamma_2(s) = \frac{12f'^2 f'' - 12f''^2 + 12f''h'' - 12f'f''' + 12f'h''' + 4f'''' + 12k''''}{f'''}$$

$$\Gamma_3(s) = \frac{12f'^2 f'' - 12f''^2 + 12f''h'' - 12f'f''' + 12f'h''' + 4f'''' + 12k''''}{f'f''}$$

$$\Gamma_4(s) = \frac{12f'^2 f'' - 12f''^2 + 12f''h'' - 12f'f''' + 12f'h''' + 4f'''' + 12k''''}{f''}$$

$$\Gamma_5(s) = \frac{12f'^2 f'' - 12f''^2 + 12f''h'' - 12f'f''' + 12f'h''' + 4f'''' + 12k''''}{h''}$$

$$\Gamma_6(s) = \frac{12f'^2 f'' - 12f''^2 + 12f''h'' - 12f'f''' + 12f'h''' + 4f'''' + 12k''''}{k''}$$

$$\Gamma_7(s) = \frac{12f'^2 f'' - 12f''^2 + 12f''h'' - 12f'f''' + 12f'h''' + 4f'''' + 12k''''}{f'^2}$$

$$\Gamma_8(s) = \frac{12f'^2 f'' - 12f''^2 + 12f''h'' - 12f'f''' + 12f'h''' + 4f'''' + 12k''''}{12f'h'}$$

$$\Gamma_9(s) = \frac{12f'^2 f'' - 12f''^2 + 12f''h'' - 12f'f''' + 12f'h''' + 4f'''' + 12k''''}{f'}$$

$$\Gamma_{10}(s) = \frac{12f'^2 f'' - 12f''^2 + 12f''h'' - 12f'f''' + 12f'h''' + 4f'''' + 12k''''}{h'}$$

$$\Gamma_{11}(s) = \frac{12f'^2 f'' - 12f''^2 + 12f''h'' - 12f'f''' + 12f'h''' + 4f'''' + 12k''''}{k'}$$

$$\Gamma_{12}(s) = 12f'^2 f'' - 12f''^2 + 12f''h'' - 12f'f''' + 12f'h''' + 4f'''' + 12k''''$$

$$\Gamma_{13}(s) = \frac{24ff''g'' + 12f'2g''' + 4f''g''' + 12h''g''' + 12g''h'' + 4f'g'''' + 4g''''}{24f'g'f'' + 60f'^2 g'' + 72g''h'' + 12h'g''' + 12g'h'''}$$

$$\Gamma_{14}(s) = \frac{24ff''g'' + 12f'2g''' + 4f''g''' + 12h''g''' + 12g''h'' + 4f'g'''' + 4g''''}{g''''}$$

$$\Gamma_{15}(s) = \frac{24ff''g'' + 12f'2g''' + 4f''g''' + 12h''g''' + 12g''h'' + 4f'g'''' + 4g''''}{f'g'''}$$

$$\Gamma_{16}(s) = \frac{24ff''g'' + 12f'2g''' + 4f''g''' + 12h''g''' + 12g''h'' + 4f'g'''' + 4g''''}{f''g''}$$

$$\Gamma_{17}(s) = \frac{24ff''g'' + 12f'2g''' + 4f''g''' + 12h''g''' + 12g''h'' + 4f'g'''' + 4g''''}{g'''}$$

$$\Gamma_{18}(s) = \frac{24ff''g'' + 12f'2g''' + 4f''g''' + 12h''g''' + 12g''h'' + 4f'g'''' + 4g''''}{f'g''}$$

$$\Gamma_{19}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{f'^2g'}$$

$$\Gamma_{20}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{f''g'}$$

$$\Gamma_{21}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{g''h'}$$

$$\Gamma_{22}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{g'h''}$$

$$\Gamma_{23}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{(24ff''+12h''')}$$

$$\Gamma_{24}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{f''}$$

$$\Gamma_{25}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{g''}$$

$$\Gamma_{26}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{h''}$$

$$\Gamma_{27}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{f'^2}$$

$$\Gamma_{28}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{f'g'}$$

$$\Gamma_{29}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{g'h'}$$

$$\Gamma_{30}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{f'}$$

$$\Gamma_{31}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{g'}$$

$$\Gamma_{32}(s)=\frac{24ff''g''+12f'2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''}{h'}$$

$$\Gamma_{33}(s)=24ff''g''+12f'^2g'''+4f''g'''+12h''g'''+12g''h''+4f'g''''+4g''''$$

$$\Gamma_{34}(s)=\frac{g'''}{h'''}\quad \Gamma_{35}(s)=\frac{g'''}{h''}\quad \Gamma_{36}(s)=\frac{g'''}{g''}\quad \Gamma_{37}(s)=\frac{g'''}{h'}\quad \Gamma_{38}(s)=\frac{g'''}{g'}\quad \Gamma_{39}(s)=g'''$$

$$\Gamma_{40}(s)=\frac{f''g''+f'g'''}{g''}\quad \Gamma_{41}(s)=\frac{f''g''+f'g'''}{f'g'}\quad \Gamma_{42}(s)=\frac{f''g''+f'g'''}{k''}\quad \Gamma_{43}(s)=\frac{f''g''+f'g'''}{f''}$$

$$\Gamma_{44}(s)=\frac{f''g''+f'g'''}{k'''}\quad \Gamma_{45}(s)=\frac{f''g''+f'g'''}{g'f''+4f'g''}\quad \Gamma_{46}(s)=\frac{f''g''+f'g'''}{g'''}\quad \Gamma_{47}(s)=\frac{f''g''+f'g'''}{f'}$$

$$\Gamma_{48}(s)=\frac{f''g''+f'g'''}{g'}\quad \Gamma_{49}(s)=\frac{f''g''+f'g'''}{k'}\quad \Gamma_{50}(s)=f''g''+f'g''' \tag{86}$$

There are freedoms in the determination of u_0, v_0, w_0, p_0, s which can exploit the following rules, without loss of generality:

If u_0 has the form $u_0 = u'(x,y,t) + \frac{\partial}{\partial x}\Omega$ then we can assume that $\Omega = 0$ $\left(\text{make the transformation } f(s) \to f(s) - \Omega\right)$,

If v_0 has the form $v_0 = v'(x,y,t) + \frac{\partial^2}{\partial x^2}\Omega$ then we can assume that $\Omega = 0$ $\left(\text{make the transformation } f(s) \to f(s) - \Omega\right)$,

If w_0 has the form $w_0 = w'(x,y,t) + \frac{\partial}{\partial x}\Omega$ then we can assume that $\Omega = 0$ $\left(\text{make the transformation } f(s) \to f(s) - \Omega\right)$,

If p_0 has the form $p_0 = p'(x,y,t) + \frac{\partial^3}{\partial x^3}\Omega$ then we can assume that $\Omega = 0$ $\left(\text{make the transformation } f(s) \to f(s) - \Omega\right)$.

If $s(x,y,t)$ is defined by an equation of the form $\Omega(s) = s_0(x,y,t)$, we can also assume that $\Omega = s$ (make the transformation $s \to \Omega^{-1}(s)$).

From Equation (17), we get

$$\frac{s_{xx}}{s_x} = s_x \Gamma_1(s) \tag{87}$$

integrating Equation (87) with respect to x, we get

$$Lns_x + Ln\Gamma(s) = Ln\theta(y,t), \tag{88}$$

$$s_x \Gamma(s) = \theta(y,t). \tag{89}$$

integrating Equation (89) with respect to x, we obtain

$$\gamma(s) = \theta(y,t)x + \sigma(y,t). \tag{90}$$

By using the rule (e) into Equation (90), we obtain a function $s(x,y,t)$ in the form

$$s(x,y,t) = \theta(y,t)x + \sigma(y,t) \tag{91}$$

substituting from Equation (90) into Equations (17), (20), (24), (25), (26), (53), (54), (55), (56), (63), (64), (71), (72), (73), (76), (77), (80), (83)and (84) we obtain

$$\begin{gathered}\Gamma_1(s) = \Gamma_6(s) = \Gamma_8(s) = \Gamma_{10}(s) = \Gamma_{11}(s) = \Gamma_{19}(s) = \Gamma_{20}(s) = \Gamma_{21}(s) = \Gamma_{22}(s) \\ \Gamma_{28}(s) = \Gamma_{29}(s) = \Gamma_{31}(s) = \Gamma_{32}(s) = \Gamma_{36}(s) = \Gamma_{37}(s) = \Gamma_{38}(s) = \Gamma_{41}(s) = \Gamma_{42}(s) \\ \Gamma_{45}(s) = \Gamma_{48}(s) = \Gamma_{49}(s) = 0,\end{gathered} \tag{92}$$

where $s_{xx} = 0$ which clear in Equation (91).

By using Equation (91) into Equation (18), we get

$$u_0 = -\frac{1}{12}\theta\Gamma_2(s) = -\frac{1}{12}\frac{\partial}{\partial x}\left[\gamma_2(s)\right] \tag{93}$$

where $\Gamma_2(s) = \frac{\mathrm{d}}{\mathrm{d}s}\gamma_2(s), \quad \theta = \frac{\partial s}{\partial x}$.

By apply the rule (a) on Equation (93), we obtain

$$\gamma_2(s) = 0 \Rightarrow \Gamma_2(s) = 0,\ u_0 = 0. \tag{94}$$

substituting from Equations (91), (94) into Equation (20), we obtain

$$\Gamma_3(s) = 0. \tag{95}$$

using Equations (91)and (94) into Equation (19), we get

$$w_0 = \frac{1}{12}\theta^2\Gamma_4(s) - \frac{\theta_t}{12\theta}x - \frac{\sigma_t}{12\theta} = \frac{1}{12}\frac{\partial^2}{\partial x^2}\left[\gamma_4(s)\right] - \frac{\theta_t}{12\theta}x - \frac{\sigma_t}{12\theta}, \tag{96}$$

where $\Gamma_4(s)=\frac{\mathrm{d}^2}{\mathrm{d}s^2}\gamma_2(s), \quad \theta^2=\frac{\partial^2 s}{\partial x^2}.$

By using the rule (c) into the above Equation (96) to become in the form

$$\gamma_4(s)=0 \Rightarrow \Gamma_4(s)=0,$$

then Equation (96) take the form

$$w_0=-\frac{1}{12}\left[\theta_t x+\sigma_t\right]. \tag{97}$$

substituting from Eqs.(91), (94), (97) into Equations (20), (23), (25), (29), (30), (31), (32), (33)and (34), we obtain

$$\Gamma_5(s)=\Gamma_7(s)=\Gamma_9(s)=\Gamma_{13}(s)=\Gamma_{14}(s)=\Gamma_{15}(s)=\Gamma_{16}(s)=\Gamma_{17}(s)=\Gamma_{18}(s)=0 \tag{98}$$

from Equation (91) into Equation (57), we get

$$v_0=\theta^2\Gamma_{23}(s)=\frac{\partial^2}{\partial x^2}\left[\gamma_{23}(s)\right] \tag{99}$$

where $\Gamma_{23}(s)=\frac{\mathrm{d}^2}{\mathrm{d}s^2}\gamma_{23}(s), \quad \theta^2=\frac{\partial^2 s}{\partial x^2}.$

By using the rule (b) into the above Equation (99)

$$\gamma_{23}(s)=0 \Rightarrow \Gamma_{23}(s)=0, \quad v_0=0. \tag{100}$$

substituting from Equations.(91), (94) and (97), into Equations.(61), (62), (65), (68), (75), (78), (81)and (82) then,we obtain

$$\Gamma_{26}(s)=\Gamma_{27}(s)=\Gamma_{30}(s)=\Gamma_{33}(s)=\Gamma_{40}(s)=\Gamma_{43}(s)=\Gamma_{46}(s)=\Gamma_{47}(s)=0. \tag{101}$$

Substituting from Equations (91), (94), (97) and (101) into Equations (60), (69), (70), we obtain

$$\Gamma_{25}(s)=\frac{A}{\theta^2}=C, \tag{102}$$

$$\Gamma_{34}(s)=-(Ds+F), \tag{103}$$

$$\Gamma_{35}(s)=-2D. \tag{104}$$

where $\theta=\theta(y,t), \sigma=\sigma(y,t).$
and

$$A=\frac{\theta_t}{\theta^2}, C=\frac{A}{\theta^2}, \tag{105}$$

$$D=\frac{\theta_y}{\theta^2}, F=\frac{\sigma_y}{\theta}-D\sigma, \tag{106}$$

$$B=\frac{\sigma_t}{\theta}-A\sigma. \tag{107}$$

Using this notation, the Equation (97) take the following form

$$w_0=-\frac{1}{12}(As+B), \tag{(108)}$$

substituting Equations (97) and (100) into Equation (74), we obtain

$$\Gamma_{39}(s)=\frac{1}{12}C(Ds+F), \tag{109}$$

using any equation which we need into Equation (28), we obtain

$$12p_{0x}=\theta^4\Gamma_{12}(s)=\frac{\partial^4}{\partial x^4}\gamma_{12}(s) \tag{110}$$

where $\Gamma_{12}(s)=\frac{\mathrm{d}^4}{\mathrm{d}s^4}\gamma_{12}(s), \quad \theta^4=\frac{\partial^4 s}{\partial x^4}$.

By using the rule (d) after differential with respect to x into Equations (110), we obtain

$$\gamma_{12}(s)=0 \Rightarrow \Gamma_{12}(s)=0, \ p_0=0. \tag{111}$$

Substituting into Equations (9)-(12), we obtain the similarity solutions of the Broer-Kaup system Equations (1)-(4) in the form

$$u(x,y,t)=\theta P(s), \quad v(x,y,t)=\theta^2 Q(s), \quad w(x,y,t)=\theta^2 R(s)-\frac{1}{12}(As+B), \quad p(x,y,t)=\theta^3 H(s). \tag{112}$$

where

$$P(s)=f', Q(s)=g'', R(s)=h'' \text{ and } H(s)=k'''. \tag{113}$$

with $s(x,y,t)=\theta(y,t)x+\sigma(y,t)$.

Substituting from Equation (112) to obtain an ordinary differential equations from the origin system (1)-(4), we get

$$P'''-3PP''+3PR'-3P'^2+3RP'+3P^2P'+3H'=0, \tag{114}$$

$$4Q'''+24QPP'+12P^2Q'+12RQ'+4P'Q'+12QR'+4PQ''-CQ=0, \tag{115}$$

$$-12Q'-ADs+12R'Ds+12FR'+24DR-FC=0, \tag{116}$$

$$(PQ)'-3DH-FH'-DsH'=0. \tag{117}$$

where $" \, ' \, "=\frac{\mathrm{d}}{\mathrm{d}s}$.

The general solution for the variable $\theta(y,t),\sigma(y,t)$ which satisfy Equations (105)-(107) are

$$\theta(y,t)=-\frac{1}{At+Dy+c_1}, \quad \sigma(y,t)=\frac{1}{A}\left[\frac{(BD-AF)y+c_2}{At+Dy+c_1}-B\right] \tag{118}$$

where c_1,c_2 are arbitrary constants.

There some subcases for the constants A,B,D,F

$D=F=0, A\neq 0,$ the solutions of Equations (105)-(107) are

$$\theta(y,t)=\theta(t)=-\frac{1}{At+c_4}, \quad \sigma(y,t)=\sigma(t)=\frac{1}{A}\left[\frac{c_5}{At+c_4}-B\right] \tag{119}$$

where c_4,c_5 are arbitrary constants. In this case the Equations (114)-(117) take the form

$$P'''-3PP''+3PR'-3P'^2+3RP'+3P^2P'+3H'=0, \tag{120}$$

$$4Q'''+24QPP'+12P^2Q'+12(RQ)'+4P'Q'+4PQ''-CQ=0, \tag{121}$$

$$Q'=0, \tag{122}$$

$$(PQ)'=0. \tag{123}$$

the solutions for Equations (120)-(123) are

$$P(s)=\frac{c_6}{c_7}, \quad Q(s)=c_7, \tag{124}$$

$$R(s)=\frac{1}{12}cs+c_8, \quad H(s)=-\frac{c_6 c}{c_7}s^2-\frac{c_6 c_8}{c_7}s+c_9.$$

To obtain the solutions for the original system (1)-(4), we substituting from the Equations (119), (124) into Equations (112), we get

$$u(x,y,t)=-\frac{c_6}{c_7(At+c_4)},\ v(x,y,t)=-\frac{c_7}{(At+c_4)^2},$$
$$w(x,y,t)=-\frac{cs+12c_8}{12(At+c_4)^2}-\frac{1}{12}(As+B),\ p(x,y,t)=\frac{(c_6cs^2+c_6c_8s-c_7c_9)}{c_7(At+c_4)^3}. \tag{125}$$

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