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A Ranking Method of Extreme Efficient DMUs Using Super-Efficiency Model*

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Abstract

In this paper, we present a method for ranking extreme efficient decision making units (DMUs) in data envelopment analysis (DEA) models based on measuring distance between them and new PPS (after omission extreme efficient DMUs) along the input-axis or output axis.

Keywords: Data envelopment analysis; Efficiency; Ranking.

1. Introduction

Data envelopment analysis (DEA) is a non-parametric method for measuring efficiency of a set of Decision Making Units (DMUs) such as firms or a public sector agencies, first introduced by Charnes, Cooper and Rhodes (CCR) [1] and extended by Banker, Charnes, and Cooper (BCC) [2]. One important issue in DEA which has been studied by many DEA researchers, is to discriminate between efficient DMUs. Several authors have proposed methods for ranking the best performers ([3]-[10] among others). Ying-Ming Wang et al. [10] proposed a ranking methodology for DMUs by imposing an appropriate minimum weight restriction on all inputs and outputs, which is decided by a decision maker (DM) or an assessor in terms of the solutions to a series of linear programming (LP) models that are specially constructed to determine a maximin weight for each DEA efficient unit. Jahanshahloo et al. [4] proposed a ranking system based on changing reference set. in the proposed ranking system, the evaluation for efficient DMUs is dependent of the efficiency changes of all inefficient units due to its absence in the reference set while the estimate for inefficient DMUs depends on the influence of the exclusion of each efficient unit from the reference set. For a review of ranking methods, readers are referred to Adler et al. [8]; in which the previous methods were divided into six categories. One of the six areas is well-known as the super-efficiency approach, which was first proposed by Andersen and Petersen (AP) [9] to rank extreme efficient DMUs. The main idea of this approach is to evaluate a DMU after this performer itself is excluded from the reference set. However, in some cases, especially under the condition of variable returns to scale (VRS), the method may fail due to the infeasibility problem associated with the super-efficiency models. In this paper, we intend to introduce a new ranking system for extreme efficient DMUs under the condition of VRS and CRS. For this aim, we use a variance of super-efficiency models (see models (6) and (7)) and obtain the most distance between them and new PPS (after omission extreme efficient DMUs) *along the input-axis or output-axis*. Also, our proposed method is able to rank extreme efficient DMUs even in presence of infeasibility. This paper is organized as follows. Section 2 presents some basic DEA models. Section 3 introduces our proposal and states and proves some facts related to properties and characteristics of it. A numerical example is given in Section 4 and Section 5 comprehends our conclusions.

2. Background

Consider a set of *n* DMUs which is associated with *m* inputs and *s* outputs. Particularly, DMU_j ($j \in J = \{1, ..., n\}$) consumes amount x_{ij} of input *i* and produces amount y_{rj} of output *r*. Let $X_j = (x_{1j}, ..., x_{mj})$ in which $X_j \geq 0$ & $X_j \neq 0$ and $Y_j = (y_{1j}, ..., y_{mj})$ in which $Y_j \geq 0$ & $Y_j \neq 0$. The production possibility set (PPS) of CCR model define as follows:

$$T_c = \{(X, Y) | X \geq \sum_{j=1}^{n} \lambda_j X_j, Y \leq \sum_{j=1}^{n} \lambda_j Y_j, \lambda_j \geq 0, j \in J\}$$

and similarly the production possibility set of BCC model define as follows:

$$T_v = \{(X,Y)| X \geq \sum_{j=1}^{n} \lambda_j X_j, Y \leq \sum_{j=1}^{n} \lambda_j X_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j \in J\}$$

By omitting (X_p, Y_p) from T_c, the new production possibility set is as follows:

$$T_c = \{(X,Y) \mid X \geq \sum\nolimits_{j \in J-\{p\}}^{n} \lambda_j X_j, Y \leq \sum\nolimits_{j \in J-\{p\}}^{n} \lambda_j Y_j, \lambda_j \geq 0, j \in J-\{p\}\}.$$

In figure (1) the polyhedral $ZABCR$ and $ZACR$ are T_v and T_v', respectively.
The input-oriented BCC and input-oriented CCR models, corresponds to$DMU_p, p \in J$, is given by (1) and (2), respectively:

$$\min \theta - \in (\sum_{i=a}^{m} s_i^+ + \sum_{r=1}^{s} s_r^-)$$
$$s.t. \sum_{j \in J} \lambda_j x_{ij} + s_i^- = \theta x_{ik} i = 1, \ldots, m$$
$$\sum_{j \in J} \lambda_j y_{rj} - s_r^+ = y_{rk} r = 1, \ldots, s (1)$$
$$\sum_{j \in J} \lambda_j = 1$$
$$\lambda_j \geq 0 \qquad j \in J$$
$$s_i^- \geq 0 \qquad i = 1, \ldots, m$$
$$s_r^+ \geq 0 \qquad r = 1, \ldots, s$$
$$\theta \qquad free$$

$$\min \min \theta - \in (\sum_{i=1}^{m} t_i^+ + \sum_{r=1}^{s} t_r^-)$$
$$s.t. \sum_{j \in J} \lambda_j x_{ij} + t_i^- = \theta x_{ik} i = 1, \ldots, m$$
$$\sum_{j \in J} \lambda_j y_{rj} - t_r^+ = y_{rk} r = 1, \ldots, s (2)$$
$$\lambda_j \geq 0 \qquad j \in J$$
$$t_i^- \geq 0 \qquad i = 1, \ldots, m$$
$$t_r^+ \geq 0 \qquad r = 1, \ldots, s$$
$$\theta \qquad free$$

where ϵ is non-Archimedean small and positive number and $s_i^+, s_r^-, t_i^+$and t_r^-, $i = 1, \ldots, m$, $r = 1, \ldots, s$are called slack variables belong to $\mathbb{R}^{\geq 0}$. Note that $s_i^-$and t_r^- represent input excesses; also $s_r^+$and t_r^+ represent output shortfalls. The models (1) and (2) are called envelopment forms (with non-Archimedean number).
DMU_p is said to be *strong efficient* (CCR-efficient) if and only if: $\theta^* = 1$ and $t^{*+} = 0$, $t^{*-} = 0$. Where the superscript (*) indicates optimality. In similar manner the BCC- *efficient* DMUs can be defined.
The AP model is as follows [9]:

$$s.t. \sum_{j \in J-\{p\}} \lambda_j x_{ij} \leq \theta x_{ip} i = 1, \ldots, m$$
$$\sum_{j \in J-\{p\}} \lambda_j y_{rj} \geq y_{rp} r = 1, \ldots, s (3)$$
$$\lambda_j \geq 0 \qquad j \in J$$
$$\theta \qquad free$$

The Jahanshahloo's method corresponding inefficient DMU_a is as follows: [4]

$$\min \partial_{a,b} \min \partial_{a,b} = \theta - \epsilon (\sum_{i=1}^{m} s_i^+ + \sum_{r=1}^{s} s_r^-)$$
$$s.t. \sum_{j \in J-\{b\}} \lambda_j x_{ij} + s_i^- = \theta x_{ia} i = 1, \ldots, m$$
$$\sum_{j \in J-\{b\}} \lambda_j y_{rj} - s_r^+ = y_{ra} r = 1, \ldots, s (4)$$
$$\lambda_j \geq 0 \qquad j \in J - \{b\}$$
$$s_i^- \geq 0 \qquad i = 1, \ldots, m$$
$$s_r^+ \geq 0 \qquad r = 1, \ldots, s$$

The efficiency of strong efficiency DMU_b will be denoted by Ω and will be given by:

$$\Omega_b = \frac{\sum_{a \in J_n} \partial_{a,b}}{\tilde{n}}$$

in which $J_n$is the set of non-strong efficiency DMUs and $\tilde{n}$ is the number of non-strong efficiency DMUs.
Jahanshahloo et al. [14] used l_1-norm in order to rank the extremely efficient DMUs in DEA models with constant and variable returns to scale, and the proposed method can remove the difficulties arising from AP and MAJ models. Their proposed model is as follows:

$$\min \Gamma(_p, X, Y) = \sum_{i=1}^{m} |x_i - x_{ip}| + \sum_{r=1}^{s} |y_r - y_{rp}|$$
$$s.t. \sum_{j \in J-\{p\}} \lambda_j x_{ij} \leq x_i i = 1, \ldots, m$$
$$\sum_{j \in J-\{p\}} \lambda_j y_{rj} \geq y_r r = 1, \ldots, s (5)$$
$$x_i \geq 0 \qquad i = 1, \ldots, m$$
$$y_r \geq 0 \qquad r = 1, \ldots, s$$
$$\lambda_j \geq 0 \qquad j \in J - \{p\}$$

In this paper we rank DMUs in CCR model; in a similar way one can also rank DMUs in BCC model. The following super-efficiency models are used for ranking extreme efficient DMUs [12]:

$$\begin{aligned}&\min\ \theta_l^p\\ &s.t. \sum_{j\in J-\{p\}} \lambda_j^p x_{lj} \le \theta_l^p x_{lk}\\ &\sum_{j\in J-\{p\}} \lambda_j^p x_{ij} \le x_{ik} i=1,\ldots m\ i\neq l (6)\\ &\sum_{j\in J-\{k\}} \lambda_j^p y_{rj} \ge y_{rk} r=1,\ldots,s\\ &\lambda_j^p \ge 0 \qquad j\in J\{p\}\end{aligned}$$

$$\begin{aligned}&\min \varphi_q^p\\ &s.t. \sum_{j\in J-\{k\}} \mu_j^p x_{ij} \le x_{ip} i=1,\ldots,m\\ &\sum_{j\in J-\{p\}} \mu_j^p y_{rj} \le y_{rp}, r=1,\ldots,s\ r\neq q (7)\\ &\sum_{j\in J-\{p\}} \mu_j^p y_{qj} \ge \varphi_q^p y_{qp}\\ &\mu_j^p \ge 0\end{aligned}$$

in which $l = 1, \ldots, m$ and $q = 1, \ldots, s$.
In order to rank DMUs in BCC model the constraint $\sum_{j\in J-\{p\}} \lambda_j^p = 1$ is is added to (6) and (7) (see [13]).
Remark 1: In (6) and (7), for each l and q, $\theta_l^p = \varphi_q^p = 1$ if and only if strong efficient DMU_k lies on the strong defining hyperplane of PPS. In fact it is an interior point of strong defining hyperplane.

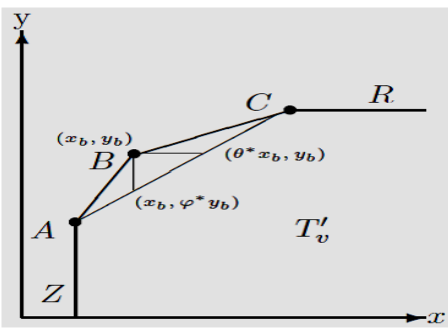

Figure 1: The value distance of DMU_b from new PPS (T'_v) along the x-axis and y-axis

Remark 2: In (6) (or (7)), if for some l (or q), $\theta_l^p > 1$ (or $\varphi_q^p < 1$) or if for some l(or q), model (6) (or model (7)) is infeasible then, strong efficient DMU_k lies on the extreme ray (edge) of PPS and vice versa. (For more details on models (6) and (7) see [12].)
We call all efficient DMUs lying on extreme ray (edge) of PPS of CCR model as extreme efficient DMUs, hereafter.
Remark 3: In multiple output case, if for some q model (7) is infeasible then, virtual DMU $DMU_k^{'} = (x_{1k}, \ldots, x_{mk}, y_{1k}, \ldots, y_{qk} - \gamma, \ldots, y_{sk})$ in which $\gamma > 0$, is on the weak defining hyperplane of PPS vertical to hyperplane y_q=0.
Remark 4: In multiple inputs case, if for at least one l, model (6) is infeasible then virtual DMU $DMU_k^{'} = (x_{1k}, \ldots, x_{lk} + \alpha, \ldots, x_{mk}, y_{1k}, \ldots, y_{sk})$ in which $\alpha > 0$, is on the weak defining hyperplane which passes through lth axis of input.[1]

We state the following theorem without proof.

Theorem 1: *If there exist at least two DEA-efficient DMUs then, there is at least one l (or q) so that model (6) (or model (7)) is feasible.*

3. A proposed method for ranking by super-efficiency model

First, we evaluate each DMU_k, ($k \in J$), by models (2). Suppose that L DMUs are strong efficient. Without lose of generality we can assume that these efficient DMUs are DMU_1 ,…, DMU_L . Consider the set E = {1, ..., L}. Then, corresponding to each DMU_p, ($p \in E$), we solve the models (6) and (7). In view of remarks 1,2 we can identify all extreme efficient DMUs.

Corresponding each extreme efficient DMU_p we obtain θ_l^{p*} and φ_q^{p*}, i = 1,…, m, q = 1,..., s.
Note that for some l and q the models (6) and (7) may be infeasible. But by theorem 1 for DMU_p there exist at least for one l or q so that the models (6) or (7) have finite optimal solution.
We have:
$x_{lp}(\theta_l^{p*} - 1)$ =*The value distance of DMU_p from new PPS* ('T$_c$) *along the lth axis of input.*
$y_{qp}(1 - \varphi_q^{p*})$ =*The value distance of DMU_p from new PPS* (T$_c$) *along the qth axis of output.*

Where it is understood that the above value distances are taken over existing $\theta_l^{p*}$and φ_q^{p*}(see Fig. 1).
Let

$$\gamma_p^* = \max_{l,q}\{x_{lp}(\theta_l^{p*} - 1), y_{qp}(1 - \varphi_q^{p*})\}$$

In order to judge which DMU has better rank in comparison with other DMUs, the following definition is given:

Definition. DMU_p has a better rank in comparison with DMU_k if $\gamma_p^* > \gamma_k^*$.

The following theorem shows that our proposed method has more influence to the PPS of DEA models than l_1-method has.

1For more detail see [12].

Theorem 2: $\Gamma_p^c(X,Y) \le \gamma_p^*$.

Proof. The proof is straightforward.

Table 1: DMUs data (extracted from [[11], p. 689]).

Branch	input: Staff	Computer terminals	Space m2	output: Deposits	Loans	Charge
1	0.9503	0.70	0.1550	0.1900	0.5214	0.2926
2	0.7962	0.60	1.0000	0.2266	0.6274	0.4624
3	0.7982	0.75	0.5125	0.2283	0.9703	0.2606
4	0.8651	0.55	0.2100	0.1927	0.6324	1.0000
5	0.8151	0.85	0.2675	0.2333	0.7221	0.2463
6	0.8416	0.65	0.5000	0.2069	0.6025	0.5689
7	0.7189	0.60	0.3500	0.1824	0.9000	0.7158
8	0.7853	0.75	0.1200	0.1250	0.2340	0.2977
9	0.4756	0.60	0.1350	0.0801	0.3643	0.2439
10	0.6782	0.55	0.5100	0.0818	0.1835	0.0486
11	0.7112	1.00	0.3050	0.2117	0.3179	0.4031
12	0.8113	0.65	0.2550	0.1227	0.9225	0.6279
13	0.6586	0.85	0.3400	0.1755	0.6452	0.2605
14	0.9763	0.80	0.5400	0.1443	0.5143	0.2433
15	0.6845	0.95	0.4500	1.0000	0.2617	0.0982
16	0.6127	0.90	0.5250	0.1151	0.4021	0.4641
17	1.0000	0.60	0.2050	0.0900	1.0000	0.1614
18	0.6337	0.65	0.2350	0.0591	0.3492	0.0678
19	0.3715	0.70	0.2375	0.0385	0.1898	0.1112
20	0.5827	0.55	0.5000	0.1101	0.6145	0.7643

Table 2: The results of evaluation extreme CCR-efficient DMUs by models (6) and (7)

DMU_k	θ_1^{k*}	θ_2^{k*}	θ_3^{k*}	φ_1^{k*}	φ_2^{k*}	φ_3^{k*}	γ_k^*
DMU_1	infs	infs	1.1009	.7964	.8171	infs	0.0954
DMU_4	infs	infs	infs	infs	infs	.3804	0.3804
DMU_7	infs	infs	infs	infs	.7729	infs	0.2044
DMU_{12}	1.2085	infs	1.2308	infs	.8782	infs	0.1692
DMU_{15}	infs	infs	infs	.2035	infs	infs	0.7965
DMU_{17}	infs	infs	infs	infs	.7416	infs	0.0549
DMU_{20}	1.1849	infs	infs	infs	infs	.8054	0.1487

Table 3: Results for comparing

	Proposed method		Model(4)		Model(3) (AP)		Model(5) l_1-norm	
DMU_k	γ_k^*	Rank	Ω_k	Rank	θ^*	Rank	$\Gamma(X,Y)$	Rank
DMU_1	0.0954	6	0.675	6	1.1009	7	0.0156	7
DMU_4	0.6196	2	0.708	2	1.9336	2	0.4802	2
DMU_7	0.2044	3	0.698	3	1.1737	5	0.1915	3
DMU_{12}	0.1692	4	0.677	5	1.1095	6	0.0589	6
DMU_{15}	0.7965	1	0.790	1	4.9139	1	0.7965	1
DMU_{17}	0.0549	7	0.677	5	1.3484	3	0.1760	4
DMU_{20}	0.1487	5	0.680	4	1.1849	4	0.1077	5

4. Numerical Example

We evaluated with our method the data of 20 branch banks of Iran. This data was previously analyzed by Amirteimoori and Kordrostami [11] and is listed in Table 1. The use of our method generated the analysis shown in Table 2, in which, the statement "infs" means "infeasible". Table 3 shows a comparison of our proposal and some other ranking approaches. All these approaches are implemented in input-oriented version under the condition of CRS. As reported in Table 3, DMU_{15} is the most efficient one in our method and other methods. According to the results, the rankings of DMUs by the four methods are almost similar; in particular, the results of our method are more similar to the method [4].

5. Conclusion

In this paper we propose a method for ranking extreme efficient DMUs based on measuring distance between extreme efficient DMUs and new PPS (after omission extreme efficient DMUs) *along the input-axis or output-axis*, using supper efficiency models (6) and (7). It seems that our approach is more robust than other method [14]; because, as it was shown in theorem 2, our proposed method has more influence in new PPS than the proposed method by Jahanshahloo et al. [14]. Initial studies had shown that our approach also can be applied with BCC model. We suggest as future works a deeper analysis in this subject.

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