

Analytical Approach to Differential Equations with Piecewise Continuous Arguments via Modified Piecewise Variational Iteration Method

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ABSTRACT

In the present article, we apply the modified piecewise variational iteration method to obtain the approximate analytical solutions of the differential equations with piecewise continuous arguments. This technique provides a sequence of functions which converges to the exact solution of the problem. Moreover, this method reduces the volume of calculations because it does not need discretization of the variables, linearization or small perturbations. The results seem to show that the method is very reliable and convenient for solving such equations.

KEYWORDS

Delay Differential Equations; Piecewise Continuous Arguments; Variational Iteration Method; Approximation

1. Introduction

Differential equations with piecewise continuous arguments (EPCA) are special type of delay differential equations (DDEs). The theory of EPCA was initiated in [1,2] and developed by many authors [3-7]. These systems have been under intensive investigation for the last twenty years. EPCA describe hybrid dynamical systems and combine properties of both differential and difference equations. They are appeared in modeling of various problems in real life such as biology, mechanics, and electronics. For some applications of this equation we refer the interested reader to [1,8-10]. Several important properties of the analytic solution of EPCA as well as numerical methods have been studied in [11-16].

In this paper, we consider the following two EPCA:

$$\begin{cases} u'(t) = a_0 u(t) + a_1 u([t]), & t \geq 0, \\ u(0) = u_0, \end{cases} \quad (1)$$

and the coupled system

$$\begin{cases} x'(t) = a_2 x(t) + a_3 y([t]), \\ y'(t) = a_4 y(t) + a_5 x([t]), \end{cases} \quad (2)$$

with initial value $X(0) = (x_0, y_0)^T$, where a_i ($i = 0, 1, \dots, 5$) are real constants and $[.]$ denotes the greatest integer function and $X(t) = (x(t), y(t))^T$.

In this work, we apply the modified piecewise variational iteration method (MPVIM) to systems (1) and (2) to obtain approximate analytical solutions. The VIM gives several successive approximations by using the iteration of the correction functional. This method was proposed by the Chinese researcher Jihuan He [17-19] as a modification of a general Lagrange multiplier method [20]. VIM is one of the non-perturbation methods that does not require any small or large parameter. An elementary introduction of VIM is given in [21]. The main concepts in VIM, such as general Lagrange multiplier, restricted variation, correction functional are explained systematically. For more comprehensive survey on this method and its applications, the reader is referred to the review articles [22,23] and the references therein.

The VIM has been favorably applied to various kinds of linear and nonlinear problems. The main property of the method is in flexibility and ability to solve linear and nonlinear equations accurately and conveniently. The flexibility and adaptation provided by this method have made the method a strong candidate for approximate analytical solutions. The VIM plays an important role in recent researches for solving various kinds of problems (see for example [24-28] and the references therein). However, the researches on the application of VIM on DDE are relatively fewer. As far as we know, only delay Burgers equation [29], delay logistic equation [30] and pantograph equation [31-33] are considered. As for the analytical study of EPCA with VIM, up to now, there are almost no results published. Therefore, we will conduct this study.

The organization of this paper is as follows. In Section 2, we simply provide the mathematical framework of the VIM. In Section 3, we apply the modified piecewise variational iteration method on the systems (1) and (2) after analyzing the conventional VIM and piecewise variational iteration method. Some numerical results are given in Section 4. Finally, in Section 5, a brief conclusion is provided.

2. He's Variational Iteration Method

In this section, we introduce the basic idea underlying the VIM for solving nonlinear equations. Consider the general differential equation

$$Lu + Nu = g(x), \tag{3}$$

where L and N are linear and nonlinear operators, respectively, and $g(x)$ is the inhomogeneous term. In VIM, a correction functional for (3) can be written as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s) [Lu_n(s) + N\tilde{u}_n(s) - g(s)] ds \tag{4}$$

where λ is a general Lagrange's multiplier, which can be identified optimally via integration by parts and the variational theory, and \tilde{u}_n denotes the restricted variation, *i.e.* $\delta\tilde{u}_n = 0$. It is to be noted that the Lagrange multiplier λ can be a constant or a function. After determining the Lagrange multiplier λ , an iteration formula, without restricted variation, should be used for the determination of the successive approximations $u_{n+1}(x)$ of the solution $u(x)$. The zeroth approximation u_0 can be selected freely. Consequently, the solution is given by

$$u(x) = \lim_{n \rightarrow \infty} u_n(x) \tag{5}$$

3. The Application of VIM

In this section the application of VIM is discussed for solving systems (1) and (2).

3.1. System (1)

We consider system (1), according to the VIM, the correction function is given by

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s) (u_n'(s) - a_0 u_n(s) - a_1 u_n([s])) ds \tag{6}$$

To find the optimal value of λ we have

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda(s) u_n'(s) ds, \tag{7}$$

that results

$$\delta u_{n+1}(t) = (1 + \lambda|_{s=t}) \delta u_n(t) - \delta \int_0^t \lambda'(s) u_n(s) ds. \tag{8}$$

Thus we have the following stationary conditions

$$\begin{cases} 1 + \lambda|_{s=t} = 0, \\ \lambda'(s)|_{s=t} = 0. \end{cases} \tag{9}$$

This in turn gives $\lambda = -1$. So we obtain the following iteration formula

$$u_{n+1}(t) = u_n(t) - \int_0^t (u_n'(s) - a_0 u_n(s) - a_1 u_n([s])) ds \tag{10}$$

and the approximation solution is given by

$$u(t) = \lim_{n \rightarrow \infty} u_{n+1}(t). \quad (11)$$

During the process of computation, the greatest integer function $[\cdot]$ causes us many problems. To overcome them, we recall a modified VIM: the piecewise variational iteration method (PVIM), which was introduced by Geng [34,35]. In PVIM, the interval $[0, X]$ is divided into some equal subintervals, then the n_i -order approximation $u_{i,n_i}(x)$ are obtained on these subintervals. Following this way, we introduce the modified piecewise variational iteration method (MPVIM). In our method, the interval $[0, \infty)$ is divided into lots of subintervals $[k, k+1)$ with unit length, where $k \in \mathbb{N}$.

On the interval $[0, 1]$, let

$$\begin{aligned} u_{1,n+1}(t) &= u_{1,0}(t) - \int_0^t (u'_{1,n}(s) - a_0 u_{1,n}(s) - a_1 u_{1,n}([s])) ds \\ u_{1,0}(t) &= u(0), \end{aligned} \quad (12)$$

where $t \in [0, 1]$. Then we can obtain the n_1 -order approximation $u_{1,n_1}(t)$ on $[0, 1]$.

On the interval $[1, 2]$, let

$$\begin{aligned} u_{2,n+1}(t) &= u_{2,0}(t) - \int_0^t (u'_{2,n}(s) - a_0 u_{2,n}(s) - a_1 u_{2,n}([s])) ds \\ u_{2,0}(t) &= u_{1,n_1}(1). \end{aligned} \quad (13)$$

The integration in (13) can be computed in $[0, 1]$ and $[1, t]$, respectively. Then the n_2 -order approximation $u_{2,n_2}(t)$ on $[1, 2]$ can be obtained.

In a similar way, on the interval $[k-1, k]$, $k = 3, 4, \dots$ let

$$\begin{aligned} u_{k,n+1}(t) &= u_{k,0}(t) - \int_0^t (u'_{k,n}(s) - a_0 u_{k,n}(s) - a_1 u_{k,n}([s])) ds \\ u_{k,0}(t) &= u_{k-1,n_{k-1}}(k-1). \end{aligned} \quad (14)$$

The integration in (14) can be computed in a series of subintervals: $[0, 1], [1, 2], \dots, [k-1, t]$. Then we can obtain the n_k -order approximation $u_{k,n_k}(t)$ on $[k-1, k]$.

Therefore, according to (12)-(14), the approximation of (1) on the entire interval $[0, \infty)$ can be obtained.

3.2. System (2)

According to VIM, the iteration formula for (2) can be constructed as follows

$$\begin{aligned} x_{n+1}(t) &= x_n(t) - \int_0^t (x'_n(s) - a_2 x_n(s) - a_3 y_n([s])) ds, \\ y_{n+1}(t) &= y_n(t) - \int_0^t (y'_n(s) - a_4 y_n(s) - a_5 x_n([s])) ds. \end{aligned} \quad (15)$$

Similar to Subsection 3.1, in view of MPVIM we have the following formulas.

On the interval $[0, 1]$, let

$$\begin{aligned} x_{1,n+1}(t) &= x_{1,0}(t) - \int_0^t (x'_{1,n}(s) - a_2 x_{1,n}(s) - a_3 y_{1,n}([s])) ds, \\ y_{1,n+1}(t) &= y_{1,0}(t) - \int_0^t (y'_{1,n}(s) - a_4 y_{1,n}(s) - a_5 x_{1,n}([s])) ds, \\ x_{1,0}(t) &= x(0), \\ y_{1,0}(t) &= y(0). \end{aligned}$$

Then we can obtain the n_1 -order approximation $X_{1,n_1}(t)$ on $[0, 1]$, where $X(t) = (x(t), y(t))^T$.

On the interval $[1, 2]$, let

$$\begin{aligned} x_{2,n+1}(t) &= x_{2,0}(t) - \int_0^t (x'_{2,n}(s) - a_2 x_{2,n}(s) - a_3 y_{2,n}([s])) ds, \\ y_{2,n+1}(t) &= y_{2,0}(t) - \int_0^t (y'_{2,n}(s) - a_4 y_{2,n}(s) - a_5 x_{2,n}([s])) ds, \end{aligned}$$

$$\begin{aligned} x_{2,0}(t) &= x_{1,n_1}(1), \\ y_{2,0}(t) &= y_{1,n_1}(1). \end{aligned}$$

Then we can obtain the n_2 -order approximation $X_{2,n_2}(t)$ on $[1, 2]$. Similarly, on the interval $[k-1, k]$, $k = 3, 4, \dots$ let

$$\begin{aligned} x_{k,n+1}(t) &= x_{k,0}(t) - \int_0^t (x'_{k,n}(s) - a_2 x_{k,n}(s) - a_3 y_{k,n}(s)) ds, \\ y_{k,n+1}(t) &= y_{k,0}(t) - \int_0^t (y'_{k,n}(s) - a_4 y_{k,n}(s) - a_5 x_{k,n}(s)) ds, \\ x_{k,0}(t) &= x_{k-1,n_{k-1}}(k-1), \\ y_{k,0}(t) &= y_{k-1,n_{k-1}}(k-1). \end{aligned}$$

Then we can obtain the n_k -order approximation $X_{k,n_k}(t)$ on $[k-1, k]$.

Therefore, according to (16)-(18), the approximation of coupled system (2) on the entire interval $[0, \infty)$ can be obtained.

4. Results and Discussion

In this section, we apply the MPVIM presented in Section 3 and the classical θ -methods to two concrete EPCA. Numerical results show that the MPVIM is very effective.

For (1), we choose $a_0 = 2$, $a_1 = -1$ and $u_0 = 1$. According to (12)-(14), taking $k = 3$ and $n_i = 5$, $i = 1, \dots, k$. We can obtain the approximations of (1) on $[0, 3]$. The numerical results are depicted in **Figure 1**. This figure shows the comparison of approximation obtained by using the present method with the exact solution and the numerical solution. Moreover, for (2), we choose $a_2 = 1$, $a_3 = -2$, $a_4 = 2$, $a_5 = -1$ and $x_0 = y_0 = 1$. In **Figure 2** we compare the 5th-order approximation of MPVIM with the numerical solution.

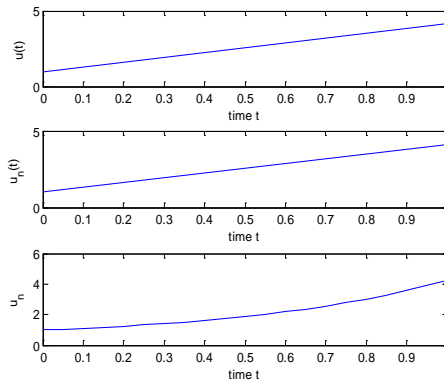


Figure 1. A comparison of the results of the exact solution (upper), the 5th-order MPVIM solution (middle) and the numerical solution (lower) with $\theta = 0.6$ and $m = 20$ to (1).

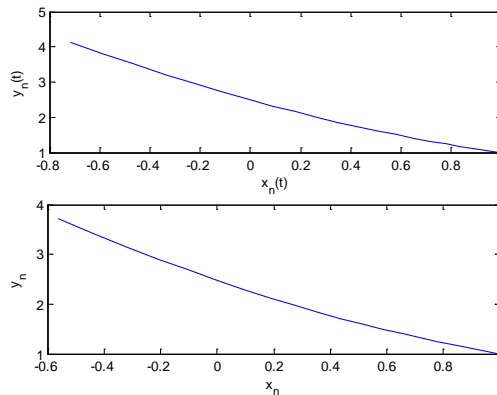


Figure 2. A comparison of the results of the 5th-order MPVIM solution (upper) and the numerical solution (lower) with $\theta = 0.3$ and $m = 20$ to (2).

The above numerical examples demonstrate that the present method is quite effective and simple.

5. Conclusions

An efficient algorithm based on the VIM has been successfully applied to the EPCA. As can be seen from the numerical results, implementing only a few steps in the MPVIM, the approximate analytical solutions with high accuracy can be obtained.

It can be concluded that the MPVIM is a powerful and promising tool for solving such kinds of delay differential equations. This method can also be extended to the EPCA of the advanced type and mixed type, which are our future research issues.

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