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Wronskian Determinant Solutions for the (3 + 1)-Dimensional Boiti-Leon-Manna-Pempinelli Equation

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ABSTRACT

In this paper, we consider (3 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation. Based on the bilinear form, we derive exact solutions of (3 + 1)-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation by using the Wronskian technique, which include rational solutions, soliton solutions, positons and negatons.

Keywords: (3 + 1)-Dimensional Boiti-Leon-Manna-Pempinelli Equation; The Wronskian Technique; Soliton; Negaton; Positon

1. Introduction

The Wronskian technique is introduced by Freeman and Nimmo [1]. After that, many researches are based on the Wronskian technique.

The (2 + 1)-dimensional BLMP equation was first derived in [2]:

$$u_{yt}+u_{xxxy}-3u_{xx}u_y-3u_xu_y=0 \tag{1}$$

where $u=u(x,y,t)$ and subscripts represent partial differentiation with respect to the given variable. This equation was used to describe the (2 + 1)-dimensional interaction of the Riemann wave propagated along the y-axis with a long wave propagated along the x-axis. The Painlevé analysis, Lax pairs, Bäcklund transformation, symmetry, similarity reductions and new exact solutions of the (2 + 1)-dimensional BLMP equation are given in [2-4]. In [5], based on the binary Bell polynomials, the bilinear form for the BLMP equation is obtained. New solutions of (2 + 1)-dimensional BLMP equation from Wronskian formalism and the Hirota method are obtained in [6,7].

The (3 + 1)-dimensional BLMP equation

$$\begin{aligned}&u_{yt}+u_{zt}+u_{xxxy}+u_{xxxz}-3u_x\left(u_{xy}+u_{xz}\right)\\&-3u_{xx}\left(u_y+u_z\right)=0\end{aligned} \tag{2}$$

which was introduced in [8] has the bilinear form

$$\left(D_yD_t+D_zD_t+D_yD_x^3+D_zD_x^3\right)f\cdot f=0 \tag{3}$$

just by substituting $u=-2\left(\ln f(x,y,z,t)\right)_x$ into equation (2), where the bilinear differential operator D is defined by Hirota [9] as

$$\begin{aligned}&D_t^mD_x^na(t,x)\cdot b(t,x)\\&=\frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n}a(t+s,x+y)b(t-s,x-y)\Big|_{s=0,y=0}\end{aligned}$$

2. Wronskian Formulation

Solutions determined by $u=-2(\ln f)_x$ to the Equation (2) are called Wronskian solutions, where

$$\begin{aligned}f&=W(\phi_1,\phi_2,\cdots,\phi_N)=\left|\widehat{N-1}\right|\\&=\begin{bmatrix}\phi_1^{(0)}&\phi_1^{(1)}&\cdots&\phi_1^{(N-1)}\\\phi_2^{(0)}&\phi_2^{(1)}&\cdots&\phi_2^{(N-1)}\\\vdots&\vdots&\ddots&\vdots\\\phi_N^{(0)}&\phi_N^{(1)}&\cdots&\phi_N^{(N-1)}\end{bmatrix},N>1,\end{aligned} \tag{4}$$

and

$$\phi_i^{(0)}=\phi_i,\phi_i^{(j)}=\frac{\partial^2}{\partial x^j}\phi_i,\ j\ge 1,1\le i\le N. \tag{5}$$

Lemma 1

$$\left|D,a,b\right|\left|D,c,d\right|-\left|D,a,c\right|\left|D,b,d\right|+\left|D,a,d\right|\left|D,b,c\right|=0\ , \quad (6)$$

where D is $N\times(N-2)$ matrix, and a,b,c,d are n-dimensional column vectors.

Lemma 2 Set $b_j\ (j=1,\cdots,n)$ to be an n-dimensional column vector, and $r_j\ (j=1,\cdots,n)$ to be a real constant but not to be zero. Then we have

$$\sum_{i=1}^{N} r_i\left|b_1,b_2,\cdots,b_N\right|=\sum_{j=1}^{N}\left|b_1,b_2,\cdots,rb_j,\cdots,a_N\right|, \quad (7)$$

where $rb_j=\left(r_1b_{1j},r_2b_{2j},\cdots,r_Nb_{Nj}\right)^{\mathrm{T}}$.

Lemma 3 The following equalities hold:

$$\left[\sum_{i=1}^{N}\lambda_{ii}\left(\sum_{i=1}^{n}\lambda_{ii}\left|\widehat{N-1}\right|\right)\right]\left|\widehat{N-1}\right|=\left(\sum_{j=1}^{N}\lambda_{ii}\left|\widehat{N-1}\right|\right)^2. \quad (8)$$

Proposition. Assuming that $\phi_i=\phi_i(x,y,z,t)$ (where $t\ge 0,-\infty<x,y,z<+\infty,i=1,2,\cdots,N$) has continuous derivative up to any order and satisfies the following linear differential conditions

$$\phi_{i,t}=-4\phi_{i,xxx},\ \phi_{i,xx}=\sum_{i=1}^{N}\lambda_{ij}\phi_j,\ \phi_{i,y}=\phi_{i,x},\ \phi_{i,z}=\phi_{i,x} \quad (9)$$

then $f=\left|\widehat{N-1}\right|$ defined by Equation (4) solves the bilinear Equation (2).

Proof. Using the conditions (9), we get that

$$f_z=f_y=f_x=\left|\widehat{N-2},N\right|,$$

$$f_{xz}=f_{xy}=f_{xx}=\left|\widehat{N-3},N-1,N\right|+\left|\widehat{N-2},N+1\right|,$$

$$f_{xxz}=f_{xxy}=f_{xxx}=\left|\widehat{N-4},N-2,N-1,N\right|+2\left|\widehat{N-3},N-1,N+1\right|+\left|\widehat{N-2},N+2\right|$$

$$\begin{aligned}f_{xxxz}=f_{xxxy}=f_{xxxx}&=\left|\widehat{N-5},N-3,,N-2,,N-1,N\right|+3\left|\widehat{N-4},N-2,N-1,N+1\right|\\&+3\left|\widehat{N-3},N-1,N+2\right|+2\left|\widehat{N-3},N,N+1\right|+\left|\widehat{N-2},N+3\right|\end{aligned}$$

$$f_t=-\left(4\left|\widehat{N-4},N-2,N-1,N\right|-4\left|\widehat{N-3},N-1,N+1\right|+4\left|\widehat{N-2},N+2\right|\right),$$

$$f_{yt}=f_{zt}=f_{xt}=-\left(4\left|\widehat{N-5},N-3,N-2,N-1,N\right|-4\left|\widehat{N-3},N,N+1\right|+4\left|\widehat{N-2},N+3\right|\right).$$

Hence, we have

$$\begin{aligned}&\left(D_yD_t+D_zD_t+D_yD_x^3+D_zD_x^3\right)f\cdot f\\&=6\left|\widehat{N-1}\right|\left(\left|\widehat{N-5},N-3,,N-2,,N-1,N\right|\right.\\&\quad-\left|\widehat{N-4},N-2,N-1,N+1\right|-2\left|\widehat{N-3},N,N+1\right|\\&\quad\left.-\left|\widehat{N-3},N-1,N+2\right|+\left|\widehat{N-2},N+3\right|\right)\\&\quad+24\left|\widehat{N-3},N-1,N+1\right|\left|\widehat{N-2},N\right|\\&\quad-6\left(\left|\widehat{N-3},N-1,N\right|+\left|\widehat{N-2},N+1\right|\right)^2.\end{aligned} \quad (10)$$

With the help of Lemma 2 and Lemma 3, we obtain

$$\begin{aligned}&6\left|\widehat{N-1}\right|\left(\left|\widehat{N-5},N-3,,N-2,,N-1,N\right|\right.\\&-\left|\widehat{N-4},N-2,N-1,N+1\right|+2\left|\widehat{N-3},N,N+1\right|\\&\left.-\left|\widehat{N-3},N-1,N+2\right|+\left|\widehat{N-2},N+3\right|\right)\\&=6\left(-\left|\widehat{N-3},N-1,N\right|+\left|\widehat{N-2},N+1\right|\right)^2.\end{aligned} \quad (11)$$

Substituting Equation (11) into Equation (10) and using lemma 1, we get

$$\begin{aligned}&\left(D_yD_t+D_zD_t+D_yD_x^3+D_zD_x^3\right)f\cdot f\\&=-24\left(\left|\widehat{N-3},N-2,N+1\right|\left|\widehat{N-3},N-1,N\right|\right.\\&-\left|\widehat{N-3},N-2,N-1\right|\left|\widehat{N-3},N+1,N\right|\\&\left.+\left|\widehat{N-3},N-2,N\right|\left|\widehat{N-3},N+1,N-1\right|\right)=0.\end{aligned}$$

Therefore, we have shown that $f=\left|\widehat{N-1}\right|$ solves Equation (4) under the linear differential conditions (9), The corresponding solution of Equation (2) is

$$u=-2\frac{f_x}{f}=-2\frac{\left|\widehat{N-2},N\right|}{\left|\widehat{N-1}\right|}. \quad (12)$$

3. Wronskian Solutions

In what follows, according to [10-12], we would like to present a few special Wronskian solutions to the (3 +

1)-dimensional Boiti-Leon-Manna-Pempinelli equation by solving the linear conditions (9).

It is well known that the corresponding Jordan form of a real marix

$$A=\begin{bmatrix} J(\lambda_1) & & & 0 \\ 1 & J(\lambda_2) & & \\ & \ddots & \ddots & \\ 0 & & 1 & J(\lambda_m) \end{bmatrix}_{N\times N}, \tag{13}$$

have the following two type of blocks:

1)

$$J(\lambda_i)=\begin{bmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{bmatrix}_{k_i\times k_i}, \tag{14}$$

2)

$$J(\lambda_i)=\begin{bmatrix} A_i & & & 0 \\ I_2 & A_i & & \\ & \ddots & \ddots & \\ 0 & & I_2 & A_i \end{bmatrix}_{l_i\times l_i}, \quad A_i=\begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}, I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{15}$$

where λ_i, α_i and β_i are all real constants. The first type of blocks have the real eigenvalue λ_i with algebraic multiplicity $k_i\left(\sum_{i=1}^{n} k_i = N\right)$, and the second type of blocks have the complex eigenvalue $\lambda_i^{\pm}=\alpha_i \pm \beta_i\sqrt{-1}$ with algebraic multiplicity l_i.

3.1. Rational Solutions

Suppose *A* have the first type of Jordan blocks

$$A=\begin{bmatrix} \lambda_1 & & & 0 \\ 1 & \lambda_1 & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_1 \end{bmatrix}_{N\times N}, \tag{16}$$

In this case, if the eigenvalue $\lambda_1 = 0$, corresponding to the following form:

$$A=\begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}_{N\times N}, \tag{17}$$

from the condition (9), we get

$$\phi_{i,xx}=0, \phi_{i,t}=-4\phi_{i,xxx}, \phi_{i,y}=\phi_{i,x}, \phi_{i,z}=\phi_{i,x}, i\ge 1. \tag{18}$$

where $\phi_i(i\ge 1)$ are all polynomials in x, y, z and t, and a general Wronskian solution to the (3 + 1) dimensional Boiti-Leon-Manna-Pempinelli Equation (2)

$$u=-2\partial_x \ln W\left(\phi_1,\phi_2,\cdots,\phi_{k_1}\right), \tag{19}$$

is called a rational Wronskian solution.

From Equation (18), we get

$$\phi_{1,xx}=0, \phi_{1,t}=-4\phi_{i,xxx}, \phi_{1,y}=\phi_{1,x}, \phi_{1,z}=\phi_{1,x}. \tag{20}$$

Solving Equation (20) by using Maple, we get the following formulas:

$$\phi_1 = C_1(x+y+z)+C_2. \tag{21}$$

Similarly, by solving

$$\begin{aligned} &\phi_{i+1,xx}=0, \phi_{i+1,t}=-4\phi_{i+1,xxx}, \\ &\phi_{i+1,y}=\phi_{i+1,x}, \phi_{i+1,z}=\phi_{i+1,x}, i\ge 1, \end{aligned} \tag{22}$$

then two special rational solution of lower-order are obtained after setting some integral constants to be zero.

1) Zero-order: Taking $\phi_1 = C_1(x+y+z)+C_2$, the corresponding Wronskian determinant and the associated rational Wronskian solution of zero-order read

$$f=W(\phi_1)=\phi_1=C_1(x+y+z)+C_2, \tag{23}$$

$$u=-2\partial_x \ln W(\phi_1)=-\frac{2C_1}{C_1(x+y+z)+C_2}, \tag{24}$$

where C_1, C_2 are arbitrary constants.

2) First-order: Taking $\phi_1 = C_1(x+y+z)+C_2$, we can have

$$\begin{aligned} \phi_2 = &\frac{C_1}{6}\Big(x^3+(3z+3y)x^2+3(y+z)^2 x-24t+z^3+3zy^2 \\ &+3z^2y+y^3\Big)+\frac{1}{2}C_2x^2+\frac{1}{6}(6C_2z+6C_3+6C_2y)x \\ &+\frac{1}{2}C_2y^2+\frac{1}{6}(6C_2z+6C_3y)+C_4+\frac{1}{2}C_2z^2+C_3z. \end{aligned} \tag{25}$$

Then, the corresponding Wronskian determinant and rational Wronskian solution of first-order are

$$\begin{aligned} f&=W(\phi_1,\phi_2)=P, \\ u&=-2\partial_x \ln W(\phi_1,\phi_2) \\ &=\frac{C_1^2\left(2yz+2xy+2xz+x^2+y^2+z^2\right)}{P} \\ &\quad+\frac{2C_1C_2(x+y+z)+C_2^2}{P}, \end{aligned}$$

where

$$\begin{aligned} P=&C_1^2\Big(xy^2+xz^2+x^2z+x^2y+z^2y+zy^2+2xyz+\frac{1}{3}x^3 \\ &+\frac{1}{3}y^3+\frac{1}{3}z^3+4t\Big)+C_1C_2\big(2xy+2yz+2xy \\ &+x^2+y^2+z^2\big)+C_2^2(x+y+z)+C_2C_3-C_1C_4 \end{aligned}$$

and C_1, C_2, C_3, C_4 are arbitrary real constants. Similarly, we can obtain more higher order rational Wronskian solutions.

3.2. Solitons, Negatons and Positons

3.2.1. Solitons

If A becomes to the following form

$$A=\begin{bmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & \ddots & \ddots & \\ 0 & & & \lambda_N\end{bmatrix}_{N\times N}, \tag{26}$$

where the eigenvaluce $\lambda_i \neq 0$. Substituting the form of expression (26) into Equation (9), the following system of differential equations is obtained

$$\begin{aligned}&\left(\phi_i(\lambda_i)\right)=\lambda_i\left(\phi_i(\lambda_i)\right),\left(\phi_i(\lambda_i)\right)_t=-4\left(\phi_i(\lambda_i)\right)_{xxx},\\ &\left(\phi_i(\lambda_i)\right)_y=\left(\phi_i(\lambda_i)\right)_x,\left(\phi_i(\lambda_i)\right)_z=\left(\phi_i(\lambda_i)\right)_x,\end{aligned} \tag{27}$$

By solving system (27), we get the *n*-soliton solution of Equation (2)

$$u=-2\partial_x \ln W\left(\phi_1,\phi_2,\cdots,\phi_N\right), \tag{28}$$

with ϕ_i being defined by

$$\begin{aligned}&\phi_i=\cosh\left(\sqrt{\lambda_i}x+\sqrt{\lambda_i}y+\sqrt{\lambda_i}z-4\lambda_i^{\frac{3}{2}}t\right),\quad i\ \ odd\\ &\phi_i=\sinh\left(\sqrt{\lambda_i}x+\sqrt{\lambda_i}y+\sqrt{\lambda_i}z-4\lambda_i^{\frac{3}{2}}t\right),\quad i\ \ even\end{aligned} \tag{29}$$

where $0<\lambda_1<\lambda_2<\cdots<\lambda_N$ are arbitrary constants.

We present the 1-soliton and 2-soliton solutions

$$\begin{aligned}u_1&=-2\partial_x \ln\left(\cosh\left(\sqrt{\lambda_1}x+\sqrt{\lambda_1}y+\sqrt{\lambda_1}z-4\lambda_1^{\frac{3}{2}}t\right)\right)\\ &=-2\sqrt{\lambda_1}\tanh\left(\sqrt{\lambda_1}x+\sqrt{\lambda_1}y+\sqrt{\lambda_1}z-4\lambda_1^{\frac{3}{2}}t\right)\end{aligned}$$

$$\begin{aligned}u_2&=-2\partial_x \ln W\left(\cosh\left(\sqrt{\lambda_1}x+\sqrt{\lambda_1}y+\sqrt{\lambda_1}z-4\lambda_1^{\frac{3}{2}}t\right),\right.\\ &\qquad\left.\sinh\left(\sqrt{\lambda_2}x+\sqrt{\lambda_2}y+\sqrt{\lambda_2}z-4\lambda_2^{\frac{3}{2}}t\right)\right)\\ &=\frac{2\left(\lambda_1-\lambda_2\right)}{P-Q}\end{aligned}$$

where

$$\begin{aligned}P&=\sqrt{\lambda_2}\coth\left(\sqrt{\lambda_2}x+\sqrt{\lambda_2}y+\sqrt{\lambda_2}z-4\lambda_2^{\frac{3}{2}}t\right)\\ Q&=\sqrt{\lambda_1}\tanh\left(\sqrt{\lambda_1}x+\sqrt{\lambda_1}y+\sqrt{\lambda_1}z-4\lambda_1^{\frac{3}{2}}t\right)\end{aligned}$$

Similarly, we can obtain 3-soliton, 4-soliton solution and *n*-soliton.

3.2.2. Negatons and Positons

If the eigenvalue $\lambda_1 \neq 0, J(\lambda_1)$ becomes to the following form

$$J(\lambda_1)=\begin{bmatrix}\lambda_1 & & & 0\\ 1 & \lambda_1 & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_1\end{bmatrix}_{k_1\times k_1}, \tag{30}$$

We start from the eigenfuction $\phi_1(\lambda_1)$, which is determined by

$$\begin{aligned}&\left(\phi_1(\lambda_1)\right)=\lambda_1\left(\phi_1(\lambda_1)\right),\left(\phi_1(\lambda_1)\right)_t=-4\left(\phi_1(\lambda_1)\right)_{xxx},\\ &\left(\phi_1(\lambda_1)\right)_y=\left(\phi_1(\lambda_1)\right)_x,\left(\phi_1(\lambda_1)\right)_z=\left(\phi_1(\lambda_1)\right)_x,\end{aligned} \tag{31}$$

General solution to this system in two cases of $\lambda_1>0$ and $\lambda_1<0$ are

$$\begin{aligned}\phi_1(\lambda_1)&=C_1\cosh\left(\sqrt{\lambda_1}x+\sqrt{\lambda_1}y+\sqrt{\lambda_1}z-4\lambda_1^{\frac{3}{2}}t\right)\\ &\quad+C_2\sinh\left(\sqrt{\lambda_1}x+\sqrt{\lambda_1}y+\sqrt{\lambda_1}z-4\lambda_1^{\frac{3}{2}}t\right),\lambda_1>0,\\ \phi_1(\lambda_1)&=C_3\cos\left(\sqrt{-\lambda_1}x+\sqrt{-\lambda_1}y+\sqrt{-\lambda_1}z+4\left(-\lambda_1\right)^{\frac{3}{2}}t\right)\\ &\quad+C_4\sin\left(\sqrt{-\lambda_1}x+\sqrt{-\lambda_1}y+\sqrt{-\lambda_1}z+4\left(-\lambda_1\right)^{\frac{3}{2}}t\right),\lambda_1<0,\end{aligned} \tag{32}$$

respectively, where C_1, C_2, C_3 and C_4 are arbitrary real constants. When $\lambda_1>0$, we get negaton solution and when $\lambda_1<0$, we get positon solutions.

To construct Wronskian solutions corresponding to Jordan blocks of higher-order, we use the basic idea developed for the KdV equation [10,11].

Differentiating (9) with respect to λ_1, we can find that the vector function

$$\begin{aligned}\Phi_1&=\Phi_1(\lambda_1)\\ &=\left(\phi_1(\lambda_1),\frac{1}{1!}\partial_{\lambda_1}\phi_1(\lambda_1),\cdots,\frac{1}{(k_1-1)!}\partial_{\lambda_1}^{k_1-1}\phi_1(\lambda_1)\right)^{\mathrm{T}},\end{aligned} \tag{33}$$

satisfies

$$\Phi_{1,xxx}=\begin{bmatrix}\lambda_1 & & & 0\\ 1 & \lambda_1 & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_1\end{bmatrix}_{k_1\times k_1}\Phi_1, \tag{34}$$

$$\Phi_{1,t} = -4\Phi_{1,xxx}, \Phi_{1,y} = \Phi_{1,x}, \Phi_{1,z} = \Phi_{1,x} \quad (35)$$

where ∂_{λ_1} denotes the derivative with respect to λ_1 and k_1 is an arbitrary nonnegative integer. Therefore, through this set of eigenfunctions and Equation (12), a Wronskian solution of order $k_1 - 1$ to Equation (2) is presented as:

$$u = -2\partial_x \ln W\left(\phi_1(\lambda_1), \frac{1}{1!}\partial_{\lambda_1}\phi_1(\lambda_1), \cdots, \frac{1}{(k_1-1)!}\partial_{\lambda_1}^{k_1-1}\phi_1(\lambda_1)\right), \quad (36)$$

which corresponds to the first type of Jordan blocks with a nonzero real eigenvalue.

In what follows, several exact solutions of lower-order are presented to the (3 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation as

where

$$\eta_1 = \sqrt{\lambda_1}x + \sqrt{\lambda_1}y + \sqrt{\lambda_1}z - 4\lambda_1^{\frac{3}{2}}t.,$$

$$\theta_1 = \sqrt{-\lambda_1}x + \sqrt{-\lambda_1}y + \sqrt{-\lambda_1}z + 4(-\lambda_1)^{\frac{3}{2}}t.$$

$$u_{\text{1-negaton}} = -2\partial_x \ln\left(\cosh\left(\sqrt{\lambda_1}x + \sqrt{\lambda_1}y + \sqrt{\lambda_1}z - 4\lambda_1^{\frac{3}{2}}t\right)\right)$$
$$= -2\sqrt{\lambda_1}\tanh\left(\sqrt{\lambda_1}x + \sqrt{\lambda_1}y + \sqrt{\lambda_1}z - 4\lambda_1^{\frac{3}{2}}t\right),$$
$$u_{\text{1-positon}} = -2\partial_x \ln\left(\cos\left(\sqrt{-\lambda_1}x + \sqrt{-\lambda_1}y + \sqrt{-\lambda_1}z + 4(-\lambda_1)^{\frac{3}{2}}t\right)\right)$$
$$= 2\sqrt{-\lambda_1}\tan\left(\sqrt{-\lambda_1}x + \sqrt{-\lambda_1}y + \sqrt{-\lambda_1}z + 4(-\lambda_1)^{\frac{3}{2}}t\right),$$
$$u_{\text{2-negaton}} = -2\partial_x \ln W\left(\cosh(\eta_1), \partial_{\lambda_1}\cosh(\eta_1)\right)$$
$$= \frac{4\sqrt{\lambda_1}\cosh(\eta_1)}{-\cosh(\eta_1)\sinh(\eta_1) - \sqrt{\lambda_1}x - \sqrt{\lambda_1}y - \sqrt{\lambda_1}z + 12\lambda_1^{\frac{3}{2}}t},$$
$$u_{\text{2-positon}} = -2\partial_x \ln W\left(\cos(\theta_1), \partial_{\lambda_1}\cos(\theta_1)\right)$$
$$= \frac{4\lambda_1\cosh(\theta_1)}{\sqrt{-\lambda_1}\cos(\theta_1)\sin(\theta_1) - \lambda_1 x - \lambda_1 y - \lambda_1 z + 12\lambda_1^{\frac{3}{2}}t},$$

3.3. Interaction Solutions

We are now presenting examples of Wronskian interaction solutions among different kinds of Wronskian solutions to the (3 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation.

Let us assume that there are two sets of eigenfunctions

$$\phi_1(\lambda), \phi_2(\lambda), \cdots, \phi_k(\lambda); \psi_1(\mu)\psi_2(\mu), \cdots, \psi_l(\mu), \quad (37)$$

associated with two different eigenvalues λ and μ, respectively. A Wronskian solution

$$u = -2\partial_x^2 \ln W\left(\phi_1(\lambda), \phi_2(\lambda), \cdots, \phi_k(\lambda); \psi_1(\mu)\ \psi_2(\mu), \cdots, \psi_l(\mu)\right) \quad (38)$$

is said to be a Wronskian interaction solution between two solutions determined by the two sets of eigenfunctions in (37). In fact, we can have more general Wronskian interaction solutions among three or more kinds of solutions such as rational solutions, positons, solitons, negatons, breathers and complexitons.

In what follows, we would like to show a few special Wronskian interaction solutions depending on rational solution, positons and solitons. Firstly, we choose three different sets of special eigenfunctions:

$$\phi_{\text{rational}} = x + y + z,$$
$$\phi_{\text{soliton}} = \cosh\left(\sqrt{\lambda_1}x + \sqrt{\lambda_1}y + \sqrt{\lambda_1}z - 4\lambda_1^{\frac{3}{2}}t\right),$$
$$\phi_{\text{positon}} = \cos\left(\sqrt{-\lambda_2}x + \sqrt{-\lambda_2}y + \sqrt{-\lambda_2}z + 4(-\lambda_2)^{\frac{3}{2}}t\right),$$

where $\lambda_1 > 0$, $\lambda_2 < 0$ are constants.

Three Wronskian interaction determinants between any two of a rational solution, a single soliton and a single positon are obtained as

$$W\left(\phi_{\text{rational}},\phi_{\text{soliton}}\right)$$
$$=\sqrt{\lambda_1}\left(x+y+z\right)\sinh\left(\eta_1\right)-\cosh\left(\eta_1\right),$$
$$W\left(\phi_{\text{rational}},\phi_{\text{positon}}\right)$$
$$=-\sqrt{-\lambda_2}\left(x+y+z\right)\sin\left(\eta_2\right)-\cos\left(\eta_2\right)$$
$$W\left(\phi_{\text{soliton}},\phi_{\text{positon}}\right)$$
$$=-\sqrt{-\lambda_2}\cosh\left(\eta_1\right)\sinh\left(\eta_2\right)-\sqrt{\lambda_1}\cos\left(\eta_2\right)\sinh\left(\eta_1\right),$$

where

$$\eta_1=\sqrt{\lambda_1}x+\sqrt{\lambda_1}y+\sqrt{\lambda_1}z-4\lambda_1^{\frac{3}{2}}t$$

$$\eta_2=\sqrt{-\lambda_1}x+\sqrt{-\lambda_1}y+\sqrt{-\lambda_1}z+4\left(-\lambda_1\right)^{\frac{3}{2}}t.$$

Further, the corresponding Wronskian interaction solutions are

$$u_{rs}=-2\partial_x\ln W\left(\phi_{\text{rational}},\phi_{\text{soliton}}\right)$$
$$=\frac{2\lambda_1\left(x+y+z\right)\cosh\left(\eta_1\right)}{\sqrt{\lambda_1}\left(x+y+z\right)\sinh\left(\eta_1\right)-\cosh\left(\eta_1\right)},$$
$$u_{rp}=-2\partial_x\ln W\left(\phi_{\text{rational}},\phi_{\text{positon}}\right)$$
$$=\frac{2\lambda_2\left(x+y+z\right)\cos\left(\eta_2\right)}{-\sqrt{-\lambda_2}\left(x+y+z\right)\sin\left(\eta_2\right)-\cos\left(\eta_2\right)}$$

$$u_{sp}=-2\partial_x\ln W\left(\phi_{\text{soliton}},\phi_{\text{positon}}\right)$$
$$=\frac{2\left(\lambda_2-\lambda_1\right)\cosh\left(\eta_1\right)\cos\left(\eta_2\right)}{-\sqrt{-\lambda_2}\cosh\left(\eta_1\right)\sinh\left(\eta_2\right)-\sqrt{\lambda_1}\cos\left(\eta_2\right)\sinh\left(\eta_1\right)},$$

where

$$\eta_1=\sqrt{\lambda_1}x+\sqrt{\lambda_1}y+\sqrt{\lambda_1}z-4\lambda_1^{\frac{3}{2}}t,$$

$$\eta_2=\sqrt{-\lambda_2}x+\sqrt{-\lambda_2}y+\sqrt{-\lambda_2}z+4\left(-\lambda_2\right)^{\frac{3}{2}}t.$$

The following is one Wronskian interaction determinant and solution involving the three eigenfunctions. The Wronskian determinant is

$$W\left(\phi_{\text{rational}},\phi_{\text{soliton}},\phi_{\text{positon}}\right)$$
$$=\left(x+y+z\right)\Big(\lambda_2\sqrt{\lambda_1}\sinh\left(\eta_1\right)\cos\left(\eta_2\right)$$
$$+\lambda_1\sqrt{-\lambda_2}\sin\left(\eta_2\right)\cosh\left(\eta_1\right)\Big),$$

so that its corresponding Wronskian solution reads as

$$u_{rsp}=-2\partial_x\ln W\left(\phi_{\text{rational}},\phi_{\text{soliton}},\phi_{\text{positon}}\right)=\frac{-2q_3}{p_3},$$

where

$$p_3=\left(x+y+z\right)\Big(\lambda_2\sqrt{\lambda_1}\sinh\left(\eta_1\right)\cos\left(\eta_2\right)$$
$$+\lambda_1\sqrt{-\lambda_2}\sin\left(\eta_2\right)\cosh\left(\eta_1\right)\Big),$$
$$q_3=\left(x+y+z\right)\sqrt{-\lambda_1\lambda_2}\left(\lambda_1-\lambda_2\right)\sinh\left(\eta_1\right)\sin\left(\eta_2\right)$$
$$+\lambda_1\sqrt{\lambda_1}\sinh\left(\eta_1\right)\cos\left(\eta_2\right)+\lambda_2\sqrt{-\lambda_2}\cosh\left(\eta_1\right)\sin\left(\eta_2\right)$$

with

$$\eta_1=\sqrt{\lambda_1}x+\sqrt{\lambda_1}y+\sqrt{\lambda_1}z-4\lambda_1^{\frac{3}{2}}t,$$

$$\eta_2=\sqrt{-\lambda_2}x+\sqrt{-\lambda_2}y+\sqrt{-\lambda_2}z+4\left(-\lambda_2\right)^{\frac{3}{2}}t.$$

4. Conclusion

In this paper, by using the Wronskian technique, we have derived the Wronskian determinant solution for the (3 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation which describes the fluid propagating and can be considered as a model for an incompressible fluid. Moreover, we obtained some rational solutions, soliton solutions, positons and negatons of this equation by solving the resultant systems of linear partial differential equations which guarantee that the Wronskian determinant solves the equation in the bilinear form. The presented solutions show the remarkable richness of the solution space of the (3 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation.

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