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Necessity of Oversampling Theorem for Affine Frames

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ABSTRACT

Let $a,n\ge 2$ be two natural numbers. C. K. Chui and X. L. Shi proved that for any affine frame $\left\{\psi_{b;j,k}(x)=a^{j/2}\psi\left(a^j x-kb\right), j,k\in Z\right\}$ of $L^2(R)$, and the family $\left\{n^{-1/2}\psi_{j,k/n}, j,k\in Z\right\}$ is also a frame with the same bounds if n is relatively prime to a. In this paper we prove that n is relatively prime to a which is also necessary.

KEYWORDS

Affine Frame; Oversampling

1. Introduction

Let $L^2=L^2(R)$ denote, as usual, the space of all complex-valued square integrable functions on the real line with inner product $\langle,\rangle$ and norm $\| \|$. For any $\psi\in L^2=L^2(R)$, we will use the notation

$$\psi_{b;j,k}(x)=2^{j/a}\psi\left(a^j x-kb\right), j,k\in Z, \tag{1}$$

where $a>1$ and $b>0$. A function $\psi\in L^2$ is said to generate an affine frame

$$\left\{\psi_{b;j,k}: j,k\in Z\right\} \tag{2}$$

of L^2, with frame bounds A and B, where $0<A\le B<\infty$, if it satisfies

$$A\|f\|^2\le\sum_{j,k\in Z}\left|\left\langle f,\psi_{b;j,k}\right\rangle\right|^2\le B\|f\|^2, \forall f\in L^2. \tag{3}$$

The frame (2) of L^2 is called a tight frame, if (3) holds with $A=B$, see [1] and [2]. In 1993, C. K.Chui and X. L. Shi [3] proved the following oversampling theorem:

Theorem A. Let $a\ge 2$ be any positive integer and $b>0$. Also, let $\psi\in L^2$ generate a frame $\left\{\psi_{b;j,k}: j,k\in Z\right\}$ with frame bounds A and B as given by (3). Then for any positive integer n which is relatively prime to a, the family

$$\left\{n^{-1/2}\psi_{b/n;j,k}: j,k\in Z\right\} \tag{4}$$

remains a frame of L^2 with the same bounds. If $(n,a)\ne 1$, this result does not hold. But they only gave a countexample for the case where $a=2, b=1, n=2$ as in [4]. For other positive integer n and a which satisfy $(n,a)\ne 1$, they did not prove. The aim of this paper is to establish the inverse proposition of Theorem A, and then we following:

Theorem 1.1. *Let* $a \ge 2$ *be any positive integer and* $b > 0$. *Also, let* $\{\psi_{b;j,k} : j,k \in Z\}$ *be any affine frame of* L^2 *with frame bounds* A *and* B. *The family* (4) *remains a frame of* L^2 *with the same bounds*: *that is,*

$$nA\|f\|^2 \le \sum_{j,k\in Z}\left|\left\langle f,\psi_{b/n;j,k}\right\rangle\right|^2 \le nB\|f\|^2,\ f \in L^2, \tag{5}$$

if and only if n and a are relatively prime.

2. Proofs

The sufficiency has been included in the theorem 4 of [3]. In the following we will prove the necessary part of the theorem.

Suppose for any affine frame (2) of L^2 with frame bounds A and B, the family (4) is also a frame of L^2 with the same bounds. Then when (1) forms an orthonormal basis, the family (4) forms a tight frame with frame bound 1. So we just need to prove that there exists a function ψ such that the family (1) forms the orthonormal basis, but for any two positive integers n and a which satisfy $(n,a) \ge 2$, there exist two functions f_1 and f_2 such that

$$S(f_1) = \sum_{j,k\in Z}\left|\left\langle f_1,\psi_{b/n;j,k}\right\rangle\right|^2$$

Doesn't equal

$$S(f_2) = \sum_{j,k\in Z}\left|\left\langle f_2,\psi_{b/n;j,k}\right\rangle\right|^2.$$

Let $\psi(x) = \psi_H(x) = \chi_{[0,1)}(x)\operatorname{sgn}\left(\frac{1}{2}-x\right)$, then $\{\psi_{j,k} : j,k \in Z\}$ forms an orthonormal basis, which is called Haar basis. Set

$$f_1(x) = \psi_H\left(x+\frac{1}{2}\right) \quad\text{and}\quad f_2(x) = \chi_{[-1/2,1/2)}(x).$$

We prove that if $(n,a) = m \ge 2, m \in N$, then

$$S(f_1) - S(f_2) \ne 0. \tag{6}$$

$$\begin{aligned}
S(f_1) &= \sum_{j,k\in Z}\left|\left\langle f_1,\psi_{H;j,k/n}\right\rangle\right|^2 \\
&= \sum_{\substack{k=0\\ j=0}}^{\lfloor n/2\rfloor}\left(\frac{1}{2}-\frac{k}{n}\right)^2 + \sum_{\substack{k=-\lfloor n/2\rfloor\\ j=0}}^{-1}\left(\frac{3k}{n}+\frac{1}{2}\right)^2 + \sum_{\substack{k=-n\\ j=0}}^{-\lfloor n/2\rfloor-1}\left(\frac{3k}{n}+\frac{5}{2}\right)^2 + \sum_{\substack{k=-\lfloor 3n/2\rfloor\\ j=0}}^{-n-1}\left(\frac{3}{2}+\frac{k}{n}\right)^2 \\
&+ \sum_{\substack{k=\lfloor a^j n-n\rfloor+1\\ j\ge 1}}^{\lfloor a^j n/2-n/2\rfloor} a^j\left(\frac{k}{a^j n}+\frac{1}{a^j}-\frac{1}{2}\right)^2 + \sum_{\substack{k=\lfloor a^j n/2-n/2\rfloor+1\\ j\ge 1}}^{\lfloor a^j n/2\rfloor} a^j\left(\frac{1}{2}-\frac{k}{a^j n}\right)^2 + \sum_{\substack{k=-\lfloor n/2\rfloor\\ j\ge 1}}^{-1} a^j\left(\frac{2k}{a^j n}\right)^2 + \sum_{\substack{k=-n+1\\ j\ge 1}}^{-\lfloor n/2\rfloor-1} 4a^j\left(\frac{1}{a^j}+\frac{k}{a^j n}\right)^2 \\
&+ \sum_{\substack{k=-\lfloor a^j n/2+n/2\rfloor\\ j\ge 1}}^{-\lfloor a^j n/2\rfloor-1} a^j\left(\frac{k}{a^j n}+\frac{1}{2}\right)^2 + \sum_{\substack{k=-\lfloor a^j n/2+n\rfloor\\ j\ge 1}}^{-\lfloor a^j n/2+n/2\rfloor-1} a^j\left(\frac{k}{a^j n}+\frac{1}{2}+\frac{1}{a^j}\right)^2 + \sum_{\substack{k=0\\ j\le -1}}^{\lfloor a^j n/2\rfloor} a^j\left(\frac{1}{2}-\frac{k}{a^j n}\right)^2 \\
&+ \sum_{\substack{k=-\lfloor a^j n/2\rfloor\\ j\le -1}}^{-1} a^j\left(\frac{1}{2}+\frac{k}{a^j n}\right)^2 + \sum_{\substack{k=-\lfloor n/2\rfloor\\ j\le -1}}^{-\lfloor n/2-a^j n/2\rfloor-1} a^j\left(1-\frac{1}{a^j}-\frac{2k}{a^j n}\right)^2 + \sum_{\substack{k=-\lfloor n+a^j n/2\rfloor\\ j\le -1}}^{-n-1} a^j\left(\frac{1}{2}+\frac{1}{a^j}+\frac{k}{a^j n}\right)^2 \\
&+ \sum_{\substack{k=-n\\ j\le -1}}^{-\lfloor n-a^j n/2\rfloor-1} a^j\left(\frac{1}{2}-\frac{1}{a^j}-\frac{k}{a^j n}\right)^2 + \sum_{\substack{k=-\lfloor n/2+a^j n/2\rfloor-1\\ j\le -1}}^{-\lfloor n/2\rfloor-1} a^j\left(1+\frac{1}{a^j}+\frac{2k}{a^j n}\right)^2,
\end{aligned}$$

and

$$
\begin{aligned}
S(f_2) = \sum_{j,k\in Z}\left|\left\langle f_2,\psi_{H;j,k/n}\right\rangle\right|^2 &= \sum_{\substack{k=0\\ j=0}}^{\lfloor n/2\rfloor}\left(\frac{1}{2}-\frac{k}{n}\right)^2 + \sum_{\substack{k=-n\\ j=0}}^{-1}\left(\frac{1}{2}+\frac{k}{n}\right)^2 \\
&+ \sum_{\substack{k=-\lfloor 3n/2\rfloor\\ j=0}}^{-n-1}\left(\frac{3}{2}+\frac{k}{n}\right)^2 + \sum_{\substack{k=\lfloor a^j n/2-n\rfloor+1\\ j\ge 1}}^{\lfloor a^j n/2-n/2\rfloor} a^j\left(\frac{k}{a^j n}+\frac{1}{a^j}-\frac{1}{2}\right)^2 \\
&+ \sum_{\substack{k=\left[a^j n/2-n/2\right]+1\\ j\ge 1}}^{\lfloor a^j n/2\rfloor} a^j\left(\frac{1}{2}-\frac{k}{a^j n}\right)^2 + \sum_{\substack{k=-\left[a^j n/2+n/2\right]\\ j\ge 1}}^{-\lfloor a^j n/2\rfloor-1} a^j\left(\frac{k}{a^j n}+\frac{1}{2}\right)^2 \\
&+ \sum_{\substack{k=-\lfloor a^j n/2+n\rfloor\\ j\ge 1}}^{-\lfloor a^j n/2+n/2\rfloor-1} a^j\left(\frac{k}{a^j n}+\frac{1}{2}+\frac{1}{a^j}\right)^2 + \sum_{\substack{k=0\\ j\le -1}}^{\lfloor a^j n/2\rfloor} a^j\left(\frac{1}{2}-\frac{k}{a^j n}\right)^2 + \sum_{\substack{k=-\lfloor a^j n/2\rfloor\\ j\le -1}}^{-1} a^j\left(\frac{1}{2}-\frac{k}{a^j n}\right)^2 \\
&+ \sum_{\substack{k=-\lfloor n/2-a^j n/2\rfloor\\ j\le -1}}^{-\lfloor a^j n/2\rfloor-1} a^j + \sum_{\substack{k=-\lfloor n/2\rfloor\\ j\le -1}}^{-\lfloor n/2-a^j n/2\rfloor-1} a^j\left(\frac{1}{a^j}+\frac{2k}{a^j n}\right)^2 + \sum_{\substack{k=-\lfloor a^j n/2+n/2\rfloor\\ j\le -1}}^{-\lfloor n/2\rfloor-1} a^j\left(\frac{1}{a^j}+\frac{2k}{a^j n}\right)^2 \\
&+ \sum_{\substack{k=-\lfloor n-a^j n/2\rfloor\\ j\le -1}}^{-\lfloor n/2+a^j n/2\rfloor-1} a^j + \sum_{\substack{k=-n\\ j\le -1}}^{-\lfloor n-a^j n/2\rfloor-1} a^j\left(\frac{1}{2}+\frac{1}{a^j}+\frac{k}{a^j n}\right)^2 + \sum_{\substack{k=-\lfloor n+a^j n/2\rfloor\\ j\le -1}}^{-n-1} a^j\left(\frac{1}{2}+\frac{1}{a^j}+\frac{k}{a^j n}\right)^2 .
\end{aligned}
$$

Denote $\Delta = S(f_1)-S(f_2)$. We have

$$\Delta = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 ,$$

where

$$A_1 = \sum_{\substack{k=-\lfloor n/2\rfloor\\ j=0}}^{-1}\left(\frac{3k}{n}+\frac{1}{2}\right)^2 + \sum_{\substack{k=-n\\ j=0}}^{-\lfloor n/2\rfloor-1}\left(\frac{3k}{n}+\frac{5}{2}\right)^2 - \sum_{\substack{k=-n\\ j=0}}^{-1}\left(\frac{1}{2}+\frac{k}{n}\right)^2, \quad A_2 = \sum_{\substack{k=-\lfloor n/2\rfloor\\ j\ge 1}}^{-1} a^j\left(\frac{2k}{a^j n}\right)^2 + \sum_{\substack{k=-n+1\\ j\ge 1}}^{-\lfloor n/2\rfloor-1} 4a^j\left(\frac{1}{a^j}+\frac{k}{a^j n}\right)^2,$$

$$A_3 = \sum_{\substack{k=-\lfloor a^j n/2\rfloor\\ j\le -1}}^{-1} a^j\left(\frac{1}{2}+\frac{k}{a^j n}\right)^2 - \sum_{\substack{k=-\lfloor a^j n/2\rfloor\\ j\le -1}}^{-1} a^j\left(\frac{1}{2}-\frac{k}{a^j n}\right)^2,$$

$$
\begin{aligned}
A_4 = &\sum_{\substack{k=-\lfloor n/2\rfloor\\ j\le -1}}^{-\lfloor n/2-a^j n/2\rfloor-1} a^j\left(1-\frac{1}{a^j}-\frac{2k}{a^j n}\right)^2 + \sum_{\substack{k=-\lfloor n/2+a^j n/2\rfloor-1\\ j\le -1}}^{-\lfloor n/2\rfloor-1} a^j\left(1+\frac{1}{a^j}+\frac{2k}{a^j n}\right)^2 \\
&- \sum_{\substack{k=-\lfloor n/2\rfloor\\ j\le -1}}^{-\lfloor n/2-a^j n/2\rfloor-1} a^j\left(\frac{1}{a^j}+\frac{2k}{a^j n}\right)^2 - \sum_{\substack{k=-\lfloor a^j n/2+n/2\rfloor\\ j\le -1}}^{-\lfloor n/2\rfloor-1} a^j\left(\frac{1}{a^j}+\frac{2k}{a^j n}\right)^2,
\end{aligned}
$$

$$A_5 = -\sum_{\substack{k=-\lfloor n/2-a^j n/2\rfloor\\ j\le -1}}^{-\lfloor a^j n/2\rfloor-1} a^j - \sum_{\substack{k=-\lfloor n-a^j n/2\rfloor\\ j\le -1}}^{-\lfloor n/2+a^j n/2\rfloor-1} a^j,$$

and

$$A_6 = \sum_{\substack{k=-n\\ j\le -1}}^{-\lfloor n-a^j n/2\rfloor-1} a^j\left(\frac{1}{2}-\frac{1}{a^j}-\frac{k}{a^j n}\right)^2 - \sum_{\substack{k=-n\\ j\le -1}}^{-\lfloor n-a^j n/2\rfloor-1} a^j\left(\frac{1}{2}+\frac{1}{a^j}+\frac{k}{a^j n}\right)^2 .$$

In order to prove the theorem, we have three cases.

Case 1. When $a = n$.
We have $\lfloor a^j n/2 \rfloor = 0$ if $j \le -1$. Thus, if n is an even integer, we can get

$$A_1 = \frac{n}{6} + \frac{4}{3n}; A_2 = \frac{n}{3(a-1)} + \frac{2}{3n(a-1)}; \quad A_3 = 0; A_4 = \sum_{j \le -1} a^j; A_5 = \sum_{j \le -1} (2-n) a^j; A_6 = 0$$

So, we have

$$\Delta = \frac{n^2 - 5n + 18}{6(n-1)} + \frac{4}{3n} + \frac{2}{3n(n-1)} > 0.$$

If n is an odd integer, we have

$$A_1 = \frac{n}{6} - \frac{1}{6n}; A_2 = \frac{n}{3(a-1)} - \frac{1}{3n(a-1)}; A_3 = 0; \quad A_4 = -\frac{1}{n}; A_5 = \sum_{j \le -2} (1-n) a^j + \frac{2-n}{n} = \frac{1}{n} - 1; A_6 = 0.$$

So, we have

$$\Delta = \frac{n^2 - 6n + 9}{6n} + \frac{n^2 - 5n + 4}{3n(n-1)} \ne 0.$$

Case 2. When $a \ge n + 1$.
If n is an even integer, we have

$$A_1 = \frac{n}{6} + \frac{4}{3n}; A_2 = \frac{n}{3(a-1)} + \frac{2}{3n(a-1)}; A_3 = 0; \quad A_3 = 0; A_4 = \sum_{j \le -1} a^j; A_5 = \sum_{j \le -1} (2-n) a^j; A_6 = 0.$$

Thus

$$\Delta = \frac{an^2 - 5n^2 + 18n + 8a - 4}{6n(a-1)} > \frac{n(n^2 - 5n + 7) + 19n - 4}{6n(a-1)} > 0.$$

If n is an odd integer, we can get $n \ge 3$ because of $(n, a) \ge 3$. As in the case 1, we also have

$$A_1 = \frac{n}{6} - \frac{1}{6n}; A_2 = \frac{n}{3(a-1)} - \frac{1}{3n(a-1)}; \quad A_3 = 0; A_4 = 0; A_5 = \frac{1-n}{a-1}; A_6 = 0.$$

So, we get

$$\Delta = \frac{an^2 - 5n^2 + 6n - a - 1}{6n(a-1)} \ge \frac{n(n-2)^2 + n - 2}{6n(a-1)} > 0.$$

Case 3. When $a < n$.
If n is an even integer. Let

$$\Lambda_1 = \{ j \le -1; j \in Z \text{ and } a^j n/2 \text{ is a positive integer} \}$$

and

$$\Lambda_2 = \{ j \le -1; j \in Z \text{ and } a^j n/2 \text{ is a not positive integer} \}$$

When $\Lambda_1 \ne \varnothing$, there exists an integer J satisfying $J \in \Lambda_1, J - 1 \in \Lambda_2$. Therefore we have

$$A_3 + A_4 + A_5 + A_6 = \sum_{J \le j \le -1} \left(3a^{2j} n/2 - a^j n\right) + \sum_{j \le J-1} f_j(x_j) + \sum_{j \le J-1} (3-n) a^j,$$

where $x_j = \lfloor a^j n/2 \rfloor$, $f_j(x_j) = -\frac{6}{n}\left[x_j^2 - (a^j n - 1) x_j\right]$. When $\frac{1}{2} a^j n < 1$, we have $\frac{1}{2} a^j n > a^j n - 1$ and $f_j\left(\frac{1}{2} a^j n\right) < 0$. Thus we have

$$\sum_{j\le J-1} f_j\left(x_j\right)=\sum_{\log_a\frac{2}{n}\le j\le J-1} f_j\left(x_j\right)\ge \sum_{\log_a\frac{2}{n}\le j\le J-1} f_j\left(\frac{1}{2}a^j n\right)>\sum_{j\le J-1} f_j\left(\frac{1}{2}a^j n\right)=\sum_{j\le J-1} -3a^j+\frac{3}{2}n\,a^j.$$

Therefore

$$\Delta\ge\frac{n^2\left(a-2\right)^2+\left(a+2\right)^2+7a^2-8}{6n\left(a^2-1\right)}>0.$$

When $\Lambda_1=\varnothing$, similar to the case $\Lambda_1\ne\varnothing$, we also have

$$\sum_{j\le -1} f_j\left(x_j\right)=\sum_{\log_a\frac{2}{n}\le j\le -1} f_j\left(x_j\right)\ge \sum_{\log_a\frac{2}{n}\le j\le -1} f_j\left(\frac{1}{2}a^j n\right)>\sum_{j\le -1} f_j\left(\frac{1}{2}a^j n\right)=\sum_{j\le -1} -3a^j+\frac{3}{2}n\,a^j.$$

So we have

$$\Delta\ge\frac{n\left(a-2\right)^2}{6\left(a^2-1\right)}+\frac{4}{3n}+\frac{2}{3n\left(a-1\right)}>0.$$

If n is an odd integer. We have

$$A_3+A_4+A_5+A_6=\sum_{\log_a\frac{1}{n}\le j\le -1} g_j\left(x_j\right)+h_j\left(y_j\right)+a^j-na^j$$

where

$$x_j=\left\lfloor a^j n/2\right\rfloor, g_j\left(x_j\right)=-\frac{2}{n}\left[x_j^2-\left(a^j n-1\right)x_j\right];$$

$$h_j=\left\lfloor a^j n/2+1/2\right\rfloor, h_j\left(y_j\right)=-\frac{4}{n}\left(y_j^2-a^j n\,y_j\right).$$

A familiar calculation shows

$$\Delta>-\frac{1}{n}\log_a n+\frac{n}{48}-\frac{7}{6n}.$$

Since $\left(a,n\right)\ge 3$ and $a<n$, we have $n\ge 9, a\ge 3$. Also when $n=9$ and $a=3$, we have

$$\Delta=\frac{14}{27}>0.$$

When $n=9$ and $a=6$, obviously we have

$$\Delta\ge\frac{10}{27}>0.$$

When $n\ge 15$, $-\frac{1}{n}\log_a n+\frac{n}{48}-\frac{7}{6n}>\frac{5}{144}$. So we have $\Delta>0$ in this case. This completes the proof of the theorem.

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