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A Note on a Kinetic Model for Rod-Like Particle Suspensions

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ABSTRACT

A system, coupled by an incompressible Navier-Stokes and a Fokker-Planck equation, is investigated. The global weak solution with small initial data is obtained.

Keywords: Fokker-Planck Equation; Navier-Stokes Equation; Small Initial Data

1. Introduction

The dilute suspensions of passive rod-like particles can be effectively modeled by a coupled microscopic Fokker-Planck equation and macroscopic Navier-Stokes equation, known as Doi model (see Doi [1]). We refer to [2] for the Doi model for suspensions of active rod-like particles without considering the effects of gravity. Recently an extended model under gravity was introduced by Hezel, Otto and Tzavaras [3], which reads

$$\begin{aligned}&\partial_t f+\nabla_x\cdot(uf)-\Delta_n f+\nabla_n\cdot\left[(Id-n\otimes n)\nabla_x unf\right]\\&=\nabla_x\cdot(Id+n\otimes n)(e_2 f+\gamma\nabla_x f)\end{aligned}\tag{1}$$

$$\sigma=\int_{S^{d-1}}\left[(dn\otimes n-Id)f\right]\mathrm{d}n\tag{2}$$

$$\begin{aligned}&\mathrm{Re}\left[\partial_t u+(u\cdot\nabla_x)u\right]-\Delta_x u+\nabla_x p\\&=\beta\gamma\nabla_x\cdot\sigma-\beta\left(\int_{S^{d-1}}f\mathrm{d}n\right)e_2\end{aligned}\tag{3}$$

$$\nabla_x\cdot u=0\tag{4}$$

where $(t,x,n)\in[0,\infty)\times\Omega\times S^{d-1},\Omega\in R^d$ is a bounded domain with $\partial\Omega$ of class C^1 and $S^{d-1}\subset R^d$ being the unit sphere; σ is a stress tensor, p is the pressure, e_2 is the unit vector in the upward direction; $\nabla_n\cdot$ and Δ_n denote the tangential divergence and Laplace-Beltrami operator on S^{d-1}, respectively. In this model, $f(t,x,n)$ is a distribution function which represents the configuration of a suspension of rod-like particles and $u(t,x)$ is the fluid velocity induced by the other particles in the suspension. $\mathrm{Re}\geq 0$ is a Reynolds number. The coefficients $\beta>0$ and $\gamma>0$ are constants (see [3], Remark 2.1 - 2.2).

If $\mathrm{Re}=0$, the model includes a Stokes equation. In this case, Chen, Li and Liu [4] obtain the global weak solution and its uniqueness to the two dimensional $(\mathrm{d}=2)$ initial-boundary problem. In Remark 3.2 of [4], they point out that it is a mathematically interesting question to ask if the above result is still valid when the Stokes equation is replaced by the Navier-Stokes equation $(\mathrm{Re}>0)$, and there are some technical difficulties in solving this problem. The main purpose of this note is to answer this question by using an assumption of small initial data. See [5-7] etc. for more results on Doi related model without considering the effects of gravity.

For conciseness in presentation, we set $\mathrm{Re}=\beta=\gamma=1$ in the rest of this paper. Define $H=\left\{u\in L^2(\Omega):\nabla_x\cdot u=0,u\cdot\nu|_{\partial\Omega}=0\right\}$ $V=\left\{u\in H_0^1(\Omega):\nabla_x\cdot u=0\right\}$, $S:=S^1$ and $F(s):=s(\log s-1)+1,s\in[0,\infty)$. Let $L>1$, define the cut-off function

$$E^L:=\begin{cases}0, & \text{if } s\leq 0,\\ s, & \text{if } s\leq L,\\ L, & \text{if } s\geq L.\end{cases}$$

Set the initial and boundary conditions as follows,

$$f|_{t=0}=f_0;u|_{t=0}=u_0;\tag{5}$$

$$(Id+n\otimes n)(e_2 f+\nabla_x f)\cdot\nu|_{\partial\Omega}=0;\quad u|_{\partial\Omega}=0.\tag{6}$$

2. The Main Result

Theorem 2.1 Let $d=2$. Suppose that $u_0\in H$, $f_0\in L^2(\Omega\times S)$, and $f_0\geq 0$ a.e. are on $\Omega\times S$. Then there exists $\varepsilon>0$, such that if

$$\|u_0\|^2_{L^2(\Omega)}+\int_{\Omega\times S}F(f_0)\mathrm{d}n\mathrm{d}x\leq\varepsilon, \tag{7}$$

the initial-boundary problem (1)-(6) has a global weak solution (u,f) which satisfies for a.e. $t\in[0,\infty)$,

$$\begin{aligned}&\|u(t)\|^2_{L^2(\Omega)}+2\int_{\Omega\times S}F(f(t))\mathrm{d}n\mathrm{d}x+2\int_0^t\|\nabla_x u(s)\|^2_{L^2(\Omega)}\mathrm{d}s\\&+4\int_0^t\left(\left\|\nabla_x\sqrt{f(s)}\right\|^2_{L^2(\Omega\times S)}+\left\|\nabla_n\sqrt{f(s)}\right\|^2_{L^2(\Omega\times S)}\right)\mathrm{d}s\\&\leq\|u_0\|^2_{L^2(\Omega)}+2\int_{\Omega\times S}F(f_0)\mathrm{d}n\mathrm{d}x+C\|f_0\|^2_{L^1(\Omega\times S)}.\end{aligned} \tag{8}$$

Definition 2.2 The weak solution (u,f) is in the following sense,

$$u\in L^\infty(0,\infty;H)\cap L^2(0,\infty;V),\ u\in H^1_{loc}(0,\infty;V'); \tag{9}$$

$$f\geq 0\ a.e.\text{ on }[0,\infty)\times\Omega\times S,\ f\in L^\infty\left(0,\infty;L^1(\Omega\times S)\right) \tag{10}$$

$$\nabla_x\sqrt{f},\nabla_n\sqrt{f}\in L^2\left(0,\infty;L^2(\Omega\times S)\right) \tag{11}$$

$$\begin{aligned}&f\in L^\infty_{loc}\left(0,\infty;L^2(\Omega\times S)\right)\cap L^2_{loc}\left(0,\infty;H^1(\Omega\times S)\right),\\&f\in H^1_{loc}\left(0,\infty;\left(H^3(\Omega\times S)\right)'\right);\end{aligned} \tag{12}$$

for any $v\in C_0^\infty([0,\infty)\times\Omega)$ with $\nabla_x\cdot v=0$.

$$\begin{aligned}&-\int_0^\infty\int_\Omega u\cdot\partial_t v\mathrm{d}x\mathrm{d}t+\int_0^\infty\int_\Omega(u\cdot\nabla_x u)\cdot v\mathrm{d}x\mathrm{d}t\\&+\int_0^\infty\int_\Omega\nabla_x u:\nabla_x v\mathrm{d}x\mathrm{d}t\\&=-\int_0^\infty\int_{\Omega\times S}(2n\otimes n-Id)f:\nabla_x v\mathrm{d}n\mathrm{d}x\mathrm{d}t\\&-\int_0^\infty\int_{\Omega\times S}fe_2\cdot v\mathrm{d}n\mathrm{d}x\mathrm{d}t+\int_\Omega u_0(x)\cdot v(0,x)\mathrm{d}x;\end{aligned} \tag{13}$$

for any $\varphi\in C_0^\infty\left([0,\infty)\times\bar\Omega\times S\right)$,

$$\begin{aligned}&-\int_0^\infty\int_{\Omega\times S}f\partial_t\varphi\mathrm{d}n\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\Omega\times S}(uf)\cdot\nabla_x\varphi\mathrm{d}n\mathrm{d}x\mathrm{d}t\\&+\int_0^\infty\int_{\Omega\times S}\nabla_n f\cdot\nabla_n\varphi\mathrm{d}n\mathrm{d}x\mathrm{d}t\\&=\int_0^\infty\int_{\Omega\times S}\left[(Id-n\otimes n)\nabla_x unf\right]\cdot\nabla_n\varphi\mathrm{d}n\mathrm{d}x\mathrm{d}t\\&-\int_0^\infty\int_{\Omega\times S}(Id+n\otimes n)(e_2f+\nabla_x f)\cdot\nabla_x\varphi\mathrm{d}n\mathrm{d}x\mathrm{d}t\\&+\int_{\Omega\times S}f_0(x,n)\varphi(0,x,n)\mathrm{d}n\mathrm{d}x.\end{aligned} \tag{14}$$

Proof. The proof follows that of [4] (some ideas and techniques come from [8]). Here we only show the different details.

Step 1. Approximate problem. For any fixed $0<\tau\ll 1$ and for any $k\in N$, given (u^{k-1},f^{k-1}), the approximate problem with cut-off reads

$$\begin{aligned}&\int_\Omega\frac{u^k-u^{k-1}}{\tau}\cdot v\mathrm{d}x+\int_\Omega\nabla_x u^k:\nabla_x v\mathrm{d}x\\&+\int_\Omega\left(u^{k-1}\cdot\nabla_x\right)u^k\cdot v\mathrm{d}x\\&=-\int_{\Omega\times S}(2n\otimes n-\mathrm{Id})f^k:\nabla_x v\mathrm{d}n\mathrm{d}x\\&-\int_{\Omega\times S}f^ke_2\cdot v\mathrm{d}n\mathrm{d}x,\ \forall v\in V;\end{aligned} \tag{15}$$

$$\begin{aligned}&\int_{\Omega\times S}\frac{f^k-f^{k-1}}{\tau}\varphi\mathrm{d}n\mathrm{d}x\\&-\int_{\Omega\times S}(u^kf^k)\cdot\nabla_x\varphi\mathrm{d}n\mathrm{d}x\\&+\int_{\Omega\times S}\nabla_n f^k\cdot\nabla_n\varphi\mathrm{d}n\mathrm{d}x\\&=\int_{\Omega\times S}\left[(Id-n\otimes n)\nabla_x u^kn\right]E^{\tau^{-1/4}}\left(f^k\right)\cdot\nabla_n\varphi\mathrm{d}n\mathrm{d}x\\&-\int_{\Omega\times S}(Id+n\otimes n)\left[e_2E^{\tau^{-1/4}}\left(f^k\right)+\nabla_x f^k\right]\cdot\nabla_x\varphi\mathrm{d}n\mathrm{d}x,\\&\forall\varphi\in H^1(\Omega\times S).\end{aligned} \tag{16}$$

Similarly as the proof of [4], we have

Lemma 2.3

Let $Z:=\left\{f\in L^2(\Omega\times S):f\geq 0\ a.e.\ \Omega\times S\right\}$.

If $(u^{k-1},f^{k-1})\in V\times Z$, then there exists $(u^k,f^k)\in V\times\left(Z\cap H^1(\Omega\times S)\right)$ which solves (15)-(16).

Step 2. Uniform estimate. Suppose that $u_0\in H$, $f_0\in L^2(\Omega\times S)$ and $f_0\geq 0$ a.e. on $\Omega\times S$. Let $u^0=u^0(\tau)$ be the solution of $u^0-\tau^{1/4}\Delta u^0=u^0$. Then

$$\left\|u^0\right\|^2_{L^2(\Omega)}+\tau^{1/4}\left\|\nabla_x u^0\right\|^2_{L^2(\Omega)}\leq\|u_0\|^2_{L^2(\Omega)} \tag{17}$$

and $u^0\to u_0$ weakly in H as $\tau\to 0$. Moreover, let $f^0=E^{\tau^{-1/4}}(f_0)$. Then $(u^0,f^0)\in V\times Z$. Using Lemma 2.3 iteratively, we obtain a sequence of approximate solutions,

$$\left(u^k,f^k\right)\in V\times\left(Z\cap H^1(\Omega\times S)\right) \tag{18}$$

to (15)-(16). Similarly as the proof of Lemma 3.5 and Lemma 3.6 in [4], we have

Lemma 2.4

$$\sup_{k\in N}\left\|f^k\right\|_{L^1(\Omega\times S)}\leq\|f_0\|_{L^1(\Omega\times S)} \tag{19}$$

For any $k\in N$,

$$\begin{aligned}&\frac{1}{2}\left\|u^k\right\|^2_{L^2(\Omega)}+\int_{\Omega\times S}F\left(f^k\right)\mathrm{d}n\mathrm{d}x\\&+\frac{1}{2}\sum_{i=1}^k\left\|u^k-u^{k-1}\right\|^2_{L^2(\Omega)}+\tau\sum_{i=1}^k\left\|\nabla_x u^i\right\|^2_{L^2(\Omega)}\\&+2\tau\sum_{i=1}^k\left(\left\|\nabla_x\sqrt{f^i}\right\|^2_{L^2(\Omega\times S)}+\left\|\nabla_n\sqrt{f^i}\right\|^2_{L^2(\Omega\times S)}\right)\\&\leq\frac{1}{2}\|u_0\|^2_{L^2(\Omega)}+\int_{\Omega\times S}F(f_0)\mathrm{d}n\mathrm{d}x+C\|f_0\|_{L^1(\Omega\times S)}.\end{aligned} \tag{20}$$

Lemma 2.5 For any $T>0$ we might as well set $N=T/\tau$. Then

$$\sup_{1\le k\le N}\left\| f^k\right\|^2_{L^2(\Omega\times S)}+\tau\sum_{k=1}^{N}\left(\left\|\nabla_x f^k\right\|^2_{L^2(\Omega\times S)}+\left\|\nabla_n f^k\right\|^2_{L^2(\Omega\times S)}\right)\le C(T). \tag{21}$$

Proof. Following the proof of (3.44) in [4], we have that

$$\left\| f^k\right\|^2_{L^2(\Omega\times S)}+\tau\sum_{i=1}^{k}\left(\left\|\nabla_x f^i\right\|^2_{L^2(\Omega\times S)}+\left\|\nabla_n f^i\right\|^2_{L^2(\Omega\times S)}\right)\le\left\| f^0\right\|^2_{L^2(\Omega\times S)}+C\tau\sum_{i=1}^{k}\left(\left\|\nabla_x u^i\right\|^2_{L^2(\Omega)}+1\right)\left\| f^i\right\|^2_{L^2(\Omega\times S)}.$$

Applying (20), one has $\varepsilon>0$, such that if $\left(u^0,f^0\right)$ satisfies $\left\|u_0\right\|^2_{L^2(\Omega)}+\int_{\Omega\times S}F\left(f_0\right)dndx\le\varepsilon$, then $C\tau\sum_{i=1}^{k}\left\|\nabla_x u^i\right\|^2_{L^2(\Omega)}\le\frac{1}{4}$. Furthermore, let $\tau\le 1/4C$, then

$$C\tau\left(\left\|\nabla_x u^k\right\|^2_{L^2(\Omega)}+1\right)\left\| f^k\right\|^2_{L^2(\Omega\times S)}\le\frac{1}{2}\left\| f^k\right\|^2_{L^2(\Omega\times S)},$$

and hence

$$\frac{1}{2}\left\| f^k\right\|^2_{L^2(\Omega\times S)}\le\left\| f_0\right\|^2_{L^2(\Omega\times S)}+C\tau\sum_{i=1}^{k-1}\left(\left\|\nabla_x u^i\right\|^2_{L^2(\Omega)}+1\right)\left\| f^i\right\|^2_{L^2(\Omega\times S)}.$$

Using (20) again, and the discrete Gronwall inequality, We finish the proof of (21).

Definition 2.6 Define the piecewise function in *t* by

$$u_\tau(t,\cdot):=u^k(\cdot),\ \pi_\tau u_\tau(t,\cdot):=u^{k-1}(\cdot),\ t\in\left((k-1)\tau,k\tau\right]$$

and the difference quotient of size τ by

$$\partial_t^\tau u_\tau(t,\cdot):=\frac{u^k(\cdot)-u^{k-1}(\cdot)}{\tau},\ t\in\left((k-1)\tau,k\tau\right]$$

Likewise, define f_τ and $\partial_t^\tau f_\tau$.

Lemma 2.7

$$f_\tau\ge 0\ a.e.\ \text{on}\ [0,T]\times\Omega\times S. \tag{22}$$

$$\left\|u_\tau;\pi_\tau u_\tau\right\|_{L^\infty\left(0,T;L^2(\Omega)\right)\cap L^2(0,T;V)}\le C. \tag{23}$$

$$\left\| f_\tau\right\|_{L^\infty\left(0,T;L^2(\Omega\times S)\right)\cap L^2\left(0,T;H^1(\Omega\times S)\right)}\le C(T). \tag{24}$$

Proof. We can use (17), (19)-(21) directly to finish the proof. Here we only show that $\pi_\tau u_\tau$ is bounded. In fact, it follows from (17) and (20) that

$$\left\|\pi_\tau u_\tau\right\|_{L^\infty\left(0,T;L^2(\Omega)\right)}\le\max\left\{\left\|u_0\right\|_{L^2(\Omega)},\left\|u_\tau\right\|_{L^\infty\left(0,T;L^2(\Omega)\right)}\right\},$$

$$\left\|\nabla_x\left(\pi_\tau u_\tau\right)\right\|^2_{L^2\left(0,T;L^2(\Omega)\right)}=\tau\left\|\nabla_x u^0\right\|^2_{L^2(\Omega)}+\tau\sum_{i=1}^{N-1}\left\|\nabla_x u^i\right\|^2_{L^2(\Omega)}\le\left\|u_0\right\|^2_{L^2(\Omega)}+\tau\sum_{i=0}^{N-1}\left\|\nabla_x u^i\right\|^2_{L^2(\Omega)}\le C.$$

Lemma 2.8

$$\left\|\partial_t^\tau u_\tau\right\|_{L^2(0,T;V')}+\left\|\partial_t^\tau f_\tau\right\|_{L^2\left(0,T;\left(H^3(\Omega\times S)\right)'\right)}\le C(T) \tag{25}$$

Proof. Observing that

$$\int_\Omega\left(u^{k-1}\cdot\nabla_x\right)u^k\cdot v\mathrm{d}x=-\int_\Omega\left(u^{k-1}\cdot\nabla_x\right)v\cdot u^k\mathrm{d}x,\ v\in V$$

we deduce from (15) that,

$$\left\|\frac{u^k-u^{k-1}}{\tau}\right\|_{V'}\le\left\|\nabla_x u^k\right\|_{L^2(\Omega)}+\left\|\left|u^k\right|\left|u^{k-1}\right|\right\|_{L^2(\Omega)}+C\left\| f^k\right\|_{L^2\left(\Omega;L^1(S)\right)}.$$

Therefore, please see the Equation (26) below.

Employing Gagliardo-Nirenberg inequality and Hölder inequality, one has from (23) that

$$\left\|u_\tau\right\|^2_{L^4((0,T)\times\Omega)}\le\left\|u_\tau\right\|_{L^2(0,T;V)}\left\|u_\tau\right\|_{L^\infty\left(0,T;L^2(\Omega)\right)}\le C.$$

Similarly, $\left\|\pi_\tau u_\tau\right\|^2_{L^4((0,T)\times\Omega)}\le C$. Then it follows from (23), (24) and (26) that $\left\|\partial_t^\tau u_\tau\right\|^2_{L^2(0,T;V')}\le C(T)$. According to (16), we have that for any $\varphi\in H^3(\Omega\times S)$,

$$\begin{aligned}\left\|\partial_t^\tau u_\tau\right\|_{L^2(0,T;V')}&=\left(\tau\sum_{k=1}^{N}\left\|\frac{u^k-u^{k-1}}{\tau}\right\|^2_{V'}\right)^{1/2}\\&\le C\left(\tau\sum_{k=1}^{N}\left[\left\|\nabla_x u^k\right\|_{L^2(\Omega)}+\left\|\left|u^k\right|\left|u^{k-1}\right|\right\|_{L^2(\Omega)}+\left\| f^k\right\|_{L^2(\Omega\times S)}\right]^2\right)^{1/2}\\&\le C\left(\sum_{k=1}^{N}\int_{(k-1)\tau}^{k\tau}\left[\left\|\nabla_x u_\tau\right\|_{L^2(\Omega)}+\left\|\left|u_\tau\right|\left|\pi_\tau u_\tau\right|\right\|_{L^2(\Omega)}+\left\| f_\tau\right\|_{L^2(\Omega\times S)}\right]^2\right)^{1/2}\\&\le C\left(\left\|\nabla_x u_\tau\right\|_{L^2((0,T)\times\Omega)}+\left\|u_\tau\right\|_{L^4((0,T)\times\Omega)}\left\|\pi_\tau u_\tau\right\|_{L^4((0,T)\times\Omega)}+\left\| f_\tau\right\|_{L^2((0,T)\times\Omega\times S)}\right).\end{aligned} \tag{26}$$

$$\left|\int_{\Omega\times S}\frac{f^k-f^{k-1}}{\tau}\cdot\varphi\,\mathrm{d}n\mathrm{d}x\right|$$
$$\le\int_{\Omega\times S}\left|u^k\right|\left|f^k\right|\left|\nabla_x\varphi\right|\mathrm{d}n\mathrm{d}x+\int_{\Omega\times S}\left|\nabla_n f^k\right|\left|\nabla_n\varphi\right|\mathrm{d}n\mathrm{d}x$$
$$+C\int_{\Omega\times S}\left|\nabla_x u^k\right|\left|f^k\right|\left|\nabla_n\varphi\right|\mathrm{d}n\mathrm{d}x$$
$$+C\int_{\Omega\times S}\left(\left|\nabla_x f^k\right|+\left|f^k\right|\right)\left|\nabla_x\varphi\right|\mathrm{d}n\mathrm{d}x.$$

Consequently

$$\left\|\frac{f^k-f^{k-1}}{\tau}\right\|_{\left(H^3(\Omega\times S)\right)'}$$
$$\le C\left(\left\|u^k\right\|_{H^1(\Omega)}\left\|f^k\right\|_{L^2(\Omega\times S)}+\left\|f^k\right\|_{H^1(\Omega\times S)}\right).$$

Similarly as the proof of (26), we have from (23) and (24) that $\left\|\partial_t^\tau f_\tau\right\|^2_{L^2\left(0,T;\left(H^3(\Omega\times S)\right)'\right)}\le C(T)$.

Step 3. Convergence. With the above uniform estimates at hand, we can use the Aubin-Lions lemma for time-piecewise functions (see [9]) to perform the compactness argument. This concludes the proof of Theorem 2.1.

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