Journal of Applied Mathematics and Physics, **2013, 1, 37-46**
http://dx.doi.org/10.4236/jamp.2013.13007 Published Online August 2013 (http://www.scirp.org/journal/jamp)

Random Attractors for the Dissipative Hamiltonian Amplitude Equation Governing Modulated Wave Instabilities with Additive Noise*

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Received June 10, 2013; revised July 15, 2013; accepted September 10, 2013

ABSTRACT

In this paper, we study the random dynamical system (RDS) generated by the dissipative Hamiltonian amplitude equation with additive noise defined on the periodic boundaries. We investigate the existence of a compact random attractor for the RDS associated with the equation through introducing two functions and one process in $E_0 = H^1 \times L^2$. The compactness of the RDS is established by the decomposition of solution semigroup.

Keywords: Random Dynamical System; Random Attractor; Hamiltonian Amplitude Equation

1. Introduction

The Hamiltonian amplitude equation

$$\mathrm{i}\phi_x + \phi_{tt} + 2\sigma|\phi|^2\phi = 0, \quad (1)$$

was first proposed by Tanaka, Yajima and Wadati as a model for the nonlinear modulation of stable plane wave in unstable media [1,2]. In 1992 as an improved version of (1), the equation

$$\mathrm{i}\psi_x + \psi_{tt} + 2\sigma|\psi|^2\psi - \varepsilon\psi_{xt} = 0, \ 0 < \varepsilon < 1, \quad (2)$$

was proposed [3], which generalized (1) in the sense that

$$\psi(x,t;\varepsilon = 0) = \phi(x,t), \quad (3)$$

but one can show that for most initial data

$$\lim_{\varepsilon\to 0}\psi(x,t;\varepsilon) \neq \phi(x,t), \quad (4)$$

even if the two functions agree at $t = 0$. Both of these models can be derived systematically from more complicated Hamiltonian systems through a particular limiting process (nearly monochromatic waves of small amplitude) corresponding to $\varepsilon \to 0$. Even so, keeping $\varepsilon \neq 0$ in (2) is crucial because (1) is formally integrable but ill-posed, whereas (2) is a generalization of it which is apparently not integrable but well-posed.

*This work is supported by National Natural Science Foundation of China (11071199) and Natural Science Foundation of Chongqing (2009BB8105).

In this paper, we consider the following dissipative Hamiltonian amplitude equation governing modulated wave instabilities perturbed by an additive white noise

$$du_t + \alpha u_t dt - \beta u_{xt} dt - \gamma u_{xx} dt + \mathrm{i}u_x dt + f\left(|u|^2\right)u dt = \sum_{j=1}^{m} h_j dW_j \quad (5)$$

$$u(x,\tau) = u_0(x), \quad u_t(x,\tau) = u_1(x), \quad (6)$$

and the periodic boundary condition

$$u(x-L,t) = u(x+L,t), \quad (7)$$

where u is an unknown complex valued function, i is the unit of imaginary number, the internal $I = (-L, L)$, α, β and γ are positive constants, which satisfy $\beta < \gamma$, the functions $h_j \in H^2(I)$, $j = 1, 2, \cdots, m$, are time independent, the random functions W_j, $j = 1, 2, \cdots, m$, are independent two-side real-valued Wiener processes on a probability space (Ω, F, P) which will be specified later, and $f(s)$ is C^1, $sf(s)$ is C^2 real valued function which satisfies that

$$\liminf_{s\to+\infty}\frac{F(s)}{s^{1+\delta}} \geq \gamma_0 > 0, \quad s \geq \tau, \delta \geq 1, \quad (8)$$

$$\liminf_{s\to+\infty}\frac{sf(s) - \vartheta F(s)}{s^{1+\delta}} \geq \gamma_0 > 0, \quad s \geq \tau, \delta \geq 1, \quad (9)$$

where $0<\vartheta<1$, γ_0 is a constant depended on δ and ϑ, and $F(s)=\int_\tau^s f(t)\mathrm{d}t$.

The deterministic case has been studied extensively, for instance, Guo, B. L. and Dai, Z. D. [4] proved that there exists a global weak attractor A_1 in $E_1=H^2\times H^1$ for (5) and it is actually a global strong attractor in E_1. Dai, Z. D. [5] proved the existence of a global attractor A_0 in $E_0=H^1\times L^2$, and obtained the equality $\mathrm{A}_0=\mathrm{A}_1$. Dai, Z. D. Yang, L. Huang, J. [6] obtained a global attractor for the unperturbed system in E_0 and E_1 respectively. Yang, L., Dai, Z. D. [7] obtained the estimate of the Hausdorff dimension and the fractal dimension of a global attractor for the perturbed and unperturbed systems separately. However, up to the best of our knowledge, the research for the dissipative Hamiltonian amplitude equation governing modulated wave instabilities with random attractors has not involved.

Recently, many authors have studied the existence of random attractors for other equations [8-10]. In this paper, for (5), we first obtain an absorbing set in E_0 and E_1 respectively through introducing two functions and one process, then by the decomposition of solution semigroup we derive the compactness in E_0. As far as we know, no one has studied stochastic equations through introducing two functions, so this method enriches the study of stochastic equations.

This paper is organized as follows. In Section 2, for convenience of the reader, we recall some basic notions on function spaces and the theory of random dynamical system. In Section 3, we solve Equation (5) and get the corresponding RDS φ. In Section 4, we prove the existence of a random attractor in E_0 for this RDS.

Throughout this paper, we adopt the following notations. We write $L^2=L^2(I)$, $H^1=H^1(I)$, $H^2=H^2(I)$ for short. We denote by $\|\cdot\|$ and $|\cdot|$ the norms, by $((\cdot,\cdot))$ and $(\cdot,\cdot)$ the inner products in H^1 and L^2 respectively. We also use $|u|$ to denote the modular or absolute value of u.

2. Preliminaries

In this section, we recall some basic notions on function spaces [4,7], the theory of RDS [11-14] and introduce the method of the existence of random attractors for the continuous RDS [8,10], which we will use in this paper.

2.1. Function Spaces and Operators

We first consider the mathematical setting for (5). Let L^2, H^1, H^2 be usual Sobolev space, $E_0=H^1\times L^2$, $E_1=H^2\times H^1$ and $\phi=(u,v)^{\mathrm{T}}$. We define the following scalar products and norms separately:

for any $\phi_i=(u_i,v_i)^{\mathrm{T}}\in E_0$ and $\phi=(u,v)^{\mathrm{T}}\in E_0$, we have

$$(\phi_1,\phi_2)_{E_0}=((u_1,u_2))+(v_1,v_2),\quad \|\phi\|_{E_0}^2=\|u\|^2+|v|^2,$$

for any $\phi_i=(u_i,v_i)^{\mathrm{T}}\in E_1$ and $\phi=(u,v)^{\mathrm{T}}\in E_1$, we have

$$(\phi_1,\phi_2)_{E_1}=(D^2u_1,D^2u_2)+((v_1,v_2)),\quad \|\phi\|_{E_1}^2=|\Delta u|^2+\|v\|^2$$

Let $A=-D^2:D(A)=H^1\cap H^2\to L^2$, then A is a positive self-adjoint operator, which has the first eigenvalue $\lambda_1=\inf_{u\in H^1}\frac{\|u\|^2}{|u|^2}$.

2.2. Random Dynamical Systems

Let (Ω,F,P) be a probability space and $\{\theta_t:\Omega\to\Omega,t\in R\}$ be a family of measure preserving transformations such that $(t,\omega)\to\theta_t\omega$ is measurable, $\theta_0=id$ and $\theta_{t+s}=\theta_t\theta_s$ for all $s,t\in R$. The flow θ_t together with the corresponding probability space $(\Omega,F,\mathrm{P},\theta_t)$ is called a measurable dynamical system.

Definition 2.2.1 A continuous random dynamical system(RDS) on a Polish space (X,d) with Borel σ-algebra on $(\Omega,F,\mathrm{P},\theta_t)$ is a measurable map

$$\varphi:R^+\times\Omega\times X\mapsto X,\quad (t,\omega,x)\mapsto\varphi(t,\omega)x$$

such that $\mathrm{P}-a.s.$

1) $\varphi(0,\omega)=id$ on X;

2) $\varphi(t+s,\omega)=\varphi(t,\theta_s\omega)\varphi(s,\omega)$, for all $s,t\in R^+$ (cocycle property);

3) $\varphi(t,\omega):\mathrm{X}\mapsto\mathrm{X}$ is continuous.

A random compact set $\{K(\omega)\}_{\omega\in\Omega}$ is a family of compact sets indexed by ω such that for every $x\in\mathrm{X}$ the mapping $x\mapsto d(x,K(\omega))$ is measurable with respect to F.

Let $\mathrm{A}(\omega)$ be a random set and $B\subset\mathrm{X}$. We say $\mathrm{A}(\omega)$ attracts B if

$$\lim_{t\to\infty}dist(\varphi(t,\theta_{-t}\omega)\mathrm{B},\mathrm{A}(\omega))=0,\ \mathrm{P}-a.s.\ \omega\in\Omega,$$

where $dist(\cdot,\cdot)$ denotes the Hausdorff semi-distance in X. We say $\mathrm{A}(\omega)$ absorbs B if there exists $t_B(\omega)>0$ such that for all $t\ge t_B(\omega)$,

$$\varphi(t,\theta_{-t}\omega)\mathrm{B}\subset\mathrm{A}(\omega),\ \mathrm{P}-a.s.\ \omega\in\Omega.$$

A random set $\mathrm{A}(\omega)$ is said to be a random attractor for the RDS φ if $\mathrm{P}-a.s.$

1) $\mathrm{A}(\omega)$ is a random compact set;

2) $\mathrm{A}(\omega)$ is invariant, that is, $\varphi(t,\omega)\mathrm{A}(\omega)=\mathrm{A}(\theta_t\omega)$ for all $t\ge 0$;

3) $\mathrm{A}(\omega)$ attracts all deterministic bounded sets $B\subset\mathrm{X}$.

Theorem 2.2.2 If there exists a random compact set absorbing every bounded set $B\subset\mathrm{X}$, then the RDS φ possesses a random attractor $\mathrm{A}(\omega)$,

$$\mathrm{A}(\omega)=\overline{\bigcup_{\mathrm{B}\subset\mathrm{X}}\Lambda_{\mathrm{B}}(\omega)},$$

where $\Lambda_{\mathrm{B}}(\omega) := \cap_{s\ge 0}\overline{\cup_{t\ge s}\varphi(t,\theta_{-t}\omega)\mathrm{B}}$ is the omega-limit set of B.

3. Solve the Equation and Generate a RDS

We consider the probability space (Ω, F, P), where

$$\Omega = \left\{\omega = (\omega_1, \omega_2, \cdots, \omega_m) \in C(R, R^m) : \omega(0) = 0\right\},$$

and F is the Borel σ-algebra induced by the compact open topology of Ω, while P is the corresponding Wiener measure on (Ω, F). Then, we identify ω with

$$W(t) = (W_1(t), W_2(t), \cdots, W_m(t)) = \omega(t) \text{ for } t \in R.$$

Finally, we define the time shift by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$, $\omega \in \Omega, t \in R$. Then $\left(\Omega, F, \mathrm{P}, (\theta_t)_{t\in R}\right)$ is a metric dynamical system.

We now want to establish a continuous random dynamical system corresponding to (5). For this purpose, we need to convert the stochastic equation with an additive noise into a deterministic equation with a random parameter.

Given $j = 1, 2, \cdots, m$, consider the stochastic stationary solution of the one-dimensional Ornstein-Uhlenbeck equation

$$dz_j + \alpha z_j dt = dW_j(t). \quad (10)$$

One may easily check that a solution to (10) is given by

$$z_j(t) = \int_{-\infty}^{t} \mathrm{e}^{-\alpha(t-s)} \mathrm{d}W_j(s), \ t \in R. \quad (11)$$

Putting $z = \sum_{j=1}^{m} h_j z_j$, by (10) we have

$$dz + \alpha z dt = \sum_{j=1}^{m} h_j dW_j.$$

We also need two facts

$$\left(E\left|z_j(0)\right|\right)^2 \le E\left|z_j(0)\right|^2 = \int_{-\infty}^{0} \mathrm{e}^{2\alpha\tau}\mathrm{d}\tau = \frac{1}{2\alpha} \to 0, \quad (12)$$

as $\alpha \to \infty$. We also have

$$\lim_{t\to\infty} \frac{z_j(t)}{t} = 0, \quad \mathrm{P}-a.s. \quad (13)$$

Assumed $h_j \in D(A) \subset H^2$. Then by Sobolev embedding theorem, $H^2(I) \subset C^1(\bar{I})$, we have $h_j \in C^1(\bar{I})$. In particular, all Dh_j are bounded continuous functions. Thus there exists a $\beta_0 > 0$ (depending only on h_j) such that

$$\sup_{x\in I}\left|Dz(x,t)\right| \le \beta_0 \sum_{j=1}^{m}\left|z_j(t)\right|, \ \forall t \in R, \ \mathrm{P}-a.s. \quad (14)$$

where $z = \sum h_j z_j$ and z_j is the Ornstein-Uhlenbeck process defined by (11). It is also easy to prove that

$$|z| + |Dz| + \left|D^2 z\right| \le \beta_1 \sum_{j=1}^{m}\left|z_j(t)\right|, \quad (15)$$

where $\beta_1 > 0$ only depends on h_j.

To show that (5) generates a random dynamical system, we let $v(t) = u_t(t) + \varepsilon u(t) - z(t)$, where u, u_t is the solution of (5), then u, v satisfies

$$\begin{cases} u_t = v - \varepsilon u + z, \\ v_t = -\gamma Au + \varepsilon(\alpha - \varepsilon)u + (\varepsilon - \alpha)v - (i + \varepsilon\beta)u_x \\ \qquad + \beta v_x + \varepsilon z + \beta z_x - f\left(|u|^2\right)u, \\ u(\tau,\omega) = u_0, \quad v(\tau,\omega) = u_1 + \varepsilon u_0 - z(\tau), \end{cases} \quad (16)$$

where $u_0 \in H^1$, $v_0 \in L^2$, and $\phi_0 = \phi(\tau,\omega) = (u_0, v_0)^{\mathrm{T}} \in E_0$.

By the same proof as deterministic case [4], one can easily get that for $\mathrm{P}-a.s.$, $\omega \in \Omega$, the following results hold

Theorem 3.1 If $(u_0, v_0)^{\mathrm{T}} \in E_0$, there exists a unique solution $\phi(t,\omega) = \left(u(t,\omega), v(t,\omega)\right)^{\mathrm{T}} \in E_0$ of (16), which satisfies

$$u(t,\omega) \in C\left([\tau,T]; H^1\right), \ v(t,\omega) \in C\left([\tau,T]; L^2\right).$$

If $(u_0, v_0)^{\mathrm{T}} \in E_1$, there exists a unique solution $\phi(t,\omega) = \left(u(t,\omega), v(t,\omega)\right)^{\mathrm{T}} \in E_1$ of (16), which satisfies

$$u(x,t) \in C\left([\tau,T]; H^2\right), \ v(t,\omega) \in C\left([\tau,T]; H^1\right).$$

From the above discussion, we denote the solution of (5) by $u(t) = u(t,\omega;\tau,u_0)$ (denote sometimes by $u(t;\tau,u_0)$, $u(t;\tau)$, $u(t,\omega)$, $u(t)$ or even u if no confusions). Then we can define a mapping $\varphi : R^+ \times \Omega \times E_0 \mapsto E_0$ by

$$\begin{aligned} &\varphi(t,\omega)\phi_0 := \phi(t,\omega;0,\phi_0) \\ &= \left(u(t,\omega;0,u_0), v(t,\omega;0,v_0)\right), \ t \ge 0, \end{aligned} \quad (17)$$

by the definition 2.2.1, it is easy to show that ϕ is a continuous RDS on E_0 with the following fact

$$\varphi(\tau, \theta_{-\tau}\omega)\phi_0 = \phi(0,\omega;-\tau,\phi_0),$$

for $\phi_0 = \phi(\tau,\omega) = (u_0, v_0)^{\mathrm{T}} \in E_0, \ \tau \ge 0.$

4. Random Attractors

4.1. Absorbing Set in E_0

In this subsection, we prove that the RDS φ defined by (17) has a bounded absorbing set $B(\omega) \subset E_0$, which absorbs, in fact, all the bounded sets $B \subset E_0$. Recall that $\phi(t,\omega;\tau,\phi_0) = \left(u(t,\omega;\tau,u_0), v(t,\omega;\tau,v_0)\right)^{\mathrm{T}}$ is the solution of (16) with $u(\tau) = u_0$ and $v(\tau) = u_1 + \varepsilon u_0 - z(\tau) = v_0$.

We then rewrite (16) as follows

$$\dot{\phi}+L\phi=F(\phi,\omega),\ \phi(\tau,\omega)=\left(u(\tau,\omega),v(\tau,\omega)\right)^{\mathrm{T}},\ t\ge\tau, \tag{18}$$

where

$$\phi=\begin{pmatrix}u\\ v\end{pmatrix},\quad L=\begin{pmatrix}\varepsilon I & -I\\ \gamma A-\varepsilon(\alpha-\varepsilon)I & (\alpha-\varepsilon)I\end{pmatrix},$$

$$F(\phi,\omega)=\begin{pmatrix}z\\ -(i+\varepsilon\beta)u_x+\beta v_x+\varepsilon z+\beta z_x-f\left(|u|^2\right)u\end{pmatrix}.$$

We now can prove the absorption of RDS φ (defined by (17)) in E_0.

Lemma 4.1 For any no random bounded set B, there exists a random variable $\rho_1(\omega)\ge 0$ satisfying the following property: for every $(u_0,u_1+\varepsilon u_0)^{\mathrm{T}}\in B$, there exists $T_B(\omega)<-1$, such that, for any $\tau\le T_B(\omega)$, the following estimate holds $\mathrm{P}-a.s.$

$$\left\|\phi(t,\omega;\tau,\phi_0)\right\|_{E_0}\le\rho_1(\omega),\ t\in[-1,0].$$

Proof. Taking the inner product of (18) with ϕ in E_0, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|^2+(L\phi,\phi)=(F(\phi,\omega),\phi). \tag{19}$$

Taking the real part of (19), we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2+|v|^2\right)+\varepsilon\|u\|^2+(\gamma-1)\mathrm{Re}((u,v))\\&-\varepsilon(\alpha-\varepsilon)\mathrm{Re}(u,v)+(\alpha-\varepsilon)|v|^2\\&=\mathrm{Re}((z,u))+\mathrm{Re}(-(i+\varepsilon\beta)u_x,v)+\beta\,\mathrm{Re}(v_x,v)\\&\quad+\varepsilon\,\mathrm{Re}(z,v)+\beta\,\mathrm{Re}(z_x,v)-\mathrm{Re}\left(f\left(|u|^2\right)u,v\right)\end{aligned}. \tag{20}$$

Since

$$\begin{aligned}&(\gamma-1)\mathrm{Re}((u,v))\\&=(\gamma-1)\mathrm{Re}((u,u_t+\varepsilon u-z))\\&=\frac{\gamma-1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\varepsilon(\gamma-1)\|u\|^2-(\gamma-1)\mathrm{Re}((u,z))\end{aligned}, \tag{21}$$

and

$$\begin{aligned}&\mathrm{Re}\left(f\left(|u|^2\right)u,v\right)\\&=\mathrm{Re}\left(f\left(|u|^2\right)u,u_t+\varepsilon u-z\right)\\&=\mathrm{Re}\left(f\left(|u|^2\right)u,u_t\right)+\varepsilon\,\mathrm{Re}\left(f\left(|u|^2\right)u,u\right)-\mathrm{Re}\left(f\left(|u|^2\right)u,z\right),\\&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int F\left(|u|^2\right)\mathrm{d}x+\varepsilon\,\mathrm{Re}\left(f\left(|u|^2\right)u,u\right)-\mathrm{Re}\left(f\left(|u|^2\right)u,z\right)\end{aligned} \tag{22}$$

it follows from (20)-(22), we get that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma}{2}\|u\|^2+\frac{1}{2}|v|^2+\frac{1}{2}\int F\left(|u|^2\right)\mathrm{d}x\right)+\varepsilon\gamma\|u\|^2\\&-\varepsilon(\alpha-\varepsilon)\mathrm{Re}(u,v)+(\alpha-\varepsilon)|v|^2\\&+\mathrm{Re}((i+\varepsilon\beta)u_x,v)+\varepsilon\,\mathrm{Re}\left(f\left(|u|^2\right)u,u\right)\\&-\gamma\,\mathrm{Re}((u,z))-\varepsilon\,\mathrm{Re}(z,v)-\beta\,\mathrm{Re}(z_x,v)\\&-\mathrm{Re}\left(f\left(|u|^2\right)u,z\right)=0\end{aligned}. \tag{23}$$

We introduce two functions

$$g_1(u,v)=\frac{\gamma}{2}\|u\|^2+\frac{1}{2}|v|^2+\frac{1}{2}\int F\left(|u|^2\right)\mathrm{d}x, \tag{24}$$

$$\begin{aligned}G_1(u,v)=&\,\varepsilon\gamma\|u\|^2-\varepsilon(\alpha-\varepsilon)\mathrm{Re}(u,v)+(\alpha-\varepsilon)|v|^2\\&+\mathrm{Re}((i+\varepsilon\beta)u_x,v)+\varepsilon\,\mathrm{Re}\left(f\left(|u|^2\right)u,u\right)\\&-\gamma\,\mathrm{Re}((u,z))-\varepsilon\,\mathrm{Re}(z,v)-\beta\,\mathrm{Re}(z_x,v)\\&-\mathrm{Re}\left(f\left(|u|^2\right)u,z\right)\end{aligned}, \tag{25}$$

and one process

$$C_1(t)=\sum_{j=1}^{m}\left|z_j(t)\right|. \tag{26}$$

So that (23) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}g_1(u,v)+G_1(u,v)=0. \tag{27}$$

In the following, we denote by c any constant depending only on the data $(\alpha,\beta,\gamma,\varepsilon,I,f)$, which can be different from line to line or even in the same line. Now we can prove there exist positive constants δ_0, d and d_1 such that

$$\begin{aligned}&G_1(u,v)-\delta_0 g_1(u,v)\\&=\kappa_1(u,v)\ge\frac{d}{2}\|\phi\|^2-2\beta_1C_1(t)\|\phi\|^2-g(t)-c\end{aligned}, \tag{28}$$

$$g_1(u,v)\ge\frac{d_1}{2}\|\phi\|^2-c, \tag{29}$$

where β_1 is defined by (15) and $g(t)$ will be defined in the following paper, hence we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}g_1(u,v)+\delta_0g_1(u,v)=-\kappa_1(u,v). \tag{30}$$

In fact we have

$$\begin{aligned}&G_1(u,v)-\delta_0g_1(u,v)=\kappa_1(u,v)\\&=\varepsilon\gamma\|u\|^2-\varepsilon(\alpha-\varepsilon)\mathrm{Re}(u,v)+(\alpha-\varepsilon)|v|^2\\&\quad+\mathrm{Re}((i+\varepsilon\beta)u_x,v)+\varepsilon\int f\left(|u|^2\right)|u|^2\mathrm{d}x\\&\quad-\frac{\delta_0}{2}\int F\left(|u|^2\right)\mathrm{d}x-\gamma\,\mathrm{Re}((u,z))-\varepsilon\,\mathrm{Re}(z,v)\\&\quad-\beta\,\mathrm{Re}(z_x,v)-\mathrm{Re}\left(f\left(|u|^2\right)u,z\right)-\frac{\gamma\delta_0}{2}\|u\|^2-\frac{\delta_0}{2}|v|^2\end{aligned}. \tag{31}$$

By estimating every terms on the right side in (31), letting

$$\varepsilon = \frac{\alpha\gamma\lambda_1}{2\alpha^2 + 3\gamma\lambda_1}, \tag{32}$$

where λ_1 is the first eigenvalue of A, then by (32), we find that

$$\begin{aligned}
&\varepsilon\gamma\|u\|^2 - \varepsilon(\alpha-\varepsilon)\operatorname{Re}(u,v) + (\alpha-\varepsilon)|v|^2 \\
&= \frac{3\varepsilon\gamma}{4}\|u\|^2 + \frac{\alpha}{2}|v|^2 + \frac{\varepsilon}{2}|v|^2 + \frac{\varepsilon\gamma}{4}\|u\|^2 - \varepsilon(\alpha-\varepsilon)\operatorname{Re}(u,v) \\
&+ \left(\frac{\alpha}{2} - \frac{3\varepsilon}{2}\right)|v|^2 \ge \frac{3\varepsilon\gamma}{4}\|u\|^2 + \frac{\alpha}{2}|v|^2 + \frac{\varepsilon}{2}|v|^2 + \frac{\varepsilon\gamma}{4}\|u\|^2 \\
&- \varepsilon(\alpha-\varepsilon)|u|\cdot|v| + \left(\frac{\alpha}{2} - \frac{3\varepsilon}{2}\right)|v|^2 \ge \frac{3\varepsilon\gamma}{4}\|u\|^2 + \frac{\alpha}{2}|v|^2 + \frac{\varepsilon}{2}|v|^2 \\
&+ \frac{\varepsilon\gamma}{4}\|u\|^2 - \frac{\varepsilon\alpha}{\sqrt{\lambda_1}}\|u\|\cdot|v| + \left(\frac{\alpha}{2} - \frac{3\varepsilon}{2}\right)|v|^2 = \frac{3\varepsilon\gamma}{4}\|u\|^2 + \frac{\alpha}{2}|v|^2 \\
&+ \frac{\varepsilon}{2}|v|^2 + \frac{\varepsilon\gamma}{4}\|u\|^2 - \frac{\varepsilon\alpha}{\sqrt{\frac{2\varepsilon\alpha^2}{\gamma(\alpha-3\varepsilon)}}}\|u\|\cdot|v| + \left(\frac{\alpha}{2} - \frac{3\varepsilon}{2}\right)|v|^2 \\
&= \frac{3\varepsilon\gamma}{4}\|u\|^2 + \frac{\alpha}{2}|v|^2 + \frac{\varepsilon}{2}|v|^2 + \left(\frac{\sqrt{\varepsilon\gamma}}{2}\|u\| - \sqrt{\frac{\alpha-3\varepsilon}{2}}|v|\right)^2 \\
&\ge \frac{3\varepsilon\gamma}{4}\|u\|^2 + \frac{\alpha}{2}|v|^2 + \frac{\varepsilon}{2}|v|^2
\end{aligned}, \tag{33}$$

and

$$\begin{aligned}
\operatorname{Re}((i+\varepsilon\beta)u_x, v) &\le \sqrt{1+\varepsilon^2\beta^2}\,\|u\|\cdot|v| \\
&\le \frac{1+\varepsilon^2\beta^2}{2\alpha}\|u\|^2 + \frac{\alpha}{2}|v|^2
\end{aligned}. \tag{34}$$

Using $|f|_\infty$ uniformly bounded, we get

$$\begin{aligned}
&-\gamma\operatorname{Re}((u,z)) - \varepsilon\operatorname{Re}(z,v) - \beta\operatorname{Re}(z_x,v) - \operatorname{Re}\left(f\left(|u|^2\right)u, z\right) \\
&\le \gamma\|z\|\cdot\|u\| + \varepsilon|z|\cdot|v| + \beta|z_x|\cdot|v| + \left\|f\left(|u|^2\right)\right\|_{L^\infty}|u|\cdot|z| \\
&\le \frac{\gamma^2\beta_1C_1(t)}{4} + \beta_1C_1(t)\|u\|^2 + \frac{\varepsilon^2\beta_1C_1(t)}{4} + \beta_1C_1(t)|v|^2 \\
&+ \frac{\beta^2\beta_1C_1(t)}{4} + \beta_1C_1(t)|v|^2 + \frac{c^2\beta_1C_1(t)}{4\lambda_1} + \beta_1C_1(t)\|u\|^2 \\
&= 2\beta_1C_1(t)\left(\|u\|^2 + |v|^2\right) + \left(\frac{\gamma^2}{4} + \frac{\varepsilon^2}{4} + \frac{\beta^2}{4} + \frac{c^2}{4\lambda_1}\right)\beta_1C_1(t) \\
&= 2\beta_1C_1(t)\|\phi\|^2 + \left(\frac{\gamma^2}{4} + \frac{\varepsilon^2}{4} + \frac{\beta^2}{4} + \frac{c^2}{4\lambda_1}\right)\beta_1C_1(t) \\
&= 2\beta_1C_1(t)\|\phi\|^2 + g(t),
\end{aligned} \tag{35}$$

where $g(t) = \left(\frac{\gamma^2}{4} + \frac{\varepsilon^2}{4} + \frac{\beta^2}{4} + \frac{c^2}{4\lambda_1}\right)\beta_1C_1(t)$.

Taking δ_0, such that $\delta_0 \le \frac{\varepsilon}{2}$ and $\frac{2\left(1+\varepsilon^2\beta^2\right)}{\alpha\gamma} \le \delta_0$, where ε is defined in (32), and letting

$$d = \min(\delta_0\gamma, \delta_0) \tag{36}$$

we find that

$$\frac{3\varepsilon\gamma}{4} - \frac{1+\varepsilon^2\beta^2}{2\alpha} - \frac{\delta_0\gamma}{2} \ge \frac{d}{2}, \tag{37}$$

$$\frac{\alpha}{2} + \frac{\varepsilon}{2} - \frac{\alpha}{2} - \frac{\delta_0}{2} \ge \frac{d}{2}. \tag{38}$$

Noting that $\frac{\delta_0}{2\varepsilon} \le \frac{1}{4} < 1$, by (8) and (9), we have

$$\varepsilon\int f\left(|u|^2\right)|u|^2\,\mathrm{d}x - \frac{\delta_0}{2}\int F\left(|u|^2\right)\mathrm{d}x \ge \varepsilon\gamma_0\int|u|^{2+2\delta}\,\mathrm{d}x, \tag{39}$$

using

$$\begin{aligned}
\int|u|^2\,\mathrm{d}x &\le (2D)^{\frac{\delta}{1+\delta}}\left(\int|u|^{2+2\delta}\,\mathrm{d}x\right)^{\frac{1}{1+\delta}} \\
&\le \frac{\varepsilon_0^{1+\delta}}{1+\delta}\int|u|^{2+2\delta}\,\mathrm{d}x + \frac{2D\delta}{1+\delta}\varepsilon_0^{-\frac{1+\delta}{\delta}},
\end{aligned}$$

where ε_0 is any positive number. Choosing

$$\varepsilon_0 \le \left(\frac{2\varepsilon\gamma_0(1+\delta)}{d}\right)^{\frac{1}{1+\delta}},$$

we find that

$$\varepsilon\gamma_0\int|u|^{2+2\delta}\,\mathrm{d}x \ge \frac{d}{2}|u|^2 - c. \tag{40}$$

Combining (31)-(40), we infer that

$$G_1(u,v) - \delta_0 g_1(u,v) \ge \frac{d}{2}\|\phi\|^2 - 2\beta_1C_1(t)\|\phi\|^2 - g(t) - c. \tag{41}$$

In order to prove (29), and similarly to (40), we have

$$g_1(u,v) \ge \frac{\gamma}{2}\|u\|^2 + \frac{1}{2}|v|^2 - c. \tag{42}$$

Taking $d_1 = \min(\gamma, 1)$, we get (29).

From (27)-(30), we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}g_1(u,v) + \delta_0 g_1(u,v) \le 2\beta_1C_1(t)\|\phi\|^2 + g(t) + c. \tag{43}$$

Taking $\mu \ge \max\left(\frac{4}{\gamma}, 4\right)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}g_1(u,v) \le \left(-\delta_0 + \mu\beta_1C_1(t)\right)g_1(u,v) + g(t) + c. \tag{44}$$

Putting

$$f_1(t) = -\delta_0 + \mu\beta_1 C_1(t) = -\delta_0 + \mu\beta_1 \sum_{j=1}^{m} \left|z_j(t)\right|, \quad (45)$$

then by the Gronwall Lemma, we get, for $t \ge \tau$

$$g_1(u,v) \le \mathrm{e}^{\int_\tau^t f_1(\eta)\mathrm{d}\eta} g_1(u_0,v_0) + \int_\tau^t \left(g(\eta)+c\right)\mathrm{e}^{\int_\eta^t f_1(s)\mathrm{d}s}\mathrm{d}\eta . \quad (46)$$

In particular, we have, for $t \in [-1,0]$, $\tau \le -1$,

$$\begin{aligned} g_1(u,v) &\le c_1 \mathrm{e}^{\int_\tau^0 f_1(\eta)\mathrm{d}\eta} g_1(u_0,v_0) \\ &\quad + c_1 \int_{-\infty}^0 \left(g(\eta)+c\right)\mathrm{e}^{\int_\eta^0 f_1(s)\mathrm{d}s}\mathrm{d}\eta, \end{aligned} \quad (47)$$

where $c_1 = \mathrm{e}^{\delta_0}$, in view of the following fact: for $-1 \le t \le 0$,

$$\begin{aligned} \mathrm{e}^{\int_\tau^t f_1(\eta)\mathrm{d}\eta} &= \mathrm{e}^{\int_\tau^0 f_1(\eta)\mathrm{d}\eta}\mathrm{e}^{-\int_t^0 f_1(\eta)\mathrm{d}\eta} \\ &\le \mathrm{e}^{\int_\tau^0 f_1(\eta)\mathrm{d}\eta}\mathrm{e}^{\int_t^0 \delta_0 \mathrm{d}\eta} \le \mathrm{e}^{\delta_0}\mathrm{e}^{\int_\tau^0 f_1(\eta)\mathrm{d}\eta} . \end{aligned}$$

To estimate all integration terms on the right side in (47), we choose $\alpha > 0$ such that

$$\mu\beta_1 E C_1(0) = \mu\beta_1 \sum_{j=1}^{m} E\left|z_j(0)\right| < \frac{\delta_0}{2} .$$

This is possible since by (12), $EC_1(0) \to 0$ as $\alpha \to \infty$. Thus, since $z_j(t)$ is stationary and ergodic, it is easy to get

$$\lim_{\tau\to-\infty} \frac{1}{-\tau}\int_\tau^0 f_1(\eta)\mathrm{d}\eta = Ef_1(0) = -\delta_0 + \mu\beta_1 EC_1(0) < -\frac{\delta_0}{2}, \quad (48)$$

which implies that

$$\lim_{\tau\to-\infty} \mathrm{e}^{\int_\tau^0 f_1(\eta)\mathrm{d}\eta} = 0, \quad \mathrm{P}-a.s. \quad (49)$$

By (13), $\dfrac{z_j(\tau)}{\tau} \to 0$ as $\tau \to -\infty$, thus $C_1(t)$ and further $g(t)$ is at most 1-times polynomoal growth at $-\infty$, which, together with (49), implies that

$$q_1(\omega) = \int_{-\infty}^0 \left(g(\eta)+c\right)\mathrm{e}^{\int_\eta^0 f_1(s)\mathrm{d}s}\mathrm{d}\eta < \infty, \quad \mathrm{P}-a.s. \quad (50)$$

and also implies that

$$q_2(\omega) = \sup_{\tau\le-1} \mathrm{e}^{\int_\tau^0 f_1(\eta)\mathrm{d}\eta}\left|z(\tau)\right|^2 < \infty . \quad (51)$$

Noting that

$$g_1(u_0,v_0) = \frac{\gamma}{2}\|u_0\|^2 + \frac{1}{2}|v_0|^2 + \frac{1}{2}\int F\left(|u_0|^2\right)\mathrm{d}x ,$$

$$\phi_0 = \left(u_0, u_1 + \varepsilon u_0 - z(\tau)\right),$$

and $u_0 \in H^1$, $u_1 \in L^2$, we get

$$g_1(u_0,v_0) \le c\left(\|\phi_0\|^2 + 1\right), \quad (52)$$

then it follows from (29) and (47)-(52), we obtain that

$$\begin{aligned} &\frac{d_1}{2}\left\|\phi(t,\omega;\tau,\phi_0)\right\|_{E_0}^2 \\ &\le cc_1 \mathrm{e}^{\int_\tau^0 f_1(\eta)\mathrm{d}\eta}\left(\|u_0\|^2 + |u_1+\varepsilon u_0|^2 + |z(\tau)|^2 + 1\right) \\ &\quad + c_1\int_{-\infty}^0 \left(g(\eta)+c\right)\mathrm{e}^{\int_\eta^0 f_1(s)\mathrm{d}s}\mathrm{d}\eta + c. \end{aligned} \quad (53)$$

We now take $\rho_1^2(\omega) = \dfrac{2\mathrm{e}^{\delta_0}}{d_1}\left(2c + q_1(\omega) + cq_2(\omega)\right) + \dfrac{2c}{d_1}$ and choose $T_B(\omega)$ such that

$$\mathrm{e}^{\int_\tau^0 f_1(\eta)\mathrm{d}\eta}\left(\|u_0\|^2 + |u_1+\varepsilon u_0|^2\right) \le 1 ,$$

for all $\tau \le T_B(\omega)$, then we get

$$\left\|\phi(t,\omega;\tau,\phi_0)\right\|_{E_0}^2 \le \rho_1^2(\omega), \ t \in [-1,0] . \quad (54)$$

4.2. Absorbing Set in E_1

In order to prove the absorption property in E_1, we also need the following change for (18).

Differentiating (18) with respect to x and letting $\eta = u_x$, $\xi = v_x$, $\psi = (\eta,\xi)^{\mathrm{T}} = (u_x,v_x)^{\mathrm{T}}$, we have

$$\begin{aligned} &\dot\psi + L\psi = F(\psi,\omega), \\ &\psi(\tau,\omega) = \left(\eta(\tau,\omega),\xi(\tau,\omega)\right)^{\mathrm{T}}, \ t \ge \tau, \end{aligned} \quad (55)$$

where

$$L = \begin{pmatrix} \varepsilon I & -I \\ \gamma A - \varepsilon(\alpha-\varepsilon)I & (\alpha-\varepsilon)I \end{pmatrix},$$

$$F(\psi,\omega) = \begin{pmatrix} z_x \\ -(i+\varepsilon\beta)\eta_x + \beta\xi_x + \varepsilon z_x + \beta z_{xx} - f'\left(|u|^2\right)|u|^2\eta \\ -f'\left(|u|^2\right)u^2\bar\eta - f\left(|u|^2\right)\eta \end{pmatrix}.$$

We now can prove the absorption of RDS φ (defined by (17)) in E_1.

Lemma 4.2 For any no random bounded set B, there exists a random variable $\rho_2(\omega) \ge 0$ satisfying the following property: for every $(\eta_0, \eta_1 + \varepsilon\eta_0)^{\mathrm{T}} \in B$, there exists $T_B(\omega) < -1$, such that, for any $\tau \le T_B(\omega)$, the following estimate holds $\mathrm{P}-a.s.$

$$\left\|\phi(t,\omega;\tau,\phi_0)\right\|_{E_1} \le \rho_2(\omega), \ t \in [-1,0].$$

Proof. Taking the inner product of (55) with ψ in E_0, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|^2+\left(L\psi,\psi\right)=\left(F\left(\psi,\omega\right),\psi\right). \quad (56)$$

Taking the real part of (56), we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\eta\|^2+|\xi|^2\right)+\varepsilon\|\eta\|^2+(\gamma-1)\mathrm{Re}\left((\eta,\xi)\right)\\&-\varepsilon(\alpha-\varepsilon)\mathrm{Re}(\eta,\xi)+(\alpha-\varepsilon)|\xi|^2\\&=\mathrm{Re}\left((z_x,\eta)\right)+\mathrm{Re}\left(-(i+\varepsilon\beta)\eta_x,\xi\right)+\beta\,\mathrm{Re}\left(\xi_x,\xi\right)\\&+\varepsilon\,\mathrm{Re}\left(z_x,\xi\right)+\beta\,\mathrm{Re}\left(z_{xx},\xi\right)-\mathrm{Re}\left(f'\left(|u|^2\right)|u|^2\eta,\xi\right)\\&-\mathrm{Re}\left(f'\left(|u|^2\right)u^2\bar{\eta},\xi\right)-\mathrm{Re}\left(f\left(|u|^2\right)\eta,\xi\right).\end{aligned} \quad (57)$$

Due to

$$\begin{aligned}&(\gamma-1)\mathrm{Re}\left((\eta,\xi)\right)=(\gamma-1)\mathrm{Re}\left((\eta,\eta_t+\varepsilon\eta-z_x)\right)\\&=\frac{\gamma-1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|^2+\varepsilon(\gamma-1)\|\eta\|^2-(\gamma-1)\mathrm{Re}\left((\eta,z_x)\right)\end{aligned}, \quad (58)$$

and

$$\begin{aligned}&\mathrm{Re}\left(f'\left(|u|^2\right)|u|^2\eta,\xi\right)=\mathrm{Re}\left(f'\left(|u|^2\right)|u|^2\eta,\eta_t+\varepsilon\eta-z_x\right)\\&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int f'\left(|u|^2\right)|u|^2|\eta|^2\,\mathrm{d}x-\int f'\left(|u|^2\right)|\eta|^2\,\mathrm{Re}\left(u\bar{u}_t\right)\mathrm{d}x\\&\quad-\int f''\left(|u|^2\right)|u|^2|\eta|^2\,\mathrm{Re}\left(u\bar{u}_t\right)\mathrm{d}x+\varepsilon\int f'\left(|u|^2\right)|u|^2|\eta|^2\,\mathrm{d}x\\&\quad-\mathrm{Re}\left(f'\left(|u|^2\right)|u|^2\eta,z_x\right),\end{aligned} \quad (59)$$

and

$$\begin{aligned}&\mathrm{Re}\left(f\left(|u|^2\right)\eta,\xi\right)=\mathrm{Re}\left(f\left(|u|^2\right)\eta,\eta_t+\varepsilon\eta-z_x\right)\\&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int f\left(|u|^2\right)|\eta|^2\,\mathrm{d}x-\int f'\left(|u|^2\right)|\eta|^2\,\mathrm{Re}\left(u\bar{u}_t\right)\mathrm{d}x\\&\quad+\varepsilon\int f\left(|u|^2\right)|\eta|^2\,\mathrm{d}x-\mathrm{Re}\left(f\left(|u|^2\right)\eta,z_x\right).\end{aligned} \quad (60)$$

In view of (57)-(60), we get that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma}{2}\|\eta\|^2+\frac{1}{2}|\xi|^2+\frac{1}{2}\int f'\left(|u|^2\right)|u|^2|\eta|^2\,dx\right.\\&\left.+\frac{1}{2}\int f\left(|u|^2\right)|\eta|^2\,dx\right)+\varepsilon\gamma\|\eta\|^2-\varepsilon(\alpha-\varepsilon)\mathrm{Re}(\eta,\xi)\\&+(\alpha-\varepsilon)|\xi|^2-\gamma\,\mathrm{Re}\left((\eta,z_x)\right)+\mathrm{Re}\left((i+\varepsilon\beta)\eta_x,\xi\right)\\&-\varepsilon\,\mathrm{Re}\left(z_x,\xi\right)-\beta\,\mathrm{Re}\left(z_{xx},\xi\right)+\mathrm{Re}\left(f'\left(|u|^2\right)u^2\bar{\eta},\xi\right)\\&-2\int f'\left(|u|^2\right)|\eta|^2\,\mathrm{Re}\left(u\bar{u}_t\right)\mathrm{d}x-\int f''\left(|u|^2\right)|u|^2|\eta|^2\,\mathrm{Re}\left(u\bar{u}_t\right)\mathrm{d}x\\&+\varepsilon\int f'\left(|u|^2\right)|u|^2|\eta|^2\,\mathrm{d}x-\mathrm{Re}\left(f'\left(|u|^2\right)|u|^2\eta,z_x\right)\\&+\varepsilon\int f\left(|u|^2\right)|\eta|^2\,\mathrm{d}x-\mathrm{Re}\left(f\left(|u|^2\right)\eta,z_x\right)=0.\end{aligned} \quad (61)$$

Letting

$$\begin{aligned}g_2(\eta,\xi)&=\frac{\gamma}{2}\|\eta\|^2+\frac{1}{2}|\xi|^2+\frac{1}{2}\int f'\left(|u|^2\right)|u|^2|\eta|^2\,\mathrm{d}x\\&\quad+\frac{1}{2}\int f\left(|u|^2\right)|\eta|^2\,\mathrm{d}x\end{aligned} \quad (62)$$

and

$$\begin{aligned}&G_2(\eta,\xi)\\&=\varepsilon\gamma\|\eta\|^2-\varepsilon(\alpha-\varepsilon)\mathrm{Re}(\eta,\xi)+(\alpha-\varepsilon)|\xi|^2-\gamma\,\mathrm{Re}\left((\eta,z_x)\right)\\&+\mathrm{Re}\left((i+\varepsilon\beta)\eta_x,\xi\right)-\varepsilon\,\mathrm{Re}\left(z_x,\xi\right)-\beta\,\mathrm{Re}\left(z_{xx},\xi\right)\\&+\mathrm{Re}\left(f'\left(|u|^2\right)u^2\bar{\eta},\xi\right)-2\int f'\left(|u|^2\right)|\eta|^2\,\mathrm{Re}\left(u\bar{u}_t\right)\mathrm{d}x\\&-\int f''\left(|u|^2\right)|u|^2|\eta|^2\,\mathrm{Re}\left(u\bar{u}_t\right)\mathrm{d}x+\varepsilon\int f'\left(|u|^2\right)|u|^2|\eta|^2\,\mathrm{d}x\\&-\mathrm{Re}\left(f'\left(|u|^2\right)|u|^2\eta,z_x\right)+\varepsilon\int f\left(|u|^2\right)|\eta|^2\,\mathrm{d}x\\&-\mathrm{Re}\left(f\left(|u|^2\right)\eta,z_x\right)\end{aligned} \quad (63)$$

Then it comes from (61)-(63) that

$$\frac{\mathrm{d}}{\mathrm{d}t}g_2(\eta,\xi)+G_2(\eta,\xi)=0. \quad (64)$$

Now similarly to the above arguments (Lemma 4.1), we can prove that there exist $\delta_2>0$, $d_2>0$ and $d_3>0$, such that

$$\begin{aligned}&G_2(\eta,\xi)-\delta_2 g_2(\eta,\xi)\\&=\kappa_2(\eta,\xi)\geq\frac{d_2}{2}\|\psi\|^2-2\beta_1C_1(t)\|\psi\|^2-g_0(t)-c\end{aligned}, \quad (65)$$

$$g_2(\eta,\xi)\geq\frac{d_3}{2}\|\psi\|^2-c, \quad (66)$$

where β_1 is defined by (15) and $g_0(t)$ will be defined in the following paper, therefore we have

$$\frac{\mathrm{d}}{\mathrm{d}t}g_2(\eta,\xi)+\delta_2 g_2(\eta,\xi)=-\kappa_2(\eta,\xi). \quad (67)$$

In fact,

$$\begin{aligned}&G_2(\eta,\xi)-\delta_2 g_2(\eta,\xi)=\kappa_2(\eta,\xi)\\&=\varepsilon\gamma\|\eta\|^2-\varepsilon(\alpha-\varepsilon)\mathrm{Re}(\eta,\xi)+(\alpha-\varepsilon)|\xi|^2\\&+\mathrm{Re}\left((i+\varepsilon\beta)\eta_x,\xi\right)-\gamma\,\mathrm{Re}\left((\eta,z_x)\right)-\varepsilon\,\mathrm{Re}\left(z_x,\xi\right)\\&-\beta\,\mathrm{Re}\left(z_{xx},\xi\right)+\mathrm{Re}\left(f'\left(|u|^2\right)u^2\bar{\eta},\xi\right)\\&-2\int f'\left(|u|^2\right)|\eta|^2\,\mathrm{Re}\left(u\bar{u}_t\right)\mathrm{d}x-\int f''\left(|u|^2\right)|u|^2|\eta|^2\,\mathrm{Re}\left(u\bar{u}_t\right)\mathrm{d}x\\&+\varepsilon\int f'\left(|u|^2\right)|u|^2|\eta|^2\,\mathrm{d}x-\mathrm{Re}\left(f'\left(|u|^2\right)|u|^2\eta,z_x\right)\\&+\varepsilon\int f\left(|u|^2\right)|\eta|^2\,\mathrm{d}x-\mathrm{Re}\left(f\left(|u|^2\right)\eta,z_x\right)-\frac{\gamma\delta_2}{2}\|\eta\|^2-\frac{\delta_2}{2}|\xi|^2\\&-\frac{\delta_2}{2}\int f'\left(|u|^2\right)|u|^2|\eta|^2\,\mathrm{d}x-\frac{\delta_2}{2}\int f\left(|u|^2\right)|\eta|^2\,\mathrm{d}x.\end{aligned} \quad (68)$$

Taking δ_2, d_2 are the same as δ_0, d respectively in (36) and using $|\eta|$, $|u_t|$, $|u|_\infty$, $|f|_\infty$, $|f'|_\infty$, $\left\| |u|^2 f''\left(|u|^2\right) \right\|_\infty$ and uniformly bounded in time, we majoring every term on the right side in (68) to get

$$\begin{aligned}
&\varepsilon\gamma\|\eta\|^2 - \varepsilon(\alpha-\varepsilon)\operatorname{Re}(\eta,\xi) + (\alpha-\varepsilon)|\xi|^2 \\
&= \frac{3\varepsilon\gamma}{4}\|\eta\|^2 + \frac{\alpha}{2}|\xi|^2 + \frac{\varepsilon}{2}|\xi|^2 + \frac{\varepsilon\gamma}{4}\|\eta\|^2 \\
&\quad -\varepsilon(\alpha-\varepsilon)\operatorname{Re}(\eta,\xi) + \left(\frac{\alpha}{2} - \frac{3\varepsilon}{2}\right)|\xi|^2 \\
&\geq \frac{3\varepsilon\gamma}{4}\|\eta\|^2 + \frac{\alpha}{2}|\xi|^2 + \frac{\varepsilon}{2}|\xi|^2 + \frac{\varepsilon\gamma}{4}\|\eta\|^2 \\
&\quad -\varepsilon(\alpha-\varepsilon)|\eta|\cdot|\xi| + \left(\frac{\alpha}{2} - \frac{3\varepsilon}{2}\right)|\xi|^2 \\
&\geq \frac{3\varepsilon\gamma}{4}\|\eta\|^2 + \frac{\alpha}{2}|\xi|^2 + \frac{\varepsilon}{2}|\xi|^2 + \frac{\varepsilon\gamma}{4}\|\eta\|^2 \\
&\quad -\frac{\varepsilon\alpha}{\sqrt{\lambda_1}}\|\eta\|\cdot|\xi| + \left(\frac{\alpha}{2} - \frac{3\varepsilon}{2}\right)|\xi|^2 \\
&\geq \frac{3\varepsilon\gamma}{4}\|\eta\|^2 + \frac{\alpha}{2}|\xi|^2 + \frac{\varepsilon}{2}|\xi|^2,
\end{aligned} \tag{69}$$

where ε is defined by (32) and λ_1 is the first eigenvalue of A, and we have

$$\begin{aligned}
\operatorname{Re}\left((i+\varepsilon\beta)\eta_x,\xi\right) &\leq \sqrt{1+\varepsilon^2\beta^2}\,\|\eta\|\cdot|\xi| \\
&\leq \frac{1+\varepsilon^2\beta^2}{\alpha}\|\eta\|^2 + \frac{\alpha}{4}|\xi|^2,
\end{aligned} \tag{70}$$

and

$$\begin{aligned}
&-\gamma\operatorname{Re}\left((\eta,z_x)\right) - \varepsilon\operatorname{Re}(z_x,\xi) - \beta\operatorname{Re}(z_{xx},\xi) \\
&-\operatorname{Re}\left(f\left(|u|^2\right)\eta, z_x\right) - \operatorname{Re}\left(f'\left(|u|^2\right)|u|^2\eta, z_x\right) \\
&\leq \gamma\|\eta\|\cdot\|z_x\| + \varepsilon|z_x|\cdot|\xi| + \beta|z_{xx}|\cdot|\xi| \\
&+\left\|f\left(|u|^2\right)\right\|_{L^\infty}|\eta|\cdot|z_x| + \left\|f'\left(|u|^2\right)|u|^2\right\|_{L^\infty}|\eta|\cdot|z_x| \\
&\leq \frac{\gamma^2\beta_1C_1(t)}{8} + 2\beta_1C_1(t)\|\eta\|^2 + \frac{\varepsilon^2\beta_1C_1(t)}{4} + \beta_1C_1(t)|\xi|^2 \\
&+\frac{\beta^2\beta_1C_1(t)}{4} + \beta_1C_1(t)|\xi|^2 + c\beta_1C_1(t) \\
&= 2\beta_1C_1(t)\|\psi\|^2 + \left(\frac{\gamma^2}{8} + \frac{\varepsilon^2}{4} + \frac{\beta^2}{4} + c\right)\beta_1C_1(t) \\
&= 2\beta_1C_1(t)\|\psi\|^2 + g_0(t),
\end{aligned} \tag{71}$$

where $g_0(t) = \left(\frac{\gamma^2}{8} + \frac{\varepsilon^2}{4} + \frac{\beta^2}{4} + c\right)\beta_1C_1(t)$.

By $|\eta|_{L^4}^2 \leq c|\eta_x|^{1/2}|\eta|^{3/2}$ and young inequality, we have

$$\begin{aligned}
&\operatorname{Re}\left(f'\left(|u|^2\right)u^2\bar{\eta},\xi\right) \\
&\leq \left\|f'\left(|u|^2\right)|u|^2\right\|_{L\infty}|\eta|\cdot|\xi| \leq c|\eta|^2 + \frac{\alpha}{4}|\xi|^2,
\end{aligned} \tag{72}$$

$$\begin{aligned}
&-2\int f'\left(|u|^2\right)|\eta|^2\operatorname{Re}\left(u\bar{u}_t\right)\mathrm{d}x \\
&\leq 2\left\|f'\left(|u|^2\right)\right\|_{L^\infty}|u|_{L^\infty}|\eta|_{L^4}^2|u_t| \leq \frac{d_2}{8}\|\eta\|^2 + c|\eta|^2,
\end{aligned} \tag{73}$$

$$\begin{aligned}
&-\int f''\left(|u|^2\right)|u|^2|\eta|^2\operatorname{Re}\left(u\bar{u}_t\right)\mathrm{d}x \\
&\leq \left\|f''\left(|u|^2\right)|u|^2\right\|_{L^\infty}|u|_{L^\infty}|\eta|_{L^4}^2|u_t| \leq \frac{d_2}{8}\|\eta\|^2 + c|\eta|^2,
\end{aligned} \tag{74}$$

$$\begin{aligned}
&\left(\varepsilon - \frac{\delta_2}{2}\right)\int f'\left(|u|^2\right)|u|^2|\eta|^2\,\mathrm{d}x \\
&\leq \left(\varepsilon - \frac{\delta_2}{2}\right)\left\|f'\left(|u|^2\right)|u|^2\right\|_{L^\infty}|\eta|_{L^4}^2 \leq \frac{d_2}{8}\|\eta\|^2 + c|\eta|^2,
\end{aligned} \tag{75}$$

$$\begin{aligned}
&\left(\varepsilon - \frac{\delta_2}{2}\right)\int f\left(|u|^2\right)|\eta|^2\,\mathrm{d}x \\
&\leq \left(\varepsilon - \frac{\delta_2}{2}\right)\left\|f\left(|u|^2\right)\right\|_{L^\infty}|\eta|_{L^4}^2 \leq \frac{d_2}{8}\|\eta\|^2 + c|\eta|^2.
\end{aligned} \tag{76}$$

Then it follows from (68)-(76) that

$$\begin{aligned}
&G_2(\eta,\xi) - \delta_2 g_2(\eta,\xi) \\
&\geq \frac{d_2}{2}\|\psi\|^2 - 2\beta_1C_1(t)\|\psi\|^2 - g_0(t) - c|\eta|^2. \\
&\geq \frac{d_2}{2}\|\psi\|^2 - 2\beta_1C_1(t)\|\psi\|^2 - g_0(t) - c
\end{aligned} \tag{77}$$

Similarly, with $d_3 = \min(\gamma,1)$, we can easily derive that

$$g_2(\eta,\xi) \geq \frac{d_3}{2}\|\psi\|^2 - c. \tag{78}$$

From (64)-(67), we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}g_2(\eta,\xi) + \delta_2 g_2(\eta,\xi) \leq 2\beta_1C_1(t)\|\psi\|^2 + g_0(t) + c, \tag{79}$$

it follows from (44) that, for the same μ, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}g_2(\eta,\xi) \leq \left(-\delta_2 + \mu\beta_1C_1(t)\right)g_2(\eta,\xi) + g_0(t) + c. \tag{80}$$

Putting

$$f_2(t) = -\delta_2 + \mu\beta_1C_1(t) = -\delta_2 + \mu\beta_1\sum_{j=1}^{m}\left|z_j(t)\right|, \tag{81}$$

then by the Gronwall Lemma, we have, for $t \geq \tau$

$$\begin{aligned}
g_2(\eta,\xi) &\leq \mathrm{e}^{\int_\tau^t f_2(\sigma)\mathrm{d}\sigma}g_2(\eta_0,\xi_0) \\
&\quad + \int_\tau^t \left(g_0(\sigma) + c\right)\mathrm{e}^{\int_\sigma^t f_2(s)\mathrm{d}s}\mathrm{d}\sigma.
\end{aligned} \tag{82}$$

The same as (47), we have, for $t \in [-1,0]$, $\tau \le -1$,

$$g_2(\eta,\xi) \le c_1 e^{\int_\tau^0 f_2(\sigma)d\sigma} g_2(\eta_0,\xi_0) + c_1\int_{-\infty}^0 \left(g_0(\sigma)+c\right) e^{\int_\sigma^0 f_2(s)ds} d\sigma \quad (83)$$

where $c_1 = e^{\delta_2}$, and

$$q_1'(\omega) = \int_{-\infty}^0 \left(g_0(\sigma)+c\right) e^{\int_\sigma^0 f_2(s)ds} d\sigma < \infty, \quad (84)$$

$$q_2'(\omega) = \sup_{\tau\le -1} e^{\int_\tau^0 f_2(\sigma)d\sigma} \left|z_x(\tau)\right|^2 < \infty. \quad (85)$$

Noting that

$$g_2(\eta_0,\xi_0) = \frac{\gamma}{2}\|\eta_0\|^2 + \frac{1}{2}|\xi_0|^2 + \frac{1}{2}\int f'\left(|u_0|^2\right)|u_0|^2|\eta_0|^2 dx, + \frac{1}{2}\int f\left(|u_0|^2\right)|\eta_0|^2 dx$$

$$\psi_0 = \left(\eta_0, \eta_1 + \varepsilon\eta_0 - z_x(\tau)\right),$$

we get

$$g_2(\eta_0,\xi_0) \le c\left(\|\psi_0\|^2 + 1\right), \quad (86)$$

then from (66) and (83)-(86), we obtain that

$$\frac{d_2}{2}\|\psi(t,\omega;\tau,\psi_0)\|_{E_0}^2 \le cc_1 e^{\int_\tau^0 f_2(\sigma)d\sigma}\left(\|\eta_0\|^2 + |\eta_1+\varepsilon\eta_0|^2 + |z_x(\tau)|^2 + 1\right). + c_1\int_{-\infty}^0 \left(g_0(\sigma)+c\right)e^{\int_\sigma^0 f_2(s)ds}d\sigma + c \quad (87)$$

We now take $\rho_2^2(\omega) = \frac{2e^{\delta_2}}{d_2}\left(2c + q_1'(\omega) + cq_2'(\omega)\right) + \frac{2c}{d_2}$ and choose $T_B(\omega)$ such that

$$e^{\int_\tau^0 f_2(\sigma)d\sigma}\left(\|\eta_0\|^2 + |\eta_1+\varepsilon\eta_0|^2\right) \le 1,$$

for all $\tau \le T_B(\omega)$, then we get

$$\|\phi(t,\omega;\tau,\phi_0)\|_{E_1}^2 \le \rho_2^2(\omega), \quad t\in[-1,0]. \quad (88)$$

4.3. The Compactness in E_0

In this subsection, we prove the compactness in E_0 through the decomposition of solution semigroup.

Let $u(t)$ be a solution of (5) with initial value $\left(u_0, u_1+\varepsilon u_0 - z(\tau)\right)^T$. We make the decomposition $u(t) = y_1(t) + y_2(t)$, where $y_1(t)$ and $y_2(t)$ satisfy

$$\begin{cases} dy_{1t} + \alpha y_{1t}dt - \beta y_{1xt}dt - \gamma y_{1xx}dt + iy_{1x}dt = 0, \\ y_1(x,\tau) = u_0(x), \quad y_{1t}(x,\tau) = u_1(x), \\ y_1(x-L,t) = y_1(x+L,t). \end{cases} \quad (89)$$

and

$$\begin{cases} dy_{2t} + \alpha y_{2t}dt - \beta y_{2xt}dt - \gamma y_{2xx}dt + iy_{2x}dt \\ \qquad + f\left(|u|^2\right)udt = \sum_{j=1}^m h_j dW_j, \\ y_2(x,\tau) = 0, \quad y_{2t}(x,\tau) = 0, \\ y_2(x-L,t) = y_2(x+L,t). \end{cases} \quad (90)$$

Lemma 4.3 For any no random bounded set B, we have, for any $\left(u_0, u_1+\varepsilon u_0\right)^T \in B$

$$\|Y_1(0)\|_{E_0}^2 = \|y_1(0)\|^2 + |y_{1t}(0) + \varepsilon y_1(0)|^2 \le \frac{2}{d_1}\left(\|u_0\|^2 + |u_1+\varepsilon u_0|^2\right)e^{\delta_3\tau}, \quad (91)$$

and there exists a random variable $\rho_3(\omega) \ge 0$ such that for $P-a.s.$ $\omega\in\Omega$

$$\|DY_2(0,\omega;\tau,Y_2(\tau,\omega))\|_{E_0}^2 \le \rho_3^2(\omega), \quad (92)$$

where $Y_1 = \left(y_1, y_{1t}+\varepsilon y_1\right)^T$ and $Y_2 = \left(y_2, y_{2t}+\varepsilon y_2 - z\right)^T$ satisfy (89) and (90) respectively.

Proof. Taking the inner product of (89) with Y_1 in E_0 whose initial value is $\left(u_0, u_1+\varepsilon u_0\right)^T$, after a simple computation similarly as Lemma 4.1, we obtain (91).

Taking the inner product of (90) with AY_2 in E_0 whose initial value is $\left(0, -z(\tau)\right)^T$, after a simple computation similarly as Lemma 4.2, we obtain (92).

Let $B_{1/2}(\omega)$ be the ball of $E_1 = H^2\times H^1$ of random variable $\rho_3(\omega)\ge 0$. From the compact embedding $E_1 = H^2\times H^1 \to E_0 = H^1\times L^2$, we see that $B_{1/2}(\omega)$ is compact. For any no random bounded set B of E_0, pick any $\phi(0)\in\varphi(t,\theta_{-t}\omega)B$. From Lemma 4.3, we have $Y_2(0) = \phi(0) - Y_1(0) \in B_{1/2}(\omega)$, where $Y_2(t,\omega)$ is given by Lemma 4.3. Therefore, again by Lemma 4.3,

$$\inf_{\ell(0)\in B_{1/2}(\omega)} \|\phi(0) - \ell(0)\|_{E_0}^2 \le \|Y_1(0)\|_{E_0}^2 \le \frac{2}{d_1}\left(\|u_0\|^2 + |u_1+\varepsilon u_0|^2\right)e^{\delta_3\tau}, \quad \tau\le 0.$$

So

$$dist\left(\varphi(t,\theta_{-t}\omega)B, B_{1/2}(\omega)\right) \to 0, \quad \text{as } t\to+\infty.$$

Corollary 4.4 The RDS $\varphi(t,\omega)$ associated with (17) possesses a uniformly attracting compact set $B_{1/2}(\omega)\subset E_0$, so the RDS $\varphi(t,\omega)$ is uniformly asymptotically compact in E_0.

By applying Theorem 2.2.2, Lemma 4.1 and corollary 4.4, we obtain the final conclusion of this whole paper.

Theorem 4.5 Assume $h_j \in D(A) = H^1\cap H^2$, then the RDS φ modeling the dissipative Hamiltonian amplitude equation governing modulated wave instabilities possesses a compact random attractor $A(\omega)$ which attracts all bounded sets of $E_0 = H^1 \times L^2$.

5. Acknowledgements

The authors would like to express their sincere thanks to the anonymous referee for his/her valuable comments and suggestions to improve the paper.

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