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Chaos in Time Series of Sakarya River Daily Flow Rate

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Abstract
In this study, possible low dimensional chaotic behavior of Sakarya river flow rates is investigated via nonlinear time series techniques. To reveal the chaotic dynamics, the maximal positive Lyapunov exponent is calculated from the reconstructed phase space, which is obtained using the phase space reconstruction method. The method reconstructs a phase space from the scalar time series, which depicts the real system’s invariants Positive values, because the Lyapunov exponent values calculated using the appropriate software program indicate possibility of chaotic behavior. Analyzed data involve the monthly average flow rates of eleven main branches of Sakarya River through the years 1960-2000.

Keywords
Chaos Theory, Time Series Analysis, Lyapunov Exponent, Mutual Information, False Nearest Neighbors

1. Introduction: Sakarya River Flow Using Low Dimensional Deterministic Techniques

Sakarya River is one of the longest rivers in western region of Turkey. It originates in the western part of Central Anatolia, but predominantly traverses the Marmara region and flows to the Black Sea. Its basin is 58,160 km²; its length is approximately 810 km and width is about 60 - 150 m [1]. Figure 1 shows its flow map. It has lots of tributaries, for example, Porsuk, Ankara, Goyunuk and Kirmir rivulets.

A natural phenomenon like river flow is a highly complicated one and usually treated as nondeterministic. Understanding the behavior of its underlying dynamics will lead to a more reliable base for choosing an appropriate modeling and prediction method.
Some recent studies [2] have shown that low-dimensional deterministic techniques can be applied as an alternative method for modeling and the results are encouraging. A complicated behavior in nature can be identified as deterministic and chaotic or non-deterministic and random, subject to its underlying dynamics. The use of low-dimensional deterministic and chaotic time series techniques in studying, modeling and possibly predicting river flow dynamics is increasing; encouraging results are being obtained. As long as a sufficient amount of care is exercised in applying low-dimensional deterministic techniques and in interpreting the findings, such techniques can be useful in studying dynamics of river flow [2]. In addition, recent articles [3] [4] on flow prediction using chaos theory can reveal the number of variables that influence the river flow dynamics. This shows nonlinear analysis is gaining importance and usefulness for river flow analysis.

Observational data obtained from the natural phenomena add another complication by way of measurement errors and scalar values which purely represent the underlying
A well-known and widely used approach to overcome these difficulties is the phase space reconstruction method. Based on the theorem of Takens, one can construct a phase space which successively resembles the global behavior of the original dynamical system from scalar measurements. A brief outline of the technique is given in Figure 2. When one observes complicated behavior in nature, one seeks a simple underlying cause. If we have only experimental or observational data at our disposal and in most cases, the data are one dimension, involving a single sequence of measurements at equal time intervals (a time series), and one would try to extract information from it to ascertain whether the dynamics is deterministic and chaotic or nondeterministic and random. In this study, the flow discharge data obtained from monthly averages of Sakarya River and its eleven tributaries are analyzed to reveal the characteristics of its flow dynamics which will be a guideline for modeling the flow discharge.

The experimental data involve the average monthly discharge rate of Sakarya River. The data have only scalar values and are taken from the [5] EIE (General Directorate of Electrical Power Resources Survey and Development Administration). Fifty-four stream flow observation stations have been set on the Sakarya River by the EIE and the observation period spans the period 1960 to 2000.

In Figure 2, from 1960 to 2000 every months’ average flow rate in m$^3$/s unit are collected from EIE. The flow rates for each tributary show similar patterns in spite of the fact that Sakarya River covers a relatively large and varied region involving two different climatic regions. Hence, the Dogancay tributary has characteristics similar to the Sakarya River. In Figure 3, similar data for the Aktas River which is another tributary of Sakarya River are shown. The similarity is apparent.

The time series analysis method applied in this work can be divided into the following steps; observing a one dimensional signal in uniform time interval $x(0), x(T), \ldots, x(n, T)$, phase space reconstruction, and calculation of invariants of the reconstructed dynamics.
2. Phase Space Reconstruction

As the scalar measurements are taken at arbitrary time intervals, a suitable delay time is the key point to preserve the global behavior of the dynamics.

In order to start the phase space reconstruction from the scalar flow rate $s_n(k)$, where $k$ is the time step, we need to construct the delay vector $\tilde{y}_n(k)$ given by;

$$\tilde{y}_n(k) = [s_n(k), s_n(k+\tau), s_n(k+d-1\tau)]$$  \hspace{1cm} (1)

$\tau$ is the delay time and $d$ means the embedding dimension. The time delay can be found from the first zero of the correlation function (linear criterion) or first minimum of the average mutual information [6].

A small delay time can lead to a strongly correlated phase space vectors; on the other hand, information loss is inevitable if a large delay time value; the delay time can be estimated from either the mutual information or the autocorrelation.

One can see both periodic and irregular behavior in Figure 4. A study of the correlation function confirms this conjecture. For example, the correlation function for the Aktas tributary, shows a decrease up to about 7 - 8 months. But the correlation function never reaches zero. It then reveals a periodic behavior involving approximately 40 months. If multiple time scales are involved, a choice must be made between the zero of the correlation function and the first minimum of the mutual information. Although there is no clear indication of consistent success, the latter is usually preferred.

2.1. Mutual Information Is Basically the Information Carried from One Random Variable to Another One

Mutual information is information between two random variables. We can only see the information sent to a given channel by receiving the corresponding information from the same channel.

In Figure 5 a brief outline of the technique is given. One can see demonstrating steps...
Let $X$ and $Y$ be random variables having a joint probability distribution given by $p(X, Y)$. If $X$ and $Y$ have individual probability distributions given by $p(X)$ and $p(Y)$ respectively, the entropy is calculated as the distance between the mutual information assuming equal distribution and the actual multiple distributions, given by the equations below:

$$I(X, Y) = D(p(x, y) \| p(x) p(y))$$

(2)

Mutual information is usually calculated using time delayed vectors reconstructed from the scalar time series as suggested by Fraser and Swinney [6] as a tool to determine a conceivable delay. Besides, the mutual information considers nonlinear correlations. For doing this one has to compute;

$$S = -\sum p_{ij}(\tau) \ln \frac{p_{ij}(\tau)}{p_i p_j}$$

(3)
Here, $p_i$ and $p_j$ are the probabilities to find a given value in the $i$-th and $j$-th intervals of the time series and $p_{ij}(\tau)$ is joint probability that a given observation falls into the $i$-th interval at a given time and in the $j$-th one after a delay time $\tau$. Theoretically there should be no relevance on the amount of separation if there is no correlation. This value can easily be calculated. There are acceptable arguments that a marked minimum at a certain value of $j$, in the time delayed mutual information gives a good estimate for a reasonable time delay. However, if we use a too long time delay, the correlations between the components of reconstructed vectors will be lost and signals will be mistakenly recognized as if it were a random signal, rather than coming from a possibly finite prediction horizon related to the maximal Lyapunov exponent. Mutual information is important for our determining maximal Lyapunov exponent as mentioned in [7].

2.2. False Nearest Neighbors

One of the main problems of reconstructing a phase space from a scalar time series is choosing a suitable embedding dimension, which will at least topologically preserve the global properties of the dynamical system. Embedding dimension directly affects the attractor trajectory in the phase space, which alters the neighborhood of the points. If the embedding dimension is chosen to be smaller than the actual attractor dimension, projection of the trajectory will map false values into other neighborhoods of values; these are called the false neighbors. The calculation goes as follows: Choose a vector $\mathbf{R}_i$ constructed using the delay time suggested by mutual information and calculate the distance between its nearest neighbors $\mathbf{R}_j$ in an arbitrary dimension. Iterate this procedure for all the successive vectors and calculate $R_i$ using the following equation.

$$R_i = \frac{|R_{i+1} - R_{j+1}|}{\|R_i - R_j\|}$$

(4)

A point of data is selected as a false neighbor if the distance, $R_i$ exceeds a given threshold. A typical false neighbor’s calculation is shown in Figure 6.

3. Calculation of the Maximal Lyapunov Exponent

Lyapunov exponent is a measure of divergence or convergence of orbits in a phase space, which can also be calculated for a time series. As the reconstructed phase space preserves the topology of the underlying dynamics, Lyapunov exponents calculated for the embedded phase space will show chaotic nature of the original attractor. The rate of exponential growth between the nearby trajectories is called as the maximal Lyapunov exponent and a positive rate indicates chaotic behavior. The following equation is used to calculate the stretching of the trajectories;

$$S(c,m,t) = \left( \frac{1}{u} \sigma s_{\epsilon} c u_n |s_{n+\epsilon} - s_n| \right)$$

(5)

$s_{n+\epsilon}$ is a neighboring point to $s_n$ in the phase space in the course of the attractor; $\epsilon$ is the
box size. At a future time \( t \) the distance between these points will be \( s_{x,t} - s_{y,t} \). So the formula measures the growth of this distance in time from the initial distance. If \( S(\epsilon, m, t) \) is linear in the separation \( t \) for a range of iterations, and this is insensitive to the embedding dimension \( m \), we can get an estimate of the value of the maximal Lyapunov exponent from the slope of this line. If a robust increase, which is sufficient to determine its sign, is observed, this can be taken as an indicator of chaotic behavior.

4. Results

Table 1 shows us mutual information values, embedding dimension values and Lyap, values which are the largest Lyapunov exponent of a given scalar data set using the

![False-Nearest neighbors for botbasi](image)

**Figure 6.** Graph of False-Nearest neighbors for botbasi tributary of Sakarya river.

**Table 1.** Nonlinear time series analysis results of Sakarya river’s tributaries.

<table>
<thead>
<tr>
<th>Branches of Sakarya river</th>
<th>Mutual information</th>
<th>Embedding dimension</th>
<th>Lyap values</th>
<th>Lyap values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aktas</td>
<td>5</td>
<td>6</td>
<td>0.012</td>
<td>0.008</td>
</tr>
<tr>
<td>Besdegirmen</td>
<td>4</td>
<td>6</td>
<td>0.018</td>
<td>0.021</td>
</tr>
<tr>
<td>Botbasi</td>
<td>4</td>
<td>5</td>
<td>0.014</td>
<td>0.016</td>
</tr>
<tr>
<td>Dogancay</td>
<td>4</td>
<td>4</td>
<td>0.016</td>
<td>0.012</td>
</tr>
<tr>
<td>Dokurcan</td>
<td>4</td>
<td>5</td>
<td>0.016</td>
<td>0.016</td>
</tr>
<tr>
<td>Hamidiye</td>
<td>7</td>
<td>6</td>
<td>0.015</td>
<td>0.008</td>
</tr>
<tr>
<td>Karakoy</td>
<td>4</td>
<td>6</td>
<td>0.021</td>
<td>0.034</td>
</tr>
<tr>
<td>Kargi</td>
<td>9</td>
<td>4</td>
<td>0.010</td>
<td>0.012</td>
</tr>
<tr>
<td>Kocasu</td>
<td>4</td>
<td>6</td>
<td>0.018</td>
<td>0.030</td>
</tr>
<tr>
<td>Mesecik</td>
<td>8</td>
<td>6</td>
<td>0.009</td>
<td>0.020</td>
</tr>
<tr>
<td>Taksirkopru</td>
<td>4</td>
<td>11</td>
<td>0.015</td>
<td>0.039</td>
</tr>
</tbody>
</table>
algorithm of Rosenstein et al. [8] (this is claimed to be a better estimate for smaller data sets as explained below). Lyapk values are largest Lyapunov exponent of a given scalar data set using the algorithm of Kantz [9], with further explanation in [9]. In order to find Lyapunov exponent, we use the mutual information values calculated by the mutual package in TISEAN by observing its first minimum. We get the embedding dimension by the false-nearest technique. In order to calculate the Lyapunov exponent, we calculate the distance between two neighboring points as a function of the separation t on a semilogarithmic plot is used. The Lyapunov exponent is estimated from the slope and their values have a standard deviation of ±0.01.

The algorithm proposed by Kantz establishes that the rate of divergence of nearby trajectories can fluctuate along the trajectory and the amount of fluctuation depends on the stretching and folding of the phase space as given by the spectrum of effective Lyapunov exponents. Rosenstein et al. [8] proposed a systematic algorithm where the distance between the trajectories is calculated by the Euclidian norm in the reconstructed phase space. They use only one neighbor trajectory which makes the method more suitable for shorter time series. Thus, the algorithm suggested by Rosenstein is more effective when the number of data is relatively small. In our study, the results obtained from each algorithm are in parallel with each other. A typical Lyapunov Exponent by stretching exponent calculation using the Rosenstein approach is illustrated in Figure 7, while the calculation using the standard Kantz approach is illustrated in Figure 8. The Kanz algorithm [9] makes use of the statistical properties of the local divergence rates of nearby trajectories. It does not need correct embedding dimension. These figures demonstrate a positive slope and as mentioned above, show positive maximal Lyapunov exponents which in turn indicates chaotic behavior. We also use Kanz algorithm to ensure positive maximal Lyapunov exponent. The statistical issues involved in the selection of the approach are discussed extensively in [10] and [11].

![Stretching factor vs. iteration graph using the Rosenstein algorithm.](image)
Table 1 shows mutual information, embedding dimension value and Lyap, value (the largest Lyapunov exponent of a given scalar data set using the algorithm of Rosenstein et al. [8]); for each tributary of Sakarya river (this is claimed to be a better estimate for smaller data sets as explained below). Lyapk values are largest Lyapunov exponent of a given scalar data set using the algorithm of Kantz [9].

5. Conclusions

Understanding the dynamics of river flow is crucial to select a feasible modeling method to forecast river discharge. In this study, phase space reconstruction method is used to obtain a depiction of the underlying dynamics, which will preserve the global invariants of the system. Maximal Lyapunov exponents which constitute a very strong evidence for chaotic behavior for eleven tributaries of the Sakarya River have been calculated. These results are encouraging for applying chaotic modeling routines instead of probabilistic methods.

As a result, monthly mean flow values of Sakarya River show chaotic behavior as quantified by the maximal Lyapunov Exponent. That may imply that the river has no long time trend, which is observed by [1]. One can say climate changes and huge amount of usage of the river’s water and dam constructions may cause to decrease trend. According to article [12], Benue River in Nigeria has a comparable trend but no low dimensional phase space chaotic dynamics has been observed there. Therefore, we can say Sakarya River has limited future for electricity from dams and other human exploitation because of its chaotic dynamics. We have studied the article by S. Isik et al. [13] that reaches the same conclusions using quasi-linear time series analysis methods from regular statistical analysis. This work corroborates the findings and finally demonstrates that the phenomenon may be better understood by nonlinear time series analysis than stochastic techniques. This work is also relevant for identifying the river’s...
complex behavior since it tries to use nonlinear techniques in river systems which come from the theory of complex systems [14].

References


Global Optimization for Solving Linear Non-Quadratic Optimal Control Problems

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Abstract
This paper presents a global optimization approach to solving linear non-quadratic optimal control problems. The main work is to construct a differential flow for finding a global minimizer of the Hamiltonian function over a Euclid space. With the Pontryagin principle, the optimal control is characterized by a function of the adjoint variable and is obtained by solving a Hamiltonian differential boundary value problem. For computing an optimal control, an algorithm for numerical practice is given with the description of an example.

Keywords
Linear Non-Quadratic Optimal Control, Pontryagin Principle, Global Optimization, Hamiltonian Differential Boundary Value Problem

1. Primal Problem.
In this paper, the notation \( \| \cdot \| \) represents a norm for the specified space concerned. The primal goal of this paper is to present a solution to the following optimal control problem (primal problem (\( \mathcal{P} \)) in short).

\[
\begin{align*}
(\mathcal{P}) \quad & \min \int_{0}^{T} \left[ F(x(t)) + P(u(t)) \right] dt \\
& \text{s.t. } \dot{x} = Ax + Bu, \ x(0) = a, \ t \in [0, T], \ x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \\
\end{align*}
\]

where \( F(x) \) is twice continuously differentiable on \( \mathbb{R}^n \), \( \nabla^2 F(x) \geq 0, \forall x \in \mathbb{R}^n \), \( P(u) \) is twice continuously differentiable on \( \mathbb{R}^m \), \( \nabla^2 P(u) > 0, \forall u \in \mathbb{R}^m \). In the control system, \( A, B \) are given matrices in \( \mathbb{R}^{n \times n} \) and \( \mathbb{R}^{n \times m} \) respectively and \( a \) stands for a given vector in \( \mathbb{R}^n \). We assume that

\[
\liminf_{u \to \infty} \frac{P(u)}{\|u\|} > 0.
\]
If \( P(u) \) is a positive definite quadratic form with respect to \( u \) and \( F(x) \) is a positive semi-definite quadratic form with respect to \( x \), then the problem \( (\mathcal{P}) \) is a classical linear-quadratic optimal control problem [1].

The rest of the paper is organized as follows. In Section 2, we focus on Pontryagin principle to yield a family of global optimizations on the adjoint variable. In Section 3, we deal with the global optimization for the Hamiltonian function. In Section 4, we show that there exists an optimal control to the primal \( (\mathcal{P}) \) and present a mathematical programming. In Section 5 and 6, we discuss how to compute the global minimizer by a differential flow and present an algorithm for the numerical practice with the description of an example.

2. Pontryagin Principle

Associated with the optimal control problem \( (\mathcal{P}) \), let’s introduce the Hamiltonian function

\[
H(x, u, \lambda) = \lambda^T (Ax + Bu) + F(x) + P(u)
\]

with the state and adjoint systems

\[
\dot{x} = H_x(x, u, \lambda) = Ax + Bu, \quad x(0) = a,
\]

\[
\dot{\lambda} = -H_\lambda(x, u, \lambda) = -A^T \lambda - \nabla^T F(x), \quad \lambda(T) = 0.
\]

We know from Pontryagin principle \([2]\) that if \( \hat{u}(.) \) is an optimal control to the problem \( (\mathcal{P}) \), then it is an extremal control. Associated with the state variable \( \hat{x}(.) \) and the adjoint variable \( \hat{\lambda}(.) \), we have

\[
\dot{\hat{x}} = H_x(\hat{x}, \hat{u}, \hat{\lambda}) = A\hat{x} + B\hat{u}, \quad \hat{x}(0) = a,
\]

\[
\dot{\hat{\lambda}} = -H_\lambda(\hat{x}, \hat{u}, \hat{\lambda}) = -A^T \hat{\lambda} - \nabla^T F(\hat{x}), \quad \hat{\lambda}(T) = 0,
\]

and

\[
H(\hat{x}(t), \hat{u}(t), \hat{\lambda}(t)) = \min_{u \in \mathbb{R}^m} H(\hat{x}(t), u, \hat{\lambda}(t))
\]

\[
= \min_{u \in \mathbb{R}^m} \left[ \hat{x}^T(t) (Ax(t) + Bu) + F(\hat{x}(t)) + P(u) \right], \quad \text{a.e. } t \in [0, T].
\]

Since in (2.6) the global optimization is processed on the variable \( u \) over \( \mathbb{R}^m \) for a given \( t \), it is equivalent to deal with the optimization (for obtaining a global minimizer):

\[
\min_{u \in \mathbb{R}^m} \left[ P(u) + \hat{x}^T(t) Bu \right].
\]

Therefore we turn to consider the following optimization with respect to a given parameter vector \( \lambda \in \mathbb{R}^n \)

\[
\min_{u \in \mathbb{R}^m} \left[ P(u) + \lambda^T Bu \right].
\]

In this paper, for a given adjoint variable, we solve the optimization (2.8) to create a function \( u = h(\lambda) \). Then in Hamiltonian boundary problem (2.2), (2.3) we replace the variable \( u \) with the function \( h(\lambda) \) and solve the following equation
\[ \dot{x} = H_\lambda(x, u, \lambda) = Ax + Bh(\lambda), \quad \dot{x}(0) = a, \tag{2.9} \]
\[ \dot{\lambda} = -H_\lambda(x, u, \lambda) = -A^T \hat{\lambda} - \nabla^T F(x), \quad \lambda(T) = 0. \tag{2.10} \]

3. Global Optimization

In this section, for a given parameter vector \( \lambda \in \mathbb{R}^n \), we deal with the following global optimization problem \( Q \) \[ [3] \quad [4] \]

\[ \min_{u \in \mathbb{R}^n} \left[ P(u) + \lambda^T Bu \right] \tag{3.1} \]

to create a function \( u = h(\lambda) \in C\left( \mathbb{R}^n, \mathbb{C}^m \right) \).

It follows from (1.3) that there exist positive numbers \( \beta \) and \( r \) such that

\[ P(u) \geq \beta \|u\|^2, \|u\| > r. \tag{3.2} \]

It follows from (1.3) that there exist positive numbers \( \beta \) and \( r \), such that, when \( \|u\| > r \),

\[ P(u) \geq \beta \|u\|^2. \tag{3.3} \]

Without loss of generality, we assume that \( \beta < \min \left\{ \min_{P(u)} \left(P(0) + 1 \right)^{-1} \right\} \).

**Lemma 3.1.** For given \( \lambda \in \mathbb{R}^n \), let \( \alpha_\lambda = \frac{1}{\beta} \frac{\|B^T \lambda\|}{\beta} \), then the global problem \( Q \) is equivalent to the the following global problem \( Q^* \):

\[ \min_{u \in \mathbb{R}^n} \left[ P(u) + \lambda^T Bu \right]. \tag{3.4} \]

**proof:** Let \( \lambda \in \mathbb{R}^n \) be given. Since \( \min \left\{ \min_{P(u)} \left(P(0) + 1 \right)^{-1} \right\} > \beta > 0 \), it is clear that \( \alpha_\lambda > \|P(0)\| + 1 \) \( \alpha_\lambda > r \). Then, when \( \|u\| > \alpha_\lambda \), we have

\[ P(u) + \left( B^T \lambda \right)^T u \geq \beta \|u\|^2 - \|B^T \lambda\| \|u\| = \left( \beta \|u\| - \|B^T \lambda\| \right) \|u\| > \|u\|. \]

On the other hand, for \( \|u\| \leq \alpha_\lambda \),

\[ P(u) + \left( B^T \lambda \right)^T u \geq \min_{\mathbb{R}^n} \left[ P(u) + \lambda^T Bu \right]. \]

But, since \( \|P(0)\| + 1 > \|P(0)\| \geq \|P(0)\| \leq \min_{u \in \mathbb{R}^n} P(u) \), when \( \|u\| > \alpha_\lambda \), we have

\[ P(u) + \left( B^T \lambda \right)^T u \geq \|u\| > \alpha_\lambda \geq \|P(0)\| + 1 > \|P(0)\| + 1 > P(0) = P(0) + \left( B^T \lambda \right)^T 0 \geq \min_{\mathbb{R}^n} \left[ P(u) + \lambda^T Bu \right]. \]

Since we have shown above that, for all \( u \in \mathbb{R}^n \),

\[ P(u) + \left( B^T \lambda \right)^T u \geq \min_{\mathbb{R}^n} \left[ P(u) + \lambda^T Bu \right], \]
noting that \( \{ \|u\| \leq \alpha_\lambda \} \subset \mathbb{R}^n \), we have

\[ \min_{u \in \mathbb{R}^n} \left[ P(u) + \lambda^T Bu \right] = \min_{\mathbb{R}^n} \left[ P(u) + \lambda^T Bu \right]. \]

The lemma has been proved.

Consequently, by Lemma 3.1 we conclude the following lemma.

**Lemma 3.2.** Let \( u_\lambda \) be a minimizer of \( \min_{\mathbb{R}^n} \left[ P(u) + \lambda^T Bu \right] \). Then \( u_\lambda \) is a minimizer of \( P(u) + \lambda^T Bu \) over \( \mathbb{R}^n \). Moreover, \( \nabla P(u_\lambda) + B^T \lambda = 0 \) and
\[ \|v_i\| \leq \frac{1 + \|B^T \lambda\|}{\beta}. \]

**Remark 3.1.** Since \( \nabla^2 P(u) > 0 \), \( u \in \mathbb{R}^n \), \( u_\lambda \) is the unique minimizer of \( P(u) + \lambda^T Bu \) over \( \|u\| \leq \alpha_\lambda \). Then, it follows by Lemma 3.2 that \( u_\lambda \) is also the unique minimizer of \( P(u) + \lambda^T Bu \) over \( \mathbb{R}^n \). Therefore \( u_\lambda \) is uniquely determined by the equation

\[ \nabla \left( P(u) + \lambda^T Bu \right) = \nabla P(u) + B^T \lambda = 0. \]

By elementary calculus [5], the above equation defines an implicit function of the variable \( \lambda \), denoted by \( u = h(\lambda) := u_\lambda \) which is continuously dependent of the parameter \( \lambda \).

### 4. Hamiltonian Boundary Value Problem

In this section we solve the following Hamiltonian boundary value problem:

\[ \dot{x} = H_\lambda (x, u, \lambda) = Ax + Bh(\lambda), \quad x(0) = a, \quad (4.1) \]

\[ \dot{\lambda} = -H_\lambda (x, u, \lambda) = -A^T \lambda - \nabla F(x), \quad \lambda(T) = 0. \quad (4.2) \]

Equation (4.2) can be rewritten by the integral form

\[ \lambda(t) = \int_t^T e^{(v-t)} \nabla F(x(v)) \, dv. \quad (4.3) \]

Substituting it into Equation (4.1), we have

\[ \dot{x}(t) = Ax(t) + Bh\left( \int_t^T e^{(v-t)} \nabla F(x(v)) \, dv \right), \quad x(0) = x_0. \quad (4.4) \]

In the following we show that Equation (4.4) has a solution, then together with (4.3) we obtain a solution to Hamiltonian boundary value problem (4.1), (4.2).

Since \( F(x) \) is twice continuously differentiable on \( \mathbb{R}^n \), we may define

\[ K = \max_{1 \to \lambda \in \mathcal{F}} \|\nabla F(x)\|, \]

and

\[ M = \|A\| (\|e\| + 1) + \frac{\|B\|}{\beta} \left( 1 + \|B^T\| \left\| \int_0^T F_K \right\| \right). \]

Let

\[ \mathcal{X} = C\left([0, T], \mathbb{R}^n \right). \]

Consider the ball centered at \( a \) in \( \mathcal{X} \) (regarding \( a \) as a function constantly equal to the vector \( a \)):

\[ \Omega = \left\{ x \in \mathcal{X} : \|x(t) - a\| \leq 1, \forall t \in [0, T]\right\}. \]

For a real number \( \epsilon \) such that \( 0 < \epsilon < \min \left\{ \frac{1}{M}, T \right\} \), define an operator \( G : \Omega \to \mathcal{X} \), which acts on each element \( x \in \Omega \) to produce an image \( Gx \) satisfying (noting that the integral in (4.4) needs the information of \( x(v) \) on the whole interval
(0, T], for \( t \in [0, \epsilon] \),
\[
(G_\lambda(x))(t) := a + \int_0^t \left[ Ax(s) + Bh \left( \int_s^T e^{s-t-s} \nabla F(x(v)) \text{d}v \right) \right] \text{d}s,
\]
while for \( t \in [\epsilon, T] \),
\[
(G_\lambda(x))(t) := a + \int_0^t \left[ Ax(s) + Bh \left( \int_s^T e^{s-t-s} \nabla F(x(v)) \text{d}v \right) \right] \text{d}s.
\]

By an elementary estimation we have, for \( t \in [0, \epsilon] \),
\[
\| (G_\lambda(x)(t) - a) \| \leq \int_0^t \left[ \| A \| \| v(s) \| + \| B \| \left( 1 + \| B^T \| \| e^{T-s} \| \frac{KT}{\beta} \right) \right] \text{d}s \leq M_\epsilon (t - 0) \leq M \epsilon \leq 1, \tag{4.6}
\]
while for \( t \in [\epsilon, T] \),
\[
\| (G_\lambda(x)(t) - a) \| \leq \int_0^t \left[ \| A \| \| v(s) \| + \| B \| \left( 1 + \| B^T \| \| e^{T-s} \| \frac{KT}{\beta} \right) \right] \text{d}s \leq M \epsilon \leq 1,
\]
which implies that \( G_\lambda \in \Omega \). It is also clear that \( G \) is a continuous and compact mapping. Then by Schauder fixed-point theorem, there is an element \( \tilde{x} \in \Omega \) such that \( G\tilde{x} = \tilde{x} \). It follows that \( \tilde{x}(\cdot) \) is a solution to (4.4) for \( t \in [0, \epsilon] \). For \( t \in [0, \epsilon] \), let
\[
\tilde{x}(t) = \int_t^T e^{s-t-s} \nabla F(\tilde{x}(v)) \text{d}v.
\]

By a traditional approach in the classical theory of ordinary differential equations, we see that the solution \( (\tilde{x}(\cdot), \lambda(\cdot)), t \in [0, \epsilon] \) can be extended to \([0, T]\). Then by (4.4), (4.3) we see that \( (\tilde{x}(\cdot), \lambda(\cdot)) \) is a solution to Hamiltonian boundary value problem (4.1), (4.2). We conclude the following result.

**Theorem 4.1.** There exists a solution pair \( (\tilde{x}(\cdot), \lambda(\cdot)) \) to Hamiltonian boundary value problem (4.1), (4.2).

Let \( h(\lambda) := u_\lambda \) and \( (\tilde{x}(\cdot), \lambda(\cdot)) \) be a solution of the Hamiltonian boundary value problem (4.1), (4.2). Then by the definition of the Pontryagin extremal control, we conclude that \( \tilde{u}(t) = h(\tilde{\lambda}(t)), t \in [0, T] \) is an extremal control to the primal problem \((P)\).

**Remark 4.1.** Moreover, noting that \( \nabla^2 F(x) \succeq 0 \left( \forall x \in \mathbb{R}^n \right) \) and \( \nabla^2 P(u) > 0 \left( \forall u \in \mathbb{R}^n \right) \), by (2.1) we see that the Hamiltonian function is convex on the state and control variables respectively. Meanwhile, noting that \( u = h(\lambda) \) does not depend on the state variable, by traditional optimal control theory, we know that the extremal control \( \tilde{u}(t) = h(\tilde{\lambda}(t)) \) is also an optimal control to the optimal control problem \((P)\).

In other words, in the practice for solving \((P)\), we only need to compute a solution of the following differential boundary value problem:

\[
\tilde{x} = Ax + Bh(\lambda), \tag{4.8}
\]
\[
\tilde{\lambda} = -A^T \tilde{\lambda} - \nabla F(x), \tag{4.9}
\]
We present a numerical method to deal with the differential boundary value Equation (4.9), Equation (4.10) as follows. Define a mesh by dividing the time interval $[0, T]$ evenly

$$0 < t_0 < \cdots < t_{N-1} < T,$$

$$t_{i+1} - t_i = \frac{T}{N} := \tau, \quad i = 0, 1, \cdots, N-1.$$

Consider solving for $x_0, x_1, \cdots, x_N$, with $x_i$ the intended approximation of $x(t_i)$. For the requirement on the adjoint variable $\lambda(t_N) = 0$ (due to the boundary condition of the differential boundary value Equation (4.9), Equation (4.10)), we consider the following difference equation:

$$\frac{x_{i+1} - x_i}{\tau} = A x_i + B z_i;$$

$$\frac{\lambda_{i+1} - \lambda_i}{\tau} = -A^T \lambda_i - \nabla F(x_i);$$

$$z_i = h(\lambda_i);$$

$$x_0 = a; \quad \lambda_N = 0;$$

$$i = 0, 1, 2, \cdots, N-1.$$

Solving the difference equation above we can get the value $\lambda_0$. According to classical numerical analysis theory, the solution of above difference equation will converge to the solution of differential boundary value problem (4.8) - (4.10). Apparently, we need to compute $h(\lambda)$ numerically. It will be given in next section.

5. Computing $h(\lambda)$ by a Differential Flow

In this section we study how to compute $h(\lambda) = u_{\lambda}$. For a given parameter vector $\lambda \in \mathbb{R}^n$, we solve the following global optimization problem

$$Q(\lambda) = \min_{u \in \mathbb{R}^n} \left[ P(u) + \lambda^T B u \right]$$

(5.1)

to create a function $u = h(\lambda)$. In the following we will determine the value of $h(\lambda)$ by a differential flow.

Since the Hessian matrix function of $P(u)$ is positive definite, by the classical theory of ordinary differential equation, for given $\lambda \in \mathbb{R}^n$, the following Cauchy initial value problem [3] [6] creates a unique flow $\xi_{\lambda}(\rho), \rho \geq 0$:

$$\frac{d\xi}{d\rho} + \left[ \nabla^2 P(\xi) + \rho I \right]^{-1} \xi = 0, \quad \xi(0) = u_{\lambda}, \quad \rho \in [0, \infty),$$

(5.2)

such that

$$\nabla P(\xi_{\lambda}(\rho)) + B^T \lambda + \rho \xi_{\lambda}(\rho) = 0,$$

(5.3)

noting that $\nabla P(u_{\lambda}) + B^T \lambda = 0$ since $u_{\lambda}$ is the minimizer of $P(u) + \lambda^T B u$ over $\mathbb{R}^n$ (Lemma 3.2). To explain the uniqueness of $\xi_{\lambda}(\rho)$, we refer to the fact that
\[ \nabla^2 P(u) + \rho I > 0, \forall u \in \mathbb{R}^n \text{ for } \rho \in [0, \infty). \] Thus, combining (5.3), \( \xi_\lambda (\rho) \) is the unique solution of the equation \( \nabla P(u) + B^T \lambda + \rho u = 0. \)

In what follows we choose a real number \( \rho^* > 1 \) such that, on \( \| u \| \leq 1, \)
\[ \frac{\| -\nabla P(u) - B^T \lambda \|}{\rho^*} \leq 1. \]

By Brouwer Fixed-Point theorem ([7]), there is a point \( u^* \in \{ \| u \| \leq 1 \}, \) such that
\[ \frac{-\nabla P(u^*) - B^T \lambda}{\rho^*} = u^*. \] (5.4)

Moreover, we have
\[ \| u^* \| \leq C \rho^{-2}. \] (5.5)
where the positive constant \( C \) is only dependent of the parameters \( \lambda. \) In the following there are several times of appearing the character \( C \) which may denote different positive constants only dependent of the parameters \( \lambda. \)

It follows from (5.4) that
\[ \nabla P(u^*) + B^T \lambda + \rho^* u^* = 0. \] (5.6)

By (5.3) and the uniqueness of the flow \( \xi_\lambda (\rho), \rho \geq 0, \) we see that
\[ \xi_\lambda (\rho^*) = u^*, \] (5.7)
and that the flow \( \xi_\lambda (\rho), \rho \geq 0 \) can also be got by the following Cauchy initial value problem
\[ \frac{d\xi}{d\rho} + \left[ \nabla^2 P(\xi) + \rho I \right]^{-1} \xi = 0, \quad \xi(\rho^*) = u^*, \quad \rho \in [0, \infty). \] (5.8)

Certainly, \( \xi_\lambda (0) = u_\lambda. \) Although it is hard to get \( u_\lambda \) and \( u^* \) exactly, we can compute numerically another vector instead of \( u^* \) by the following result.

**Theorem 5.1.** Let the flow \( \overline{x}(\rho) \) be got by the following backward differential equation
\[ \frac{d\xi}{d\rho} + \left[ \nabla^2 P(\xi) + \rho I \right]^{-1} \xi = 0, \] (5.9)
\[ \xi (\rho^*) = \sigma = -\left( \nabla^2 P(0) + \rho^* I \right)^{-1} (\nabla P(0) + B^T \lambda), \quad \rho \in [0, \infty). \] (5.10)

Then
\[ \| u^* - \overline{u} \| \leq C \rho^{-1}, \] (5.11)
\[ \| \nabla P(\overline{x}(0)) + B^T \lambda \| \leq C \rho^{-2}, \quad \overline{x}(0) - u_\lambda \| \leq C \rho^{-1}, \] (5.12)
where the positive constant \( C \) is only dependent of the parameters \( \lambda \) and \( \rho^* \) is selected to be sufficiently large satisfying (5.4), (5.5).

**Proof:** When \( \rho^* > 1 \) is sufficiently large, \( u^* \) is near the origin. In a neighborhood of the origin, by (5.4), we have
\[ \nabla P(0) + \nabla^2 P(0) u^* + O\left(\|u^*\|^2\right) = \nabla P\left(u^*\right) = -\rho^* u^* - B^T \lambda, \]

and
\[ \left(\nabla^2 P(0) + \rho^* I\right) u^* = -\nabla P(0) - B^T \lambda + O\left(\|u^*\|^2\right). \]

Noting \( \nabla^2 P(0) > 0 \), consequently, we have
\[ u^* = -\left(\nabla^2 P(0) + \rho^* I\right)^{-1} \left(\nabla P(0) + B^T \lambda\right) + O\left(\|u^*\|^2\right). \]

Let
\[ \pi := -\left(\nabla^2 P(0) + \rho^* I\right)^{-1} \left(\nabla P(0) + B^T \lambda\right), \]
we have
\[ u^* - \pi = O\left(\|u^*\|^2\right). \]

Thus, by (5.5),
\[ \|u^* - \pi\| \leq C \rho^* -2, \quad (5.13) \]
where the positive constant \( C \) is only dependent of the parameters \( \lambda \). By the way, we deduce that \( \|P\| \leq \|u^*\| + \|u^* - \pi\| \leq 1 + C \), noting that \( \rho^* > 1 \).

In the following we need to keep in mind that
\[ \nabla P\left(u^*\right) + \rho^* u^* + B^T \lambda = 0. \quad (5.14) \]

By (5.13), (5.14), for sufficiently large \( \rho^* \), we have
\[
\begin{align*}
\|\nabla P(\pi) + \rho^* \pi + B^T \lambda\| & = \|\nabla P(\pi) + \rho^* \pi - \left(\nabla P\left(u^*\right) + \rho^* u^*\right)\| \\
& \leq \rho^* \|u^* - \pi\| + \left(\max_{\|u\| \leq C}\|\nabla^2 P(u)\|\right) \|u^* - \pi\| \\
& \leq \rho^* C \rho^* -2 + C \|u^* - \pi\| \\
& \leq C \rho^* -1 + C \rho^* -2 \leq C \rho^* -1, \quad (5.15)
\end{align*}
\]
noting that in the inequality process the value of the constant \( C \) has been changed several times but only dependent of given information like \( \lambda, P, B \).

Since \( \nabla P\left(\bar{\xi} (\rho)\right) + \rho \bar{\xi} (\rho) \) is a constant along the flow \( \bar{\xi} (\rho) \), noting that (5.9) (5.10) we have
\[ \nabla P\left(\bar{\xi} (\rho)\right) + \rho \bar{\xi} (\rho) = \nabla P(\bar{\pi}) + \rho^* \bar{\pi}. \]

Consequently for \( \rho = 0 \) we have
\[ \nabla P\left(\bar{\xi} (0)\right) = \nabla P(\bar{\pi}) + \rho^* \bar{\pi}, \]

Thus, by (5.15),
\[ \|\nabla P\left(\bar{\xi} (0)\right) + B^T \lambda\| < C \rho^* -1. \quad (5.16) \]

Further, noting that \( \nabla P\left(u_0\right) + B^T \lambda = 0 \), we have
\[ \| \bar{z}(0) - u_\lambda \| \leq C \| \nabla P(\bar{z}(0)) - \nabla P(u_\lambda) \| \]
\[ = C \| \nabla P(\bar{z}(0)) + B^T \lambda - \nabla P(u_\lambda) - B^T \lambda \| \]
\[ = C \| \nabla P(\bar{z}(0)) + B^T \lambda \| \leq C \rho^{-1}. \]

noting that \( \nabla P(u_\lambda) + B^T \lambda = 0 \), and also noting that in the inequality process the value of the constant \( C \) takes different positive values which are only dependent of given information like \( \lambda, P, B \). The theorem has been proved.

**Remark 5.1.** Comparing with \( \nabla P(u_\lambda) + B^T \lambda = 0 \), in the computation practice, we can solve the Cauchy initial problem (5.9) (5.10), instead of (5.8) to get \( \bar{z}(0) \) as an approximation of \( u_\lambda \).

In what follows, we give an algorithm to compute \( h(\lambda) = u_\lambda \) numerically in finding a discrete solution to Hamiltonian boundary value problem (4.1), (4.2).

**Algorithm 5.1.**

1) Given \( \lambda \in \mathbb{R}^n \);
2) Given \( \epsilon > 0, \rho^* > 1 \);
3) Get the flow \( \xi(\rho) \) by solving the following Cauchy initial problem
\begin{align*}
\frac{d\xi}{d\rho} + \left[ \nabla^2 P(\xi) + \rho I \right]^{-1} \xi = 0, \quad \xi(\rho^*) = -\left( \nabla^2 P(0) + \rho^* I \right)^{-1} \left( \nabla P(0) + B^T \lambda \right), \quad \rho \in [0, \rho^*];
\end{align*}

4) if \( \| \nabla P(\bar{z}(0)) + B^T \lambda \| \leq \epsilon, h(\lambda) = u_\lambda \approx \bar{z}(0), \) stop; Otherwise, go to 5);
5) \( \rho^* = 10\rho^* \), go to 3).

**Remark 5.2.** For the step 3) of above algorithm, we present a numerical method to deal with the Cauchy initial problem as follows. Define a mesh by dividing the time interval \([0,T]\) evenly
\[ 0 < \rho_0 < \cdots < \rho_{M-1} < \rho_M = \rho^*, \]
\[ \rho_{j+1} - \rho_j = \frac{\rho^*}{M}, \quad j = 0, 1, \cdots, M - 1. \]

Consider solving for \( \xi_0, \xi_1, \cdots, \xi_M \), with \( \xi_j \) the intended approximation of \( \xi(\rho_j) \). We deal with the following difference equation.
\begin{align*}
\frac{\xi_j - \xi_{j-1}}{s} &= -\left[ \nabla^2 P(\xi_{j-1}) + \rho_{j-1} I \right]^{-1} \xi_{j-1}, \quad j = 1, 2, \cdots, M, \\
\xi_M &= -\left( \nabla^2 P(0) + \rho^* I \right)^{-1} \left( \nabla P(0) + B^T \lambda \right). 
\end{align*}

6. A Description of an Example

Let’s consider to solve the following optimal control problem numerically:
\[ \int_0^t (x^2 + u^4 + u^2) dt \]
s.t. \( \dot{x} = x + u, \ x(0) = a, \)
where state and control variables all take values in \( \mathbb{R}^1 \). Let \( P(u) = u^4 + u^2 \). We have \( P'(u) = 4u^3 + 2u \), \( P''(u) = 12u^2 + 2 \), \( H(x, u, \lambda) = \lambda (x + u) + x^2 + u^4 + u^2 \). We have the following Hamiltonian boundary value problem and a global optimization problem:
\[ \dot{x} = x + u, \quad x(0) = a; \]
\[ \dot{\lambda} = -\left( \lambda + 2x \right), \quad \lambda(1) = 0, \quad t \in [0,1]. \]
\[ u(t) = \arg \left\{ \min_{u \in U} \left[ u^2 + \lambda(t)u \right] \right\}, \quad \text{a.e.} \ t \in [0,1]. \]

We need to solve the following differential boundary value equation:
\[ \dot{x} = x + h(\lambda), \quad x(0) = a; \]
\[ \dot{\lambda} = -\left( \lambda + 2x \right), \quad \lambda(1) = 0, \]
which yields the corresponding difference equation:
\[ \frac{x_i - x_{i+1}}{\tau} = x_{i+1} + h(\lambda_{i+1}), \]
\[ \frac{\lambda_i - \lambda_{i+1}}{\tau} = -\lambda_{i+1} - 2x_{i+1}, \quad i = 1,2,\ldots,N, \]
\[ x_0 = a, \quad \lambda_N = 0. \]

Noting that \( P''(u) + \rho = 12u^2 + 2 + \rho, \rho \geq 0, \quad \text{and} \quad P'(0) = 0 \), by Algorithm 5.1 and Remark 5.2, given positive integers \( N, M \) (properly large) and positive real numbers \( \epsilon \) (properly small), \( \rho^* > 1 \), we consider solving the following difference equation:
\[ \frac{x_{i+1} - x_i}{\tau} = x_i + z_i; \]
\[ \frac{\lambda_{i+1} - \lambda_i}{\tau} = -\lambda_i - 2x_i; \]
\[ \xi_j(i) - \xi_{j+1}(i) \frac{s}{S} = -\left( 12\left( \xi_{j+1}(i) \right)^2 + 2(j-1)s \right)^{-1} \xi_{j+1}(i), \]
\[ \xi_{iM} = -\left( 2 + \rho^* \right)^{-1} \lambda_i; \]
\[ z_i = \xi_0(i); \]
\[ x_0 = a, \quad \lambda_N = 0; \]
\[ i = 0,1,2,\ldots,N-1, \quad j = 1,2,\ldots,M. \]

If \( |\lambda_0| \leq \epsilon \), then the discrete solution of an optimal control \( \hat{u}_i \approx z_i, i = 0,1,2,\ldots,N-1 \). Otherwise, let \( \rho^* = 10\rho^* \) and \( N = 2N, M = 2M \) and do the above difference equation again.

References


Generalized Uncertainty Relations, Curved Phase-Spaces and Quantum Gravity

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Abstract

Modifications of the Weyl-Heisenberg algebra $[\hat{x}, \hat{p}] = i\hbar g^{ij}(p)$ are proposed where the classical limit $g_{ij}(p)$ corresponds to a metric in (curved) momentum spaces. In the simplest scenario, the $2D$ de Sitter metric of constant curvature in momentum space furnishes a hierarchy of modified uncertainty relations leading to a minimum value for the position uncertainty $\Delta x$. The first uncertainty relation of this hierarchy has the same functional form as the stringy modified uncertainty relation with a Planck scale minimum value for $\Delta x = L_p$ at $\Delta p = p_{\text{Planck}}$. We proceed with a discussion of the most general curved phase space scenario (cotangent bundle of spacetime) and provide the noncommuting phase space coordinates algebra in terms of the symmetric $g_{\mu\nu}$ and nonsymmetric $g^{[\mu\nu]}$ metric components of a Hermitian complex metric $g_{\mu\nu} = g_{\mu\nu} + ig^{[\mu\nu]}$, such $g_{\mu\nu} = (g_{\nu\mu})^\dagger$. Yang’s noncommuting phase-space coordinates algebra, combined with the Schrödinger-Robertson inequalities involving angular momentum eigenstates, reveals how a quantized area operator in units of $L_p$ emerges like it occurs in Loop Quantum Gravity (LQG). Some final comments are made about Fedosov deformation quantization, Noncommutative and Nonassociative gravity.

Keywords

Uncertainty Relations, Gravity, Finsler Geometry, Born Reciprocity, Phase Space

Generalized Uncertainty Relations

Recently, we studied the generalized gravitational field equations in curved phase spaces (the cotangent bundle of spacetime) [1]. A nontrivial solution generalizing the Hil-
The most relevant physical consequence is that the metric becomes momentum-dependent (observer dependent) which is what one should aim for in trying to quantize geometry (gravity): The observer must play an important role in any measurement (observation) process of the spacetime he/she lives in.

Most of the work devoted to Quantum Gravity has been focused on the geometry of spacetime rather than phase space per se. The first indication that phase space should play a role in Quantum Gravity was raised by [2]. The principle of Born’s reciprocal relativity [2] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature given by the speed of light, there should be a maximal force as well, since force is the temporal derivative of the momentum. An upper bound on the force does not imply that there is a maximum momentum. Likewise, in ordinary Special Relativity, an upper bound (speed of light) on the velocity does not imply that there is a maximum length. To sum up, a direct consequence of the Born reciprocity principle is that a maximal speed limit (speed of light) should be accompanied with a maximal proper force.

It is better understood now that the Planck-scale modifications of the particle dispersion relations can be encoded in the nontrivial geometrical properties of momentum space [3]. When both spacetime curvature and Planck-scale deformations of momentum space are present, it is expected that the nontrivial geometry of momentum space and spacetime get intertwined. The interplay between spacetime curvature and nontrivial momentum space effects was essential in the notion of “relative locality” and in the deepening of the relativity principle [3]. Recently the authors [4] described the Hamilton geometry of the phase space of particles whose motion is characterized by general dispersion relations. Explicit examples of two models for Planck-scale modified dispersion relations, inspired from the \(q\)-de Sitter and \(\kappa\)-Poincare quantum groups, were considered. In the first case they found the expressions for the momentum and position dependent curvature of spacetime and momentum space, while for the second case the manifold is flat and only the momentum space possesses a nonzero, momentum dependent curvature.

We shall focus in this work on two main points. Firstly, on solutions to the field equations in momentum space with the inclusion of the momentum analog of a cosmologically constant \(\Lambda\).

\[
R_{\mu\nu}(p) - \frac{1}{2} g_{\mu\nu}(p) R + \Lambda g_{\mu\nu}(p) = 0, \quad \mu, \nu = 1, 2, \ldots, D
\]

the solutions to the above field equations will be used in the modified uncertainty relations. The momentum-space analog \(\Lambda\) of the cosmological constant should not be confused with the spacetime one.

Secondly, on the rotationally invariant commutator of the form [5]

\[
\left[ \mathbf{S}_i, \mathbf{p}_j \right] = \hbar \left( f(p^2) \delta_{ij} + g(p^2) \mathbf{p}_i \mathbf{p}_j \right), \quad p^2 = \mathbf{p} \cdot \mathbf{p}', \quad i, j = 1, 2, \ldots, D - 1
\]
one can see that under rotations
\[ x_i \rightarrow x'_i = M^i_j x_j, \ p_j \rightarrow p'_j = M^i_j p_i, \ p_i \rightarrow p'_i = M^i_j p_j \] (3)
the left and right hand side of Equation (2) become
\[ \left[ x'_i, p'_j \right] = i\hbar \left( f \left( p^2 \right) \delta_{ij} + g \left( p^2 \right) p'_i p'_j \right), \ p^2 = p_i p^i = p'_i p'_i = p''_i p''_i \] (4)
and the commutator relations remain invariant. Consequently, if one is to set
\[ \left[ x_i, p_j \right] = i\hbar g_{ij} \left( p_k \right), \] a rotationally invariant commutator can be associated to a classical momentum space metric of the form
\[ (ds)^2 = -h\left( p^2 \right) (dE)^2 + \left( f \left( p^2 \right) \delta_{ij} + g \left( p^2 \right) p_i p_j \right) dp^i dp^j \] (5)
A close inspection reveals that the 4D momentum-space metric analog of the de Sitter metric in a 4D spacetime (written in static coordinates and using the momentum-space analog of the cosmological constant \( \Lambda \))
\[ (ds)^2 = -\left( 1 - \frac{\Lambda}{3} p^2 \right) (dE)^2 + \left( \frac{\Lambda}{1 - \frac{3}{3} p^2} \right) p^2 (d\Omega)^2 \] (6)
does not have the required form indicated by Equation (5). To verify this one simply rewrites the de Sitter metric in Cartesian coordinates. One then finds that the rotationally invariant commutation relations, leading to the metrics (5), are not compatible with a spherically symmetric momentum space de Sitter metric (6).

One may insert the metric (5) into the field equations in momentum space in order to determine whether or not there exist actual functions \( f \left( p^2 \right), g \left( p^2 \right), h \left( p^2 \right) \) which solve the field Equation (1). However, for our purposes it is not necessary to do so, and it is much simpler just to write down the momentum space analog of the de Sitter metric in 2D in natural units \( h = c = 1 \)
\[ (ds)^2 = -\left( 1 - L^2 p^2 \right) (dE)^2 + \left( 1 - L^2 p^2 \right)^{-1} \left( dp \right)^2, \ h = c = 1 \] (7)
which is trivially rotational invariant. \( \Lambda = 0 \) in 2D. There is a cosmological horizon in momentum space when \( L^2 p^2 = 1 \Rightarrow p = \frac{1}{L} \). We shall choose the length scale \( L = L_p \) to coincide with the Planck length \( L_p \) so that the momentum horizon \( p = p_{Planck} \) is the Planck momentum.

Inspired by the 2D de Sitter momentum space metric (7), and by promoting the classical momentum variable \( p \) to an operator \( \hat{p} \), such that \( g_{11} \left( p^2 \right) \rightarrow g_{11} \left( \hat{p}^2 \right) \), the Schrodinger-Robertson inequality yields the modified uncertainty relations after performing a series expansion (with \( h = c = 1 \))
\[ \Delta x \Delta p \geq \frac{1}{2} \left| \langle \Psi | [x, p] | \Psi \rangle \right| \]
\[ = \frac{1}{2} \left| \langle \Psi | g_{11} \left( \hat{p}^2 \right) | \Psi \rangle \right| = \frac{1}{2} \left| \langle \Psi | \frac{1}{1 - L^2 \hat{p}^2} | \Psi \rangle \right| \]
\[ = \frac{1}{2} \left| \langle \Psi | 1 + L^2 \hat{p}^2 + \left( L^2 \hat{p}^2 \right)^2 + \left( L^2 \hat{p}^2 \right)^3 + \cdots | \Psi \rangle \right| \] (8)
One may notice that since \( \left\langle \Psi \left| L^{2n} \mathbf{p}^{2n} \right| \Psi \right\rangle \geq 0 \) for \( n \geq 0 \), given a self-adjoint (Hermitian) momentum operator \( \mathbf{p} \), one may drop the absolute value symbol in last terms of Equation (8). The (geometric) series expansion in \( \mathbf{p}^2 \) converges as an operator if

\[
L^2 \mathbf{p}^2 < 1 \Rightarrow \left\langle \Psi \left| \mathbf{p}^2 \right| \Psi \right\rangle < \frac{1}{L^2} = p_h^2
\]  

consistent with the cosmological momentum-horizon \( p_h \) being an ultraviolet cutoff value for the momentum. The unit operator is \( \mathbf{1} \) and the states are normalized to unity \( \left\langle \Psi \right| \Psi = 1 \) so that \( \left\langle \Psi \left| \mathbf{1} \right| \Psi \right\rangle = 1 \).

Inserting the inequality of the equation below

\[
(\Delta \mathbf{p})^2 = \left\langle \Psi \left| \mathbf{p}^2 \right| \Psi \right\rangle - \left\langle \Psi \left| \mathbf{p} \right| \Psi \right\rangle^2 \Rightarrow \left\langle \Psi \left| \mathbf{p}^2 \right| \Psi \right\rangle \geq (\Delta \mathbf{p})^2
\]  

into Equation (8), yields to leading order in \( L^2 \), a modified uncertainty relation

\[
\Delta x \Delta \mathbf{p} > \frac{1}{2} \left\langle \Psi \left| \left( 1 + L^2 \mathbf{p}^2 \right) \right| \Psi \right\rangle \geq \frac{1}{2} \left( 1 + (L^2 \Delta \mathbf{p})^2 \right)
\]  

\[
\Rightarrow \Delta x > \frac{1}{2} \left( \frac{1}{\Delta \mathbf{p}} + L^2 \Delta \mathbf{p} \right), \quad \hbar = c = 1
\]  

which has the same functional form as the stringy modified uncertainty relations [6], with the main difference being that now one has the cosmological momentum-horizon \( p_h = \frac{1}{L} \) as an ultraviolet cutoff for \( \Delta \mathbf{p} \), and there is an strict inequality in Equation (11).

The minimum value for the position uncertainty is \( (\Delta x)_\text{min} = L \) at \( \Delta \mathbf{p} = \frac{1}{L} = p_h \) and which coincides with the location of the cosmological momentum horizon. If one equates the minimum value of the position uncertainty to the Planck scale length it gives \( (\Delta x)_\text{min} = L = L_p \), and which is consistent with the fact that we chose the length scale \( L \) to coincide with the Planck length \( L_p \).

To sum up, to leading order in \( L \), the de Sitter momentum space metric in 2D furnishes: 1) a cosmological momentum-horizon \( p_h = \frac{1}{L_p} = p_{\text{Planck}} \) as an ultraviolet cutoff; 2) a Planck scale minimal length uncertainty for the position coordinate \( (\Delta x)_\text{min} = L_p \) at \( \Delta \mathbf{p} = p_{\text{Planck}} \).

The next-to-leading order term can be obtained after using the inequality

\[
\left\langle \Psi \left| \mathbf{p}^4 \right| \Psi \right\rangle \geq (\Delta \mathbf{p})^4
\]  

that simply follows from

\[
(\Delta A)^2 = \left\langle \Psi \left| \left( A - \bar{A} \right)^2 \right| \Psi \right\rangle \geq 0, \quad \bar{A} = \left\langle \Psi \left| A \right| \Psi \right\rangle
\]  

after replacing \( A \rightarrow \mathbf{p}^2 \) and recurring to \( \left\langle \Psi \left| \mathbf{p}^2 \right| \Psi \right\rangle \geq (\Delta \mathbf{p})^2 \). Upon doing so one obtains another modified uncertainty relation given by

\[
\Delta x \Delta \mathbf{p} > \frac{1}{2} \left( 1 + L_p^4 (\Delta \mathbf{p})^2 + L_p^4 (\Delta \mathbf{p})^4 \right)
\]
The minimum position uncertainty now turns out to be \((\Delta x)_{\text{min}} > L_p\) at

\[
(\Delta p)_x = \left(\frac{-1 + \sqrt{13}}{6}\right)^{\frac{1}{2}} p_{\text{Planck}} < p_{\text{Planck}}
\] (12d)

The value of \((\Delta p)_x\) lies between \(\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right) p_{\text{Planck}}\). Repeating the procedure based on Equation (12a), Equation (12b), by a process of successive squaring, a hierarchy of modified uncertainty relations of the form are derived

\[
\Delta x\Delta p > \frac{1}{2} \left(1 + L_p^2 (\Delta p)^2 + L_p^4 (\Delta p)^4 + \cdots + L_p^{2k} (\Delta p)^{2k}\right), \quad k = 1, 2, \ldots
\] (12e)

The most salient feature of the modified uncertainty relations (11), (12c), (12d) is that there is a minimum value for the position uncertainty \(\Delta x\). The Scale Relativity Theory [7] is based on the postulate that the Planck scale is the minimum length resolution. Generalized uncertainty relations in spacetime were derived from the Extended Relativity Theory in Clifford spaces (C-spaces) in [8]. Such Clifford space Extended Relativity Theory has two universal parameters: the speed of light and the Planck length.

In general one can postulate the following modification of the Weyl-Heisenberg algebra

\[
\left[ x_i, p_j \right] = i\hbar g_{ij}(p)
\] (13)

combined with the additional commutation relations

\[
\left[ x_i, x_j \right] \neq 0, \quad \left[ p_i, p_j \right] = 0
\] (14)

with the provision that the above commutators obey the Jacobi identities [5]. A nonvanishing \(\left[ x_i, x_j \right] \neq 0\) is compatible with a curved momentum space. The de Sitter momentum space metric yields a constant scalar curvature \(2/L_p^2 = 2L_p^2\) in momentum space. The vanishing \(\left[ p_i, p_j \right] = 0\) commutator is consistent with a flat spacetime. A nontrivial problem is to find the most general solutions to the field equations in momentum space (with and without the \(\Lambda\)) for the metric which has the form of Equation (5), in order to yield rotationally symmetric commutators \(\left[ x_i, p_j \right]\), after promoting the classical momentum variables \(p_i, p_j\) to self-adjoint operators \(p_i, p_j\). When the momentum space is still commutative, one can find a Hilbert space representation in the spectral representation of the momentum operator [5]. The states \(\Psi(p)\) were analyzed in detail by [5]. Upon performing the expectation values \(\left\langle \Psi(p) \right| g_{ij}(p) \left| \Psi(p) \right\rangle = g_{ij}(p)\) one recovers the classical metric in momentum space. These momentum eigenstates have for momentum uncertainty \(\Delta p = 0\) so that \(\Delta x = \infty\), as expected.

The more general commutator than the one in Equation (13)

\[
\left[ x_i, p_j \right] = i\hbar g_{ij}(x, p)
\] (15)

may be chosen such that the classical limit \(g_{ij}(x, p) \rightarrow g_{ij}(x, p)\) (involving \(c\) variables) furnishes a classical phase space metric obeying the full-fledged gravitational field
equations in curved phase spaces (cotangent bundle of spacetime). This poses more difficulties due to the ordering ambiguities of the \(x, p\) operators inside \(g_{ij}(x, p)\). One can solve this ordering ambiguity by performing a Weyl ordering procedure, like 
\[
x p \rightarrow \frac{1}{2}(x p + p x)
\]
to ensure that the latter ordering is Hermitian, since the product \(x p\) is not.

An important remark is in order. By Hermitian metric one usually means \(g_{ij} = g_{ji}\). This should not be confused with performing the Hermitian (adjoint) operation to each one of the entries inside the metric matrix \(g_{ij}(x, p)\). If the entries of the metric matrix are given by polynomials in the operators \(x, p\), the Weyl ordering procedure will ensure that each one of the entries of the metric matrix will remain Hermitian. For example if \(g_{ij}(x, p) = g_{ij}(x p)\), a Weyl ordering yields 
\[
g_{ij}\left(\frac{1}{2}(x p + p x)\right)
\]
ensuring that the argument of the metric matrix is Hermitian. Similarly, by anti-Hermitian metric one usually means \(g_{ij} = -g_{ji}\).

Since the commutator of two Hermitian operators in anti-Hermitian, one may postulate the following commutators below (in a fully relativistic phase space) given in terms of of a real metric which has both symmetric \(g_{\mu \nu}\) and anti-symmetric components \(g_{[\mu \nu]}\) as follows (\(h = c = 1\))
\[
[x_\mu, x_\nu] = i \left(A g_{[\mu \nu]}(x, p) + B g_{[\mu \nu]}(x, p)\right) l
\]

\[A, B\] are real numerical coefficients.

\[
[x_\mu, x_\nu] = i L^2 g_{[\mu \nu]}(x, p) l
\]

\[
[p_\mu, p_\nu] = i (R^2) g_{[\mu \nu]}(x, p) l
\]

the right hand sides are anti-Hermitian due to \((il)^\dagger = -il\). A Taylor expansion of the metric components in powers of \(x, p\) must be followed by a Weyl ordering of all the \(x, p\) variables to ensure Hermiticity of the arguments of the metric. An UV (ultra-violet) cutoff is given by the Planck scale \(L_P\); an IR (infrared) cutoff is given by \(R_H\) (Hubble radius). The Jacobi identities will impose very strong constraints on the functional form of \(g_{\mu \nu}\) and \(g_{[\mu \nu]}\). A complex Hermitian metric can be introduced by writing \(g_{\mu \nu} = g_{(\mu \nu)} + ig_{[\mu \nu]}\) such that \(g_{\mu \nu} = g_{\nu \mu}\). This raises the possibility that complex Hermitian metrics might be relevant in Quantum Gravity.

It is at this point where the following Schrodinger-Robertson inequalities for \(2n\) observables \(A_1, A_2, \ldots, A_{2n}\) will play an important role. They are given by the inequality of the determinants below involving the covariance \(\Sigma\) and commutator \(C\) matrices

\[
\det \Sigma \geq \det C, \quad \Sigma_{ij} = \text{cov}(A_i, A_j), \quad C_{ij} = -\frac{i}{2} \langle \Psi | [A_i, A_j] | \Psi \rangle
\]

the covariance is defined as

\[
\text{cov}(A_i, A_j) = \frac{1}{2} \left( \langle \Psi | A_i A_j | \Psi \rangle + \langle \Psi | A_j A_i | \Psi \rangle \right) - \langle \Psi | A_i | \Psi \rangle \langle \Psi | A_j | \Psi \rangle
\]
uncorrelated variables have zero covariance. The uncertainty squared is 
\[
(\Delta A)^2 = \text{cov}(A, A).
\]

For the \(2n\) phase space coordinates, the \(\Sigma\) and \(C\) matrices are respectively given by
\[
\Sigma = \begin{pmatrix}
\text{cov}(x_\mu, x_\nu) & \text{cov}(x_\mu, p_\nu) \\
\text{cov}(p_\mu, x_\nu) & \text{cov}(p_\mu, p_\nu)
\end{pmatrix}
\]
\[
C = -\frac{i}{2} \begin{pmatrix}
\langle \Psi | [x_\mu, x_\nu] | \Psi' \rangle & \langle \Psi | [x_\mu, p_\nu] | \Psi' \rangle \\
\langle \Psi | [p_\mu, x_\nu] | \Psi' \rangle & \langle \Psi | [p_\mu, p_\nu] | \Psi' \rangle
\end{pmatrix}
\]

Due to the nontrivial commutation relations (16)-(18), the Schrodinger-Robertson inequalities \(\det \Sigma \geq \det C\) will lead to very complicated uncertainty relations. Furthermore, because the phase space coordinates are noncommutative [10], one must deal now with Noncommutative Quantum Mechanics; i.e. Quantum Mechanics on Noncommutative spacetimes which is the realm of Hopf algebras and Quantum Groups.

Closely related to the nontrivial commutation relations (16)-(18) is Yang’s algebra in an 8D Noncommutative phase space [11]
\[
\left[ x_\mu, p_\nu \right] = -\frac{i}{R_{lt}} \eta_{\mu \nu} J_{56}, \quad J_{56} = -J_{65}
\]
\[
\left[ x_\mu, x_\nu \right] = -iL_{p}^2 \eta_{\mu \nu} J_{56}, \quad J_{56} = -J_{65}
\]
\[
\left[ p_\mu, p_\nu \right] = -i \left( R_{lt} \right)^{-2} \eta_{\mu \nu} J_{56}
\]
\[
\left[ J_{56}, x_\mu \right] = i \left( \eta_{\mu \nu} x_\nu - \eta_{\nu \mu} x_\nu \right)
\]
\[
\left[ J_{56}, p_\mu \right] = i \left( \eta_{\mu \nu} p_\nu - \eta_{\nu \mu} p_\nu \right)
\]
\[
\left[ J_{56}, J_{56} \right] = \eta_{\mu \nu} J_{56} \pm \ldots
\]

Yang’s algebra can be obtained simply by replacing
\[
x_\mu \rightarrow L_p J_{56}; \quad p_\mu \rightarrow (R_{lt})^{-1} J_{56}, \quad \mu, \nu = 1, 2, 3, 4
\]
and recurring to the angular momentum algebra in 6D. The Jacobi identities are satisfied because the angular momentum algebra in 6D obeys them. The noncommuting coordinates and momenta are just rotations/boosts involving the extra directions. \(\eta_{55}, \eta_{66}\) may be chosen to be \(\pm 1\), depending on the signature of the extra two dimensions. \(J_{56}\) is an exchange operator which exchanges \(x \leftrightarrow p\) in Equation (28). When \(L_p \rightarrow 0\) and/or \(R_{lt} \rightarrow \infty\), \(\left[ x_\mu, p_\nu \right] \rightarrow 0\). Thus the classical commuting 8D phase space is recovered when \(L_p \rightarrow 0\) and \(R_{lt} \rightarrow \infty\).

One may notice that Yang’s algebra and the algebra of Equations (16)-(18) bears a certain resemblance if one were to set the numerical coefficient \(B\) to zero;
\[
Ag_{(\mu \nu)} \leftrightarrow \left( L_p / R_{lt} \right) \eta_{\mu \nu} J_{56}, \quad \text{and} \quad g_{(\mu \nu)} \leftrightarrow J_{\mu \nu} \]. The Schrodinger-Robertson inequalities \(\det \Sigma \geq \det C\) could be applied directly to the Yang’s algebra commutators by taking
the expectation values with respect to angular momentum eigenstates. If one were to interpret \( i\left[ x_{\mu} , x_{\nu} \right] \sim J_{\mu \nu} \) as a bivector \( x_{\mu \nu} \) Hermitian operator, and which in turn can be seen as a geometric area operator \( A_{\mu \nu} \), then the norm of the spatial area operator would be

\[
\left( \langle J | A_{\mu} A_{\mu} | J \rangle \right)^{\frac{1}{2}} = L_{P}^{2} \left( \langle J | J | J \rangle \right)^{\frac{1}{2}} = L_{P}^{2} \sqrt{J(J+1)} \quad (31)
\]

which bears a similarity to the results associated to the area operator obtained in Loop Quantum Gravity (LQG) and based on spin networks. The Planck area is the quantum of minimal area [12]. This deserves further investigation. Modified uncertainty relations also apply to the energy and time variables \( \Delta E, \Delta t \) as well. The granularity of spacetime has been interpreted from the principle of Born reciprocity by [9].

Symplectic geometry is the realm of phase spaces [13] where the symplectic form \( \omega_{\mu \nu} \) plays an essential role. Fedosov deformation quantization [14] and the generalized star products in curved phase spaces are tailor made for these generalized gravitational theories in curved phase spaces (cotangent bundle). The geometry of the cotangent bundle \( T^{*} M \) of spacetime has been rigorously studied by [15], among others. In particular, deformation quantization in Fedosov-Finsler spaces has been analyzed extensively by [16].

To conclude, we may add that non-geometric fluxes in string theory give rise to noncommutative/nonassociative structures. More recently, the differential geometry on the simplest nonassociative (phase) space arising for a constant non-geometric \( R \)-flux has been analyzed in [17]. This nonassociativity for a constant \( R \)-flux background in closed strings is captured by the commutation relations

\[
\left[ x^{\mu} , x^{\nu} \right] = R^{\mu \nu \rho \sigma} p_{\rho} \left[ x^{\sigma} , p_{\nu} \right] = i\hbar \delta_{\mu \nu} .
\]

These studies paves the road towards a Noncommutative and Nonassociative gravity which might be a key feature in the final theory of Quantum Gravity.

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**References**


High Quality of Piezoelectric Response of Nano Ni Doped Pb_{1-x}Ni_{x}TiO_{3} Ceramic

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Abstract

A series of Pb_{1-x}Ni_{x}TiO_{3} (x = 0, 0.1, 0.2, 0.3, 0.4 and 0.5) were prepared by the general ceramic and co-precipitation method. The grain size decreased with increasing Ni contents. The piezoelectric modulus d_{33} increased with increasing Ni contents. It was observed also the d_{33} of the nanocomposite Pb_{1-x}Ni_{x}TiO_{3} prepared by co-precipitation method is higher than those of the prepared by general ceramic method. The coupling factor k_{p} increased by the more doping of Ni. The ultrasonic wave velocity decreased also with increasing Ni contents. The polarized nano-sample of composition Pb_{0.5}Ni_{0.5}TiO_{3} is suitable for improving application of useful piezoelectric technology.

Keywords

Piezoelectric Ceramics, d_{33} Coefficient, Electromechanical Coupling Factor K_{p}

1. Introduction

Lead titanate PT ceramics modified by rare earth elements and alkaline earth elements have emerged as highly promising materials for several piezoelectric applications. This is due to existence of large electromechanical anisotropy in the coupling factors along and transfers direction of polarization [1]-[3].

Samarium modified PT ceramics are reported to exhibit best electric feature [4], [5]. In calcium, modified PT ceramics grain size and porosity is found to affect the behavior of permittivity and planar thickness frequency constant [6]. Additional of strontium in calcium modified PT ceramics increases piezoelectric anisotropy [7], [8]. By substitution of isovalent (Ca^{2+}, Ba^{2+}, Cd^{2+} ... etc.) ions into the Pb^{2+} sites, the lattice anisotropy is reduced [9], [10] and the samples become more dense result in a relatively large
thickness electromechanical coupling coefficient.

In the present work, a comparison between the piezoelectric modulus $d_{33}$ and the electromechanical coupling factor $k_p$ for polarized tablets of $\text{Pb}_{1-x}\text{Ni}_x\text{TiO}_3$ prepared by the usual ceramic method and those prepared by co-precipitation method. It is expected that the nano $\text{Pb}_{1-x}\text{Ni}_x\text{TiO}_3$ ceramics will have a pronounced characteristic of $d_{33}$ and $k_p$ than those of published data. This purpose can help in the improved technology.

2. Experimental

A series of $\text{Pb}_{1-x}\text{Ni}_x\text{TiO}_3$ ($x = 0, 0.1, 0.2, 0.3, 0.4$ and $0.5$) were prepared by the usual ceramic method. The raw materials were presenter at $800^\circ C$ for 2 hours. The samples were in the form of tablets and then to be cooled gradually to room temperature.

A second series of $\text{Pb}_{1-x}\text{Ni}_x\text{TiO}_3$ ($x = 0, 0.1, 0.2, 0.3, 0.4$ and $0.5$) were prepared using co-precipitation method. The chemical reagent were titanium chloride ($\text{TiCl}_3\cdot15 \text{ H}_2\text{O}$) nickel (II) chloride, ($\text{NiCl}_2\cdot6 \text{ H}_2\text{O}$), lead chloride ($\text{PbCl}_2$) and sodium hydroxide ($\text{NaOH}$). All the chemical reagents were dissolve in to $200 \text{ ml}$ of distilled water. After mixing and stirring solutions for 6 hours, chemical precipitation was achieved at room temperature vigorous stirring by adding of NaOH solution gradually the reaction system keep at $80^\circ C$ for 2 hours and PH solution ±12. After the system cooled to room temperature, the precipitates were washed with distilled water until PH-7. Finally the samples dried in oven at $200^\circ C$ for several hours and sintered at $1000^\circ C$ for 2 hours then left to be cooled gradually with the rate $50^\circ C$/hr.

The samples were examined by X-ray diffraction using a Philips model (PW-1729) diffractometer with Cu-Kα radiation ($\lambda = 1.541 \text{ A}^\circ$).

The microstructure of the sintered samples examined using SEM and TEM model Quanta 250 FEG (Field Emission Gun) attached with EDX unit (Energy Dispersive X-ray Analyses), with accelerating voltage $30 \text{ K.V.}$, magnification 14x up tp 100,000 and resolution for Gun. 1n. FEI company, Netherlands EMITECH K550X sputter coate England and (JOEL-100SX) and HRTEM model (High Resolution Transmission Electron Microscope JOEL EM 2-100) respectively. The prepared tablets with silver electrodes were polarized by exposing to $2 \text{ KV}$ per min at $150^\circ C$ for 1 Hour and then the tablets were left to be cooled under applying the dc field.

The piezoelectric modulus $d_{33}$ was determined by taking the slope of the charge dependence of the applied stress.

The coupling factor $k_p$ was estimated from the resonance frequency $f_r$ and antiresonance frequency $f_a$ for the radial mode of vibration [4], [5]:

$$k_p = \sqrt{\frac{2.55 \ast (f_u - f_r)}{f_r}}$$

The ultrasonic wave velocity $C$ was determined from the relation:

$$f_r = \frac{C}{2\pi a}$$

where $a$ is radius of polarized tablets.
3. Results and Discussion

Room temperature XRD patterns of the sintered tablets of Pb$_{1-x}$Ni$_x$TiO$_3$ for the two methods of preparation are shown in Figure 1. These have sharp and single diffraction peaks, indicating better homogeneity and crystallization of the samples. All the reflection peaks were indexed and correlated with JCPDS card file number 70 - 4057 showing the perovskite structure for all composites. The average particle size $D$ was determined from the half width of the stronger reflections of X-ray using Scherer’s equation:

$$D = \frac{0.9\lambda}{\beta \cos \theta}$$

where $\lambda$ is the wavelength of the X-ray for Cu-Kα radiation ($\lambda = 1.541$ Å) and $\beta$ is the full width at half maximum of the peak.

The particle size determined from X-ray diffraction for all samples using Scherer’s equation are found to be close nearly to be found from TEM micrographs as shown in Table 1, Figure 1 and Figure 2.

Figure 3 shows the scanning electron micrographs (SEM) of sintered and co-precipitation method of Pb$_{1-x}$Ni$_x$TiO$_3$ indicate homogenous distribution of grain brought out the surface of material. The average grain size calculated from micrograph of PbTiO$_3$ is 0.25 μm. As Ni$^{2+}$ concentration increases the grain size decreases as given in Table 1. Figure 2 shows the transmission electron micrographs (TEM) of the usual
Table 1. Indicates the value of particle size of $\text{Pb}_1-x\text{Ni}_x\text{TiO}_3$ for 1) usual method and 2) co-precipitation method.

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Figure 2. (a) TEM of co-precipitation $\text{Pb}_1-x\text{Ni}_x\text{TiO}_3$ samples; (b) TEM of ceramic $\text{Pb}_1-x\text{Ni}_x\text{TiO}_3$ samples.
Figure 3. (a) SEM of co-precipitation Pb$_{1-x}$Ni$_x$TiO$_3$ samples; (b) SEM of ceramic Pb$_{1-x}$Ni$_x$TiO$_3$ samples.

ceramic samples and nano polycrystalline samples prepared by co-precipitation method. It is obvious that the increase of Ni content decrease the particle and grain size. This is due to the following: Ni$^{2+}$ ions enter the lattice and replace some lead ions (Pb$^{2+}$). These doped ions precipitate at grain boundaries and retard its motions. The increase of Ni$^{2+}$ ions inhibit the grain growth keeping the grain size small. Our explanation in similar to previous work [10], [11].

3.1. Effect of Ni Content on the Piezoelectric Modulus $d_{33}$

The effect of Ni content on the piezoelectric modulus $d_{33}$ is shown in Figure 4. It is observed that $d_{33}$ of the polarized samples prepared by co-precipitation method are higher than those prepared by ceramic method. Doping ions concentrate near the domain boundaries in which some dislocations might act as precipitating centers for migration.
of these ions, which in turn act as pinning points causing further dislocations. Also the
coulomb attraction between the doping ions and the induced vacancies leads to conti-
nuous formation of more lattice vacancies since many of them might act as sinks for
these point defects.

Thus, the domain walls are expected be quite free in smaller particle size and grain
size considering that grain boundaries will give additional pinning points to the moving
walls. The beneficial result of the reduction in grain boundary is the decrease in couple-
ing effect between boundaries and the domain walls leading to increase in domain wall
mobility.

This mechanism may affect domain wall motion towards the increase of polarization
and hence the observed increase of $d_{33}$ and generated piezoelectric charges under ap-
plied stress. Moreover the $d_{33}$ of nanoparticles of co-precipitation method is higher due
to smaller area of grain boundaries compared to those in the samples prepared by cer-
eramic method. This small area helps the domain wall mobility to be increased giving
rise to polarization under applied stresses. The behavior of our results are similar to
those in previous work on lead titanate zirconate ceramics [12], [13].

3.2. Electromechanical Coupling Factor $k_p$

The planer-coupling factor $k_p$, as a function of Ni dopant is shown in Figure 5. for the
prepared polarized tablets by usual ceramic and co-precipitation method. As Ni addi-
tions increases, $k_p$ increases. This can be explained as follows:

$\text{Ni}^{3+}$ substituted $\text{Pb}^{2+}$ sites of the perovskite tetragonal structure. For each $\text{Pb}^{2+}$ cation
replaced with $\text{Ni}^{3+}$ that a vacancy is created to maintain the charge neutrality of the sys-
tem. It is in the tetragonal polarized ferroelectric materials, two types of domains are
Figure 5. Effect of “Pb_{1-x}Ni_xTiO_3” on the coupling factor $k_p$ for ceramic and co-precipitation methods.

The increase of $k_p$ in the nano Pb_{1-x}Ni_xTiO_3 polarized discs than those prepared by usual ceramic method is due to the lower grain boundary area which facilitate the domain wall motion under applied ac field. This lead to increase $k_p$ of the nano disc of Pb_{1-x}Ni_xTiO_3. From Table 2 it is noticed that the particle size of the co-precipitation method is smaller for the usual ceramic in all Ni contents. Ni^{3+} ions enter the lattice instead of Pb^{2+} ions. The small area of grain boundary of the nanoparticle of the co-precipitation method helped the domains wall mobility to be increased. This led to increase the piezoelectric characteristic properties in the nano crystallite of the co-precipitation method than those prepared by usual ceramic method.

3.3. Effect of Ni Contents on the Ultrasonic Velocity

Figure 6 illustrates the effect of Ni content on the velocity of ultrasonic wave for the radial mode of vibrating polarized tablets. It is obvious that the velocity decreases with increasing x. This could be explained as following:

The velocity of sound wave for poled Pb_{1-x}Ni_xTiO_3 of the two method of preparation can be evaluated in the radial mode may be due to the following:

The decrease in sound wave velocity with doping may be due to the creation of lead vacancies. These vacancies help the domain wall to vibrate early leading to decreases the ultrasonic wave velocity. Our explain is in close to previous work on PZT [11].
Table 2. $k_p$ and $d_{33}$ of the published data and our present data.

<table>
<thead>
<tr>
<th>Composition</th>
<th>$k_p$</th>
<th>$d_{33}$ (PC/N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PbTiO$_3$</td>
<td>0.3</td>
<td>56</td>
</tr>
<tr>
<td>BaTiO$_3$</td>
<td>0.35</td>
<td>191</td>
</tr>
<tr>
<td>PZT:Nb</td>
<td>0.54</td>
<td></td>
</tr>
<tr>
<td>PZT:(La + Nb)</td>
<td>0.57</td>
<td></td>
</tr>
<tr>
<td>PZT</td>
<td>0.32</td>
<td>220</td>
</tr>
<tr>
<td>Pure PZT (52/48)</td>
<td>0.47</td>
<td></td>
</tr>
<tr>
<td>Pb(Zr$<em>{52}$, Ti$</em>{48}$)O$_3$</td>
<td>0.53</td>
<td>223</td>
</tr>
<tr>
<td>Pb$<em>{0.9725}$(Zr$</em>{0.52}$Ti$<em>{0.48}$)$</em>{1−5y/4}$NbyO$_3$ (PFZTN)</td>
<td>0.45</td>
<td></td>
</tr>
<tr>
<td>Pb$<em>{0.95}$Ni$</em>{0.03}$Nb$<em>{0.02}$Zr$</em>{0.51}$Ti$_{0.49−x}$M$_x$O$_3$</td>
<td>0.665</td>
<td>625 pm/V</td>
</tr>
<tr>
<td>Pb$<em>{0.5}$Ni$</em>{0.5}$TiO$_3$ crystallite</td>
<td>0.73</td>
<td>650</td>
</tr>
</tbody>
</table>

**Figure 6.** Effect of $x$ on ultrasonic velocity for ceramic and co-precipitation methods.

4. Conclusions

In this paper, the piezoelectric $d_{33}$ for the Pb$_{1−x}$Ni$_x$TiO$_3$ ceramics prepared by co-pre-
cipation method were found to be higher than those prepared by ceramic method. This caused by inhibition of grain wall volume.

The increase of $k_p$ for samples prepared by co-precipitation method was due to entering Ni ions the lattice and substitute Pb$^{2+}$ ions. This caused creation of vacancies. These vacancies helped the dipoles to vibrate at lower frequency.

The decrease of ultrasonic waves with enhancing Ni contents was due to creation of vacancies, which scattered the propagation of the waves.

It is clearer the present sample Pb$_{0.5}$Ni$_{0.5}$TiO$_3$ possessed higher $d_{33}$ and $k_p$ than those in previous work [12] which is useful in technology.

References


The Effect of State-Dependent Control for an SIRS Epidemic Model with Varying Total Population

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Abstract

Based on the mechanism of prevention and control of infectious disease, we propose, in this paper, an SIRS epidemic model with varying total population size and state-dependent control, where the fraction of susceptible individuals in population is the detection threshold value. By the Poincaré map, theory of differential inequalities and differential equation geometry, the existence and orbital stability of the disease-free periodic solution are discussed. Theoretical results show that by state-dependent pulse vaccination we can make the proportion of infected individuals tend to zero, and control the transmission of disease in population.

Keywords

SIRS Epidemic Model, Varying Total Population, State-Dependent Pulse Control, Periodic Solution, Orbital Stability

1. Introduction

It is generally known that the spread of infectious diseases has been a threat to healthy of human beings and other species. In order to prevent and control the transmission of disease (such as hepatitis C, malaria, influenza), pulse vaccination as an effective strategy has been widely studied by many scholars in the study of mathematical epidemiology. In the classical research literature it is usually assumed that the pulse vaccination occurs at fixed moment intervals and total population size remains constant [1] [2], and so on. Although fixed time pulse vaccination strategy is better than the traditional vaccination strategies (continuous vaccination), it has a few disadvantages. For these reasons, a new vaccination strategies, state-dependent pulse vaccination is proposed when the number of the susceptible individuals or infected individuals reaches a critical value. Clearly, the latter control strategies are more rational for disease control because of its efficiency, economy, and feasibility. In recent
years, mathematical models with state-dependent pulse control strategies have been extensively applied to research fields of applied science, such as pest management model [3], tumor model [4], predator-prey model [5], and others. Particularly, Nie et al. [6] investigated an SIR epidemic model with state-dependent pulse vaccination. In it, authors obtained the existence and stability of positive order-1 and order-2 periodic solution. Tang et al. [7] proposed an SIR epidemic model with state-dependent pulse control strategies. Authors demonstrated that the combination of pulse vaccination and treatment is optimal in terms of cost under certain conditions, and studied the existence and stability of periodic solution.

On the other hand, the population sizes of all epidemic models with state-dependent pulse control are constant. These types of models have been studied extensively since they are easier to analyze than variable population size models. Obviously, the assumption that the total population size which remains constant is reasonable if negligible mortality rate and the disease spread quickly through the population. However, it fails to hold for diseases that are endemic in communities with changing populations, and for diseases which raise the mortality rate substantially. In such situation, we can hardly expect a population remaining constant, and hence more complicated epidemic models with varying population size should be considered. In fact, studies of this type of models have been become a major topic in mathematical epidemiology. For example, an general epidemiological model with vaccination and varying total population was discussed by Yang et al. [8], in which the global dynamics of this model and it’s corresponding proportionate model are investigated. The conditions between the two models in terms of disease eradication and persistence are obtained. Hui et al. [9] introduced an SEIS epidemic model with total population which is not stationary. Results are obtained in terms of three threshold which respectively determines whether or not the disease dies out and dynamics of epidemic model when births of population are throughout a year. At same time, they also discussed the existence of disease-free periodic solution when births of population are birth pulse. More related literature, we also can be found in [10] [11], and the references therein.

As far as we know, epidemic model with varying total population and state-dependent feedback control strategies had never been done in the literatures. Hence, in this paper, the dynamical behavior of an SIRS epidemic model with varying total population and state-dependent pulse control strategy is studied. The main aim is to explore how the state-dependent pulse control strategy affects the transmission of diseases. The remaining part of this paper is organized as follows. In the next section, an SIRS control model is constructed and some preliminaries are introduced, which are useful for the latter discussion. In section 3, we will focus our attention on the existence and orbital stability of disease-free periodic. Finally, some concluding remarks are presented in the last section.

2. Models and Preliminaries

In the study of the dynamic properties of infectious diseases, it was found that when the
The popularity of disease for a long time total population size change this factor should be considered. In this case, Busenberg et al. [12] proposed the following SIRS epidemic model with varying total population size.

\[
\begin{align*}
\frac{dS(t)}{dt} &= bN(t) - dS(t) - \frac{\lambda S(t)I(t)}{N(t)} + eR(t), \\
\frac{dI(t)}{dt} &= \frac{\lambda S(t)I(t)}{N(t)} - (d + c + \epsilon)I(t), \\
\frac{dR(t)}{dt} &= cI(t) - (d + e + \delta)R(t).
\end{align*}
\]  

(1)

Here \( S(t) \), \( I(t) \), and \( R(t) \) denote the numbers of susceptible, infected, and recovered individuals respectively, and \( N(t) = S(t) + I(t) + R(t) \) denote the total population size at time \( t \). The parameters in the model have the following features: \( b \) is the per capita birth rate with the assumption that all newborns are susceptible; \( d \) is the per capita disease free death rate of the population; the constants \( \epsilon \) and \( \delta \) denote the excess per capita death rate of infected individuals and recovered individuals, respectively; \( c \) is the per capita recovery rate of the infected individuals and \( e \) is the per capita loss of immunity rate for recovered individuals. It is assumed that all susceptible group becomes infected at a rate \( \lambda I(t)/N(t) \), where \( \lambda > 0 \) is the effective per capita contract rate of infective individuals. All parameter values are assumed to be non-negative and \( b, c > 0 \).

Since the susceptible individuals are immunity toward certain infectious diseases in the crowd, once infected individuals get into the susceptible groups, this will lead to the outbreak of the diseases. For this reason, we propose a pulse vaccination function as follows \( S(t^+) = (1 - p)S(t) \) where \( p \) (\( 0 < p < 1 \)) is the proportion by which the susceptible individuals numbers is reduced by pulse vaccination.

Taking into account pulse vaccination as state-dependent feedback control strategies, model (1) can be extend to the following state-dependent pulse differential equation.

\[
\begin{align*}
\frac{dS(t)}{dt} &= bN(t) - dS(t) - \frac{\lambda S(t)I(t)}{N(t)} + eR(t) \\
\frac{dI(t)}{dt} &= \frac{\lambda S(t)I(t)}{N(t)} - (d + c + \epsilon)I(t) \\
\frac{dR(t)}{dt} &= cI(t) - (d + e + \delta)R(t) \\
S(t^+) &= (1 - p)S(t) \\
I(t^+) &= I(t) \\
R(t^+) &= R(t) + pS(t).
\end{align*}
\]  

(2)

where the critical threshold \( H > 0 \) is a constant. The meaning of model (2) as following: once the fraction of the susceptible individuals in the population reaches the critical value \( H \) at time \( t_c(H) \), vaccination control strategies are carried out which lead to the
number of susceptible and recovered individuals abruptly turn to \((1 - p)S(t)\), and \(R(t) + pS(t)\) respectively.

The equation for the total population size \(N(t)\) can be determined from model (2)

\[
\frac{dN(t)}{dt} = (b - d)N(t) - \varepsilon I(t) - \delta R(t) \neq 0.
\]

It means that total population size \(N(t)\) is not constant. In such situations, to discuss the dynamics behavior of model (2) we need to consider the fraction of individuals in the three epidemiological classes, namely

\[
x(t) = \frac{S(t)}{N(t)}, \quad y(t) = \frac{I(t)}{N(t)}, \quad z(t) = \frac{R(t)}{N(t)}.
\]

It following from (3) that we can transforms model (2) into the following model for these new variables

\[
\begin{align*}
\frac{dx(t)}{dt} &= b - bx(t) + ez(t) - (\lambda - \varepsilon)x(t)y(t) + \delta x(t)z(t) \\
\frac{dy(t)}{dt} &= \lambda x(t)y(t) + \varepsilon y^2(t) + \delta y(t)z(t) - (b + c + \varepsilon)y(t) \\
\frac{dz(t)}{dt} &= cy(t) + \varepsilon y(t)z(t) + \delta z^2(t) - (b + e + \delta)z(t).
\end{align*}
\]

Define three threshold parameter as follows

\[
R_0 = \frac{\lambda}{(b + c + \varepsilon)}, \quad R_1 = b \frac{\lambda}{d + \varepsilon y_x + \delta z_x}, \quad R_2 = \frac{\lambda x_e}{c + d + \varepsilon}.
\]

On the dynamics of model (4) without pulse effect has been studied in [12]. Relevant conclusions can be summarized as the following Theorem 1.

**Theorem 1.** For model (4) without pulse control, the following result hold true.

1) The disease-free equilibrium \(E_0(1,0,0)\) always exists and is globally asymptotically stable in the feasibility region \(\mathbb{D} = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}\) whenever \(R_0 \leq 1\), and unable when \(R_0 > 1\).

2) When \(R_0 > 1\), there exist a unique endemic equilibrium \(E^e(x_e, y_e, z_e)\), which is globally asymptotically stable in the feasibility region \(\mathbb{D}_0 = \mathbb{D} / \{1,0,0\}\) where

\[
x_e = 1 - y_e - z_e, \quad y_e = \frac{(\delta - \lambda)z_e + \lambda - b - c - e}{\lambda - \varepsilon},
\]

and \(z_e\) can be found by solving equation

\[
\varphi(z_e) = \lambda(\delta - \varepsilon)z_e^2 + \left[-\lambda(b + c + \varepsilon + \delta - \varepsilon) + \varepsilon(e - c + \delta - \varepsilon) + c\delta\right]z_e + c(\lambda - b - c - \varepsilon) = 0.
\]
The total population \( N(t) \) has the asymptotic behavior \( \lim_{t \to +\infty} N(t) = 0 \) if \( R_1 < 1 \), and \( \lim_{t \to +\infty} N(t) = +\infty \) if \( R_1 > 1 \).

When \( R_1 > 1 \), the total infected population has the asymptotic behavior \( \lim_{t \to +\infty} I(t) = 0 \) if \( R_2 < 1 \), and \( \lim_{t \to +\infty} I(t) = +\infty \) if \( R_2 > 1 \).

Based on the above discussions, we just need to discuss cases (a) and (b) in Table 1.

Considering the similarities of cases (a) and (b), throughout of this paper, we discuss only the case (a). That is, in an increasing population, the number of infected individuals converges to infinity, while the fraction of infected individuals in population is tending to a nonzero constant \( y_e \).

Due to \( x(t) + y(t) + z(t) = 1 \), for model (4) we can eliminate \( z(t) \) by \( z(t) = 1 - x(t) - y(t) \) and consider the two-dimensional model.

\[
\begin{align*}
\frac{dx(t)}{dt} &= b - bx(t) - (\lambda - \varepsilon)x(t)y(t) + (\delta x(t) + \varepsilon)(1 - x(t) - y(t)) \quad \left(x(t) < H, \right. \\
\frac{dy(t)}{dt} &= \lambda x(t)y(t) + \varepsilon y^2(t) + \delta y(t)(1 - x(t) - y(t)) - (b + c + \varepsilon)y(t) \\
x(t^+) &= (1 - p)x(t) \\
y(t^+) &= y(t)
\end{align*}
\]

By the biological background, we only focus on model (5) in the biological meaning region \( \mathbb{D} = \{(x(t), y(t)) : x(t) \geq 0, y(t) \geq 0, x(t) + y(t) \leq 1\} \). Besides, the globally existence and uniqueness properties of solution of model (5) are guaranteed by the smoothness of \( f \) which is the mapping defined by right-side of model (5), for details see [13].

Let \( S \subseteq \mathbb{R}^2 \) be an arbitrary nonempty set and \( P_0 \in \mathbb{R}^2 \) be an arbitrary point. The distance between \( P_0 \) and \( S \) is defined by \( \rho(P_0, S) = \inf_{P \in S} |P - P_0| \). Set \( X(t) = (x(t), y(t)) \) be a solution of model (5) starting from initial point \( X_0 \in \mathbb{R}^2 \) at \( t = t_0 \). We define the positive orbit as follows

\[
O^+(X_0, t_0) = \{X(t) = (x(t), y(t)) : t \geq t_0, X(t_0) = X_0\}.
\]

Firstly, on the positivity of solutions of model (5), we have the following Lemma 1.

**Lemma 1.** Supposing that \( (x(t), y(t)) \) is a solution of model (5) with the initial condition \( (x(t_0), y(t_0)) \in \mathbb{D} \), then \( (x(t), y(t)) \in \mathbb{D} \) for all \( t \geq t_0 \).

**Proof.** For any initial value \( (x(t_0), y(t_0)) \in \mathbb{D} \), we will discuss all possible cases by the relation of the solution \( (x(t), y(t)) \) to the line \( L_1 : y(t) = H \) as follows.

1) The solution \( (x(t), y(t)) \in \mathbb{D} \) intersects with line \( L_1 \) finitely many times.

For this case, due to the endemic equilibrium \( (x_e, y_e) \) is globally asymptotically

**Table 1.** Threshold criteria and asymptotic behavior.

<table>
<thead>
<tr>
<th>case</th>
<th>( R_0 )</th>
<th>( R_1 )</th>
<th>( R_2 )</th>
<th>( N \to )</th>
<th>( (x, y, z) \to )</th>
<th>( (S, I, R) \to )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>( &gt; 1 )</td>
<td>( &gt; 1 )</td>
<td>( &gt; 1 )</td>
<td>( +\infty )</td>
<td>( (x_e, y_e, z_e) )</td>
<td>( (+\infty, +\infty, +\infty) )</td>
</tr>
<tr>
<td>(b)</td>
<td>( &gt; 1 )</td>
<td>( &lt; 1 )</td>
<td>( &lt; 1 )</td>
<td>( 0 )</td>
<td>( (x_e, y_e, z_e) )</td>
<td>( (0, 0, 0) )</td>
</tr>
</tbody>
</table>
stable, then \( x(t) > 0, \ y(t) > 0 \) for all \( t \geq t_0 \).

2) The solution \( (x(t), y(t)) \in \mathbb{D} \) intersects with line \( L_1 \) infinitely many times.

For second situation, assume that solution \( (x(t), y(t)) \in \mathbb{D} \) intersects with line \( L_1 \) at times \( t_i, (i = 1, 2, \cdots) \) and \( \lim_{i \to \infty} t_i = +\infty \). If the conclusion of Lemma 1 is false, we obtain that there exists a positive integer \( n \) and a \( t^* \in (t_{n-1}, t_n) \) such that

\[
\min \{x(t^*), y(t^*)\} = 0 \quad \text{and} \quad x(t) > 0, y(t) > 0 \quad \text{for} \quad t_0 < t < t_n.
\]

The first possibility is that \( x(t^*) = 0 \) and \( y(t^*) > 0 \). For this case, it follows from the first and third equation of model (5) that

\[
x(t^*) \geq (1 - p)^{n-1} x(t_0^*) \exp \left\{ \int_{t_0^*}^{t^*} \left[ -b + (\lambda - c) y(\tau) \right] d\tau \right\} \geq 0.
\]

which contradicts the fact that \( x(t^*) = 0 \).

The other case is that \( x(t^*) > 0 \) and \( y(t^*) = 0 \). In this regard, it follows from the second and fourth equation of model (5) that

\[
y(t^*) \geq y(t_0^*) \exp \left\{ \int_{t_0^*}^{t^*} \left[ -\lambda x(\tau) - (b + c + e) \right] d\tau \right\} \geq 0.
\]

which lead to a contradiction with \( y(t^*) = 0 \). Therefore, according to above discussion, we can obtain that \( x(t) \geq 0 \) and \( y(t) \geq 0 \) for all \( t \geq t_0 \). This proof is complete.

In order to address the dynamical behaviors of model (5), we could construct two sections to the vector field of model (5) by

\[
\Sigma_p := \{(x(t), y(t)) : x(t) = (1 - p) H, 0 < y(t) < 1 - (1 - p) H\},
\]

and

\[
\Sigma_H := \{(x(t), y(t)) : x(t) = H, 0 < y(t) < 1 - H\}.
\]

Choosing section \( \Sigma_p \) as a Poincaré section. Assume that for any point \( P_1^*(H, y_i) \in \Sigma_H \), the trajectory \( O^*(P_1, t_0) \) starting from the initial point \( P_1(H, y_i) \) intersects section \( \Sigma_H \) infinitely many times. That is, trajectory \( O^*(P_1, t_0) \) jumps to section \( \Sigma_p \) at point \( P_1^* (1 - p) H, y_i^* \) due to pulse effect. Moreover, trajectory \( O^*(P_1, t_0) \) will reach at section \( \Sigma_H \) at point \( P_2(H, y_j) \), and then jumps to point \( P_2^*(1 - p) H, y_j^* \) on section \( \Sigma_p \). Repeating this procedure, we get two pulse point sequences \( \{P_n^* (1 - p) H, y_n^*\} \) and \( \{P_n(H, y_n)\} \) \( (n = 1, 2, \cdots) \), where \( y_n^* \) is only determined by \( y_n, H, \) and \( p \). Therefore, we can define a Poincaré map of section \( \Sigma_p \) as

\[
y_{n+1}^* := \mathcal{F} \left( y_n^*, H, p \right) \quad (6)
\]

From the definition of Poincaré map \( \mathcal{F} \), it easy to get that

\[
y_{n+1}^* = \mathcal{F} \left( y_n^*, H, p \right) = \mathcal{F} \left( \mathcal{F} \left( y_{n-1}^*, H, p \right) \right) = \mathcal{F}^2 \left( y_{n-1}^*, H, p \right) = \cdots = \mathcal{F}^n \left( y_1^*, H, p \right).
\]

Obviously, function \( \mathcal{F} \) is continuously differential according to the Cauchy-Lipschitz theorem. If there exist positive integer \( k \) such that \( y_{k+1}^* = \mathcal{F}^k \left( y_1^*, H, p \right) \), then trajectory \( O^*(P_1^*, t_0) \) of model (5) is said to be order-\( k \) periodic solution.
3. Main Results

Our main purpose in this section is to investigate the existence and orbital stability of periodic solution of model (5). From the geometrical construction of phase space of model (5), we note that the trajectory $O^*(M, t_0)$ from any initial point $M \in \Sigma_p$ intersects section $\Sigma_H$ infinite times with $H \leq x_e$. However, if $(1 - p)H < x_e < H < 1$, then trajectory $O^*(M, t_0)$ from any initial point $M \in \Sigma_p$ may be free from pulse effects or intersects section $\Sigma_p$ infinitely times, which depend on the initial conditions. Consequently, based on different positions of section $\Sigma_H$ we need to discuss the existence and orbital stability of periodic solution of model (5) in the cases of $H \leq x_e$ and $(1 - p)H < x_e < H < 1$.

Case I: The case of $H \leq x_e$.

For this case, it will prove that model (5) possesses a disease-free periodic solution, which is orbitally asymptotically stable.

Suppose $y(t) = 0$ for all $t \in (0, +\infty)$, then model (5) degenerates into the following model

$$\begin{cases}
\frac{dx(t)}{dt} = b - bx(t) + c(1 - x(t)) + \delta x(t)(1 - x(t)), & x(t) < H, \\
x(t^+) = (1 - p)x(t), & x(t) = H.
\end{cases}$$

Integrating the first equation of model (7) with the initial condition $x(0) = x_0 = (1 - p)H$, one yields

$$\bar{x}(t) = \exp\left[\frac{(b + \delta)(t + C) - b - e}{\delta + \exp\left[\left(t + C\right)(b + e + \delta)\right]}\right],$$

where

$$C = \frac{1}{b + e + \delta} \ln\left(\frac{(1 - p)H\delta + b + e}{1 - (1 - p)H}\right).$$

Assume that $x(T) = H$ and $x(T^+) = x_0 = (1 - p)H$, then we obtain

$$T = \frac{1}{b + e + \delta} \ln\left(\frac{(\delta H + b + e)(1 - (1 - p)H)}{(1 - H)((1 - p)H\delta + b + e)}\right).$$

Therefore, model (5) possesses the following disease-free periodic solution, denoted by

$$\phi(t) = \exp\left[\frac{(b + \delta)((t - (k - 1)T) + C) - b - e}{\delta + \exp\left[\left((t - (k - 1)T) + C\right)(b + e + \delta)\right]}\right],$$

where $t \in \left((k - 1)T, kT\right]$, $k = 1, 2, \cdots$.

On the stability of this disease-free periodic solution $(\phi_1(t), \phi_2(t))$ we have the following result.

**Theorem 2.** For any $p \in (0, 1)$ and $H \leq x_e$ the disease-free periodic solution (8)
of model (5) is orbitally asymptotically stable.

Proof. We assume that section \( \Sigma_p \) intersects line \( L_2 : x + y = 1 \) and x axis at points \( P \) and \( Q \), respectively. From the geometrical structure of phase space of model (5), we know that trajectory starts from any point on set

\[
PQ = \{ (x,y) : x = (1-p)H, 0 \leq y \leq 1-(1-p)H \}
\]

will enter set \( \overline{PQ} \). Further, set

\[
\overline{PQ} = \{ (x,y) : x = (1-p)H, 0 \leq y \leq 1-(1-p)H \}
\]

is mapped to set

\[
\overline{PQ} = \{ (x,y) : x = (1-p)H, 0 < y < y_q \}
\]

by Poincaré map (6), where \( y_p > y_q > 0 \). Then, set \( \overline{PQ} \) is mapped to set \( \overline{PQ} = \{ (x,y) : x = (1-p)H, 0 < y < y_{pq} \} \) and \( y_p > y_{pq} > y_q > 0 \). Repeat above-mentioned procedure, we gain one point sequences \( \{ y_n \} \) which satisfy

\[
y_p > y_{pq} > \cdots > y_n > \cdots > 0,
\]

and

\[
|\overline{PQ}| > |\overline{PQ}| > |\overline{PQ}| > \cdots > |\overline{PQ}| > \cdots > 0,
\]

where \( y_p = 1-(1-p)H \).

From (9), it is concluded that the point sequence \( |\overline{PQ}| \) is monotonically decrease in the interval \( (0, y_p) \) and converge to a fixed point in this bound region. That is \( \lim_{n \to \infty} |\overline{PQ}| = 0 \).

Suppose that \( (\overline{x}(t), \overline{y}(t)) \) is a solution of small-amplitude perturbation of disease-free periodic solution \( (\phi(t), \phi(t)) \) with initial value \( (\overline{x}(0), \overline{y}(0)) = ((1-p)H, \overline{y}_0) \), which first intersects section \( \Sigma_{\overline{H}} \) at point \((H, \overline{y})\) and then jumps to point \((1-p)H, \overline{y}_q\). Further, solution \((\overline{x}(t), \overline{y}(t))\) insects section \( \Sigma_{\overline{H}} \) at point \((H, \overline{y}_q)\) again. Repeating the above process, we have two point sequences \( \{ \overline{y}_n \} \) and \( \{ y_n \} \), where \( \overline{y}_n = y_n \). Furthermore, by \( \lim_{n \to \infty} |\overline{PQ}| = 0 \), it is clear that \( \lim_{n \to \infty} \overline{y}(t) = 0 \). This shows that the disease-free periodic solution (8) of model (5) is orbitally asymptotically stable. This proof is complete.

Case II: The case of \( (1-p)H < x_e < H < 1 \).

For this case, we know that there a point \( E_0 (\overline{1-p})H, y_{k_0} \in \Sigma_p \) such that trajectory \( O^+ \) (\( E_0, t_0 \)) is tangent to section \( \Sigma_{\overline{H}} \) at the point

\[
E_1 (H, (1-H)(b+c+\delta H))/((\lambda - \epsilon + \delta)H + \epsilon).
\]

Then the point \( E_1 \) is jump to the point

\[
E_1^+ (\overline{(1-p)}H, (1-H)(b+c+\delta H))/((\lambda - \epsilon + \delta)H + \epsilon)
\]

on section \( \Sigma_p \) after pulse effect. According to the different positions of point \( E_1^+ \) we has the following results.

Theorem 3. For any \( p \in (0,1) \) and \( (1-p)H < x_e < H < 1 \), if

\[
y_{k_0} = (1-H)(b+c+\delta H))/((\lambda - \epsilon + \delta)H + \epsilon),
\]

then model (5) exists a positive order-1 periodic solution. Further, if

\[
y_{k_0} > \frac{(1-H)(b+c+\delta H)}{((\lambda - \epsilon + \delta)H + \epsilon)},
\]

then model (5) exists a disease-free periodic solution (8), which is orbitally asymptotically stable.

For this case, (8) is a disease-free periodic solution of model (5), and the proof of stability is similar to the proof of Theorem 2, we therefore omit here.
4. Concluding Remarks

In order to explore the effects of the state-dependent pulse control strategies on the transmission of the infectious diseases in a population of varying size, an SIRS epidemic model with varying total population and state-dependent pulse control strategy is proposed and analyzed in this paper. Theoretically analyzing this control model, we find that a disease-free periodic solution always exists and orbitally stable when condition $H \leq x_e$ holds. Theoretical results shows that the disease finally disappears if we control the fraction of susceptible individuals in relatively low levels. Furthermore, we obtained some sufficient condition on existence and stability of the positive order-1 periodic solution when $(1-p)H < x_e < H < 1$. This amounts to that we can control the fraction of susceptible individuals and infected individuals within a retain range for a long time by appropriately choose the immune strength $p$ and critical threshold $H$. Therefore, we can concluded that state-dependent pulse vaccination is a feasible, economic, and high efficient method to prevention and control spread of diseases.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Fund

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References


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Measurement of Mathematical Constant $\pi$ and Physical Quantity $Pi$

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Abstract

Instead of calculating the number $\pi$ in this article special attention is paid to the method of measuring it. It has been found that there is a direct and indirect measurement of that number. To perform such a measurement, a selection was made of some mathematical and physical quantities which within themselves contain a number $\pi$. One such quantity is a straight angle $Pi$, which may be expressed as $Pi = \pi \text{ rad}$. By measuring the angle, using the direct method, we determine the number $\pi$ as $\pi = \arccos(-1)$. To implement an indirect measurement of the number $\pi$, a system consisting of a container with liquid and equating it with the measuring pot has been conceived. The accuracy of measurement by this method depends on the precision performance of these elements of the system.

Keywords

$Pi$, Greek Letter $\pi$, Angle Measurement, Measuring Pot, Radian

1. Introduction

The number $\pi$ is a mathematical and physical constant which originated as the ratio of a circle’s circumference to its diameter. Throughout scientific history it became significant as it appeared across all fields of mathematics and physics having little to do with the geometry of circles. Some of those fields are mathematical number theory, statistics, cosmology, thermodynamics, mechanics, quantum mechanics and electromagnetism. Great efforts have been invested in finding as many decimal digits of that number as it is possible. On the contrary, in this article we focus on the measurement associated with the mathematical constant $\pi$ and the physical quantity $Pi$. So, essentially we have to distinguish two different things, the number $\pi$ and the physical quantity $Pi$. Many have focused exclusively on the calculation of the number $\pi$ and not to measure it. To be able to measure it, we need to find a link between the number $\pi$ and the physical quantity $Pi$. 

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2. Definition of the Radian

The radian is the standard unit of angular measure. By definition, the radian (symbol rad) is the plane angle unit in the International System of Units (abbreviated SI). It represents the central angle on a circumference and covers an arc \( \hat{s}_{AB} \) whose length is equal to the radius \( r \) of the circuit (Figure 1):

\[
1 \text{ rad} \triangleq \frac{\hat{s}_{AB}}{r} = \frac{\hat{r}}{r}.
\]

It is one of two supplementary units in SI; namely, radian—for plane angles, and steradian (symbol sr)—for solid angles, based on abstract geometrical concept rather than physical standards.

Any angle \( \varphi \) formed by two radii, measured in radians, is equal to the arc length between two spokes \( \hat{s}_{AX} \), divided by radius \( r \) (Figure 1). So, the full angle \( \varphi_\infty \) (360°) is:

\[
\varphi_\infty = \frac{\hat{s}_{AD}}{r} = \frac{2\pi \hat{r}}{r} = 2\pi \frac{\hat{r}}{r} = 2\pi \text{ rad},
\]

where \( \pi \) is a transcendental number, i.e., a fundamental mathematical constant with the decimal expansion 3.141592653..., tens of billions of decimal places have been calculated; just 39 places suffice to compute the circumference of the observable universe accurate to within the radius of a hydrogen atom. The number \( \pi \) is sometimes known as Archimedes’ constant or Ludolph’s constant, after Ludolph van Ceulen (1539-1610), a German-Dutch mathematician [1].

We should distinguish two different things; first, the mention number \( \pi \) and a second, the mathematical and physical quantity denoted by \( Pi \). While \( \pi \) is a number, \( Pi \) is a quantity defined as the ratio of the circumference of a circle (2\( \pi \hat{r} \)) to its diameter \( (d = 2r) \). Actually, according to the definition of radian, and using Equation (2), the quantity \( Pi \) is the angle, because is defined as any angle:

\[
Pi = \frac{\hat{s}_{AB}}{d} = \frac{2\pi \hat{r}}{2r} = \frac{\pi \hat{r}}{r} = \frac{\varphi_\infty}{2} = \varphi_\pi = \pi \text{ rad}.
\]

There are many formulas for calculating the number \( \pi \), from the simple to the very
complicated. While many researchers’ oriented to calculating the number $\pi$ in the largest possible number of decimals, we will focus here our efforts at something else, namely, how to measure the mathematical and physical quantity $Pi$.

All the measurable quantities are physical quantities. The measurement is associated with unit (rad, in our case) and numerical value, like number $\pi$ in Equation (3):

$$\frac{Pi}{\text{rad}} = \pi.$$  \hspace{1cm} (4)

In general, the quantities can be measured directly or indirectly. E.g. measuring the length of road, weight of body and so on, are the direct measurement. Measuring the mass of Earth, electron, and so on are indirect measurement. Measuring of the quantity $Pi$ can be made both ways, directly and indirectly.

3. Direct Measurement of the Quantity $Pi$

Formula for the arc length $\hat{S}_{AB}$ of the curve (circle, Figure 1)

$$y = f(x) = \pm \sqrt{r^2 - x^2}$$ \hspace{1cm} (5)

read \cite{2}:

$$\hat{S}_{AB} = \int_{x=\text{rad}}^{x=r} \sqrt{1 + \left[f'(x)\right]^2} \, dx.$$ \hspace{1cm} (6)

The derivative of Equation (5) is equal to

$$f'(x) = \mp \frac{x}{\sqrt{r^2 - x^2}},$$ \hspace{1cm} (7)

so

$$\hat{S}_{AB} = \int_{x=\text{rad}}^{x=r} \frac{r}{\sqrt{r^2 - x^2}} \, dx = r \arccos \left( \frac{X_{\text{rad}}}{r} \right).$$ \hspace{1cm} (8)

When the arc length $\hat{S}_{AB} = r \arccos \left( \frac{X_{\text{rad}}}{r} \right)$ becomes equal to the radius $r$,

$$r \arccos \left( \frac{X_{\text{rad}}}{r} \right) = r,$$ \hspace{1cm} (9)

i.e.,

$$X_{\text{rad}} = r \cos(1) = r \cdot 0.5403023058681397...,$$ \hspace{1cm} (10)

then the corresponding angle, according to the definition in Equation (1), is equal to 1 rad (Figure 1). In this way chosen is the arc that belongs to the angle of one radian. With this arc, as units of measurement ($r \neq 1$, i.e., $r$ corresponds to one), we can now measure any other arc, of course and one that belongs to a straight angle $\varphi_{\pi}$ (Figure 1):

$$\varphi_{\pi} = Pi = \pi \text{ rad}.$$ \hspace{1cm} (11)

In this way we can directly measure the length of the arc belonging to the angle $\varphi_{\pi} = Pi$. What we measure in accordance with Equation (8) is

$$\hat{S}_{AC} = \int_{x=-r}^{x=r} \frac{r}{\sqrt{r^2 - x^2}} \, dx = r \arccos \left( \frac{-r}{r} \right) = r \arccos(-1) = r \cdot 3.141592653589793....$$ \hspace{1cm} (12)
The numeric value in such measurement of the arc $\tilde{s}_{AC}$ we denoted by the Greek letter $\pi = 3.141592653589793\ldots$.

4. Indirect Measurement of the Quantity $Pi$

The quantities that cannot be measured are the aphysical quantities. They do not have any scale to measure. They are also called the abstract quantities and are considered not to be present in this physical world. The quantity $Pi$ certainly does not belong in the abstract quantities. This article presents some of the methods of indirect measurement of the physical quantity $Pi$.

To perform measurements the presented method uses liquid in a container of any shape which volume is $V_c$ (Figure 2), relying on the document which is in the process of obtaining a patent in the UK [3]. The bottom surface of the measuring pot of any shape is

$$S_m = k x_m^2,$$  \hspace{1cm} (13)

where $k$ is a constant dependent on the shape of the bottom surface; $x_m$ is a variable, which depends on both the shapes of containers and measuring pots, and which is also the calibration parameter.

When the entire content of the liquid from the container is poured into the measuring pot, and the height of the liquid reaches the amount $h$, using Equation (13) (Figure 2) we get:

$$V_c = V_m = S_m h = k x_m^2 h.$$

The shape and dimension of the measuring pot can be chosen at our will. We can choose them such that height $h$ of liquid in the measuring pot is:

$$h = x_m \pi,$$

where this relation is a requirement for calibration of the system. Now Equation (14) reads as follows:

![Figure 2. Perspective view of the container $V_c$ of any shapes and the measuring pot $V_m$ as well as of any shapes but with the bottom surface of which can be displayed as $S_m = k x_m^2$.](image)
\[ V_c = k x_m^3 \pi, \] (16)

where we get
\[ x_m = \sqrt[3]{\frac{V_c}{k \pi}}. \] (17)

To perform independent measurements of the number \( \pi \) we should get rid of it from Expression (17). This can be achieved by choosing the container which volume \( V_c \) depends on the number \( \pi \), what gives us to possibility to eliminate the number \( \pi \) from Equation (17).

5. Calibration of the Measuring System

The easiest way to measure the number \( \pi \) using the system shown in Figure 2, is by using Equation (4) and Equation (15):
\[ \pi = \frac{Pi}{rad} = \frac{h}{x_m}, \] (18)

which means that for selected \( x_m \) [according to Equation (17)] and after that fixed for a given system, the number \( \pi \) in this system is proportional to the height of the liquid column \( h \) in the measuring pot.

A unit of measurement is a magnitude of a quantity, defined an adopted by convention or by law, that is used as a standard for measurement of the same quantity. This means that in our system we can at will choose \( x_m \) as a unit of measurement. This further means that \( x_m \) corresponds to the unit; i.e., \( x_m \equiv 1 \). Then we write Equation (18):
\[ \pi = \frac{Pi}{rad} = \frac{h}{x_m} \equiv h. \] (19)

Using the above described system of calibration we can made different forms of containers and measuring pots, and they can realize a few systems of measurement (Table 1).

The two combinations of system to measure the number \( \pi \), resulting from Table 1, are shown in Figure 3 and Figure 4.

6. Conclusion

The direct and indirect measurements of the number \( \pi \) have been shown. To be able to do that, the difference between the number \( \pi \) and physical quantity \( Pi \) has been explained. It has been shown that physical quantity \( Pi \) is actually a stretched angle, while \( \pi \) is a number which characterizes this angle. The idea (in patent application process) according to which the content of the liquid from the container equated with the content of measuring pot and so allowed us to use the measuring pot scale to directly read the number \( \pi \) has been used. Measurement accuracy of such a system depends on the precision performance of containers and measuring pots, and the conditions in which such measurement is made (type of liquid, ambient temperature and tilt of the system). The
Table 1. Various exemplary embodiments measuring pots and containers, and their pairing.

<table>
<thead>
<tr>
<th>Measuring pot</th>
<th>Container ((x_m), for a given container and measuring pot)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The shape of the bottom of the measuring pot and its surface: (S_m = kx_m^2)</td>
<td>The constant (k) for a given shape: (x_m = \sqrt{\frac{V_c}{k\pi}})</td>
</tr>
<tr>
<td></td>
<td>Determination of (x_m):</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Square</td>
<td>(k = 1)</td>
</tr>
<tr>
<td>Circle</td>
<td>(k = \pi)</td>
</tr>
<tr>
<td>Equilateral triangle</td>
<td>(k = \frac{\sqrt{3}}{4})</td>
</tr>
</tbody>
</table>

\(a.\) Ref. [4], \(b.\) In that case the number \(\pi\) must be known before the measurement.

Figure 3. Perspective view of the sphere as a container \(V_c\) and rectangular cuboid as a measuring pot \(V_m\).

simplest implementation is the use of a cylinder as container and cuboid as a measuring pot (Figure 4).

Acknowledgements

The founding of this paper is, each in its own special way, contributed by Ms. Srebrenka Ursic, Mr. Branko Balon, Mr. Damir Vuk and especially Mr. Zlatko A. Voloder.
Figure 4. Perspective view of the cylinder as a container $V_c$ and rectangular cuboid as a measuring pot $V_m$.

They all have my gratitude.

References

Modeling of Noise Power Spectral Density Analysis for GaN/AlGaN HEMT

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Abstract
Nano Technology is the branch of technology that deals with dimensions and tolerances in terms of nanometers. In this paper, the electrical characteristics analysis is determined for the Nano-GaN HEMT and Micro-GaN HEMT and also power spectrum density is determined for GaN Nano-HEMT by reducing the gate length Lg in nm range. The GaN Nano HEMT is producing high current comparing to Micro GaN HEMT. Accuracy of the proposed analytical model results is verified with simulation results.

Keywords
HEMT, GaN/AlGaN, 2DEG, Drain Current, Noise Power Spectrum Density

1. Introduction
High electron mobility transistors are made up of the GaAs semiconductor material, but the research on HEMT experts says that GaN has good characteristics than GaAs because of their good breakdown voltage, less power dissipation. It can operate at high temperature, and it is having more power density and can operate at high voltage compares to GaAs based HEMT for that purpose. We are talking about GaN/AlGaN HEMT here [1]-[3].

The Applications of GaN/AlGaN HEMT are used in Space applications, Radar and in satellites because they are considered to be very promising candidates for high-speed and high-power applications. These devices offer advantages such as high breakdown voltage, high charge density, and good electron mobility [1]. The formation of the 2-dimensional electron gas (2DEG) in these devices is the heart of the device operation and has been studied in great detail in the literature [4]-[6]. Many articles are present for HEMTs, but the dimension used for gate length is in micrometer range [7]-[9]. Thus, there is scope for developing a simple accurate analytical model for GaN HEMT
by considering the gate length dimension in nanometer range (100 nm).

In this paper, we developed electrical characteristics analytical model for the GaN/AlGaN HEMT. Using the dimensions and the parameters of developed HEMT, we have compared the I-V characteristics for the Nano meter gate length and Micro meter gate length GaN-HEMT, and we have determined the noise power spectrum density for Nano GaN HEMT using Matlab and the performance is compared with the simulation results.

2. Model Formation

We call High electron mobility transistor as a Heterostructure FET because it uses two different semiconductor materials. One semiconductor should have a higher energy band gap and another one should have a lower energy band gap for the conduction takes place. We also call it as a MODFET (Modulation doping FET) he term “modulation doping” refers to the fact that the dopants are spatially in a different region from the current carrying electrons.

The schematic cross-section of the AlGaN/GaN HEMT is shown in Figure 1(a). GaN-based HEMTs employ two kinds of materials with different band gaps as the barrier and channel layer. The most popular one is AlGaN/GaN HEMTs. Due to the conduction band offset between AlGaN and GaN, an electron potential quantum well is formed at the hetero-interface between AlGaN and GaN. This heterojunction of different band-gap materials constitutes a triangular-like quantum well on the GaN side of the conduction band which allows electrons move freely parallel to the heterojunction plane without any impurity collision. This collection of high mobility electrons inside the quantum well is called two-dimensional electron gas (2DEG).

Figure 1(b) shows the 2DEG produced between GaN and AlGaN semiconductor materials. Fermi level of both the bands must be equal for the conduction takes place. This is known as the equilibrium state. When we give the gate voltage, the conduction band of AlGaN raises to communicate with the GaN conduction band and the conduction band of GaN material bends down to communicate with AlGaN material in between these two we will get the 2DEG it is like a channel in MOSFET where the electrons

![Figure 1. (a) Cross section and (b) band diagram of GaN HEMT structure.](image)
are present and they start moving for the applied gate voltage and drain voltage. The electrons are confined in this potential well to form a 2DEG. The electrons transport in a two-dimensional way, which can largely improve the electron mobility.

Table 1 shows the properties of different semiconductor materials. We can see GaN semiconductor material has a good property compared to the Si, GaAs and SIC semiconductor materials. Hence we are discussing about GaN HEMT here.


3.1. $I_d$ Characteristics of GaN HEMT

If assumed a constant mobility, then for low values of $V_{ds}$ (drain to source voltage), the drain current $I_d$ in the linear region is given by [10],

$$I_d = \varepsilon \mu \frac{\omega}{2L(d + \Delta d)} \left[2(V_{gs} - V_{off})V_{ds} - V_{ds}^2\right], \text{ for } V_{ds} \leq V_{gs} - V_{off}$$

(1)

$V_{ds}$ is further increased, then the carrier reaches the saturation voltage and the saturated drain current Drain current becomes,

$$I_d = \varepsilon \mu \frac{\omega}{2L(d + \Delta d)} (V_{gs} - V_{off})^2, \text{ for } V_{ds} \geq V_{gs} - V_{off}$$

(2)

where,

- $\varepsilon$ - permittivity of the substrate material in HEMT;
- $V_{gs}$ - The gate to source voltage;
- $V_{off}$ - Offset voltage.

The offset voltage $V_{off}$ can be calculated by the formula [10],

$$V_{off} = \phi_k - \frac{\Delta E_c}{q} - V_{p2}$$

(3)

$V_{p2}$ can be calculated by using formula [10],

$$V_{p2} = \frac{qN_{dd}^2}{2\varepsilon \varepsilon_0}$$

(4)

Table 1. Material properties of semiconductors at 300 K.

<table>
<thead>
<tr>
<th>Property</th>
<th>GaN</th>
<th>AlGaN/GaN</th>
<th>SiC</th>
<th>Diamond</th>
<th>Si</th>
<th>GaAs/AlGaAs, InGaAs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Band gap energy, $E_g$ (MV/cm)</td>
<td>3.44 eV</td>
<td>3.26 eV</td>
<td>5.45 eV</td>
<td>1.12 eV</td>
<td>1.43 eV</td>
<td></td>
</tr>
<tr>
<td>Electric breakdown field, $E_b$ (MV/cm)</td>
<td>3</td>
<td>3</td>
<td>10</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Saturated (Peak) Velocity electronics, $V_{sat}$ ($\times 10^7$ cm/s)</td>
<td>2.5</td>
<td>2.0</td>
<td>2.7</td>
<td>1.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>Electron mobility, $\mu_e$ (cm$^2$/V·s)</td>
<td>900</td>
<td>700</td>
<td>4800</td>
<td>1500</td>
<td>8500</td>
<td></td>
</tr>
<tr>
<td>2DEG density, $n_2$ (x10$^{13}$ cm$^{-2}$)</td>
<td>1.0</td>
<td>N.A</td>
<td>N.A</td>
<td>N.A</td>
<td>&lt;0.2</td>
<td></td>
</tr>
<tr>
<td>Thermal conductivity (W/cm-K)</td>
<td>1.3 - 2.1</td>
<td>3.7 - 4.5</td>
<td>22</td>
<td>1.5</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>Relative Permittivity $\varepsilon_r$</td>
<td>9.0</td>
<td>10.1</td>
<td>5.5</td>
<td>11.8</td>
<td>1.8</td>
<td></td>
</tr>
</tbody>
</table>
3.2. Noise Power Spectrum Densities

Noise power spectral densities (PSD) can be described as superposition of flicker noise, thermal noise and several generation-recombination (G-R) noise components.

- (G-R) Noise: The temperature dependence of the (G-R) noise arising from the traps was used to deduce the thermal activation energies and cross sections. The present results are compared to those of the literature to identify the Physica-chemical nature of traps responsible of the G-R noise.
- Flicker Noise: Flicker noise is found in all active devices as well as passive elements. Flicker noise dominates noise at low frequencies. The noise spectral density has a $1/f$ frequency dependence and hence the name “$1/f$” noise. This noise source is most significant at low frequencies, although in devices exhibiting high flicker noise.
- Thermal noise: This is directly proportional to the absolute temperature ($T$) and as $T$ approaches zero, the thermal noise also approaches zero. The thermal noise spectral density is also independent of the frequency and thus thermal noise can also classified as white noise.

The Noise power spectrum Density analysis of Nano-HEMT using GaN material can obtain by drain current. After obtaining the drain current of GaN Nano-HEMT and GaN Micro-HEMT, the noise PSD can be calculated by following the Hooge’s expression [11].

$$S_{id} = \frac{I_d^2 \alpha_H}{f^N}$$

where,

- $S_{id}$ = Noise power spectrum density;
- $I_d$ = Drain current;
- $f$ = operating frequency;
- $N$ = Total no of electrons in the conduction Band;
- $\alpha_H$ = Hooge’s parameter of GaN.

Hooge’s Parameter is the dimensionless parameter. Very recently, a mobility fluctuation noise theory was proposed by Musha and Tacano, suggests that energy partition among weakly coupled harmonic oscillators in an equilibrium system is subjected to $1/f$ fluctuations.

4. Results and Discussion

The accuracy of the proposed model is validated using the commercially available TCAD Sentaurus device simulator. The set of parameters used for simulation are shown in Table 2.

The Drain current characteristics plotted for GaN HEMT for Lg in 1 micrometer is shown in Figure 2 and for Lg in 100 nanometer is shown in Figure 3. From the figures it is observed that um gate length GaN HEMT produces less Drain current compared to the Nano GaN HEMT. The Drain current of GaN Nano-HEMT is hundred times more than that of GaN HEMT is observed in Figure 2 and Figure 3.
Table 2. List of parameters used in equations and their values that are used to obtain results.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value/unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W$</td>
<td>Gate width</td>
<td>1 µm</td>
</tr>
<tr>
<td>$L$</td>
<td>Gate Length</td>
<td>100 nm</td>
</tr>
<tr>
<td>$N_d$</td>
<td>Donor level concentration</td>
<td>$1 \times 10^{19}$ cm$^{-3}$</td>
</tr>
<tr>
<td>$K$</td>
<td>Boltzmann's Constant</td>
<td>$1.38 \times 10^{-23}$ J/K</td>
</tr>
<tr>
<td>$T$</td>
<td>Operating temperature</td>
<td>300 K</td>
</tr>
<tr>
<td>$Q$</td>
<td>Electronic charge</td>
<td>$1.6 \times 10^{-19}$ C</td>
</tr>
<tr>
<td>$D_d$</td>
<td>Thickness of doped layer in Nano-HEMT</td>
<td>90 nm</td>
</tr>
<tr>
<td>$D_i$</td>
<td>Thickness of undoped layer in Nano-HEMT</td>
<td>80 nm</td>
</tr>
<tr>
<td>$\Delta d = d_d + d_i$</td>
<td>Correction factor</td>
<td>170 nm</td>
</tr>
<tr>
<td>$\Phi_b$</td>
<td>Schottky barrier Height</td>
<td>0.697</td>
</tr>
<tr>
<td>$M$</td>
<td>Mobility</td>
<td>900 cm$^2$/Vs</td>
</tr>
<tr>
<td>$E_g$</td>
<td>Band gap energy of GaAs</td>
<td>3.44 eV</td>
</tr>
<tr>
<td>$N_v$</td>
<td>Density of state in valance band</td>
<td>$9 \times 10^{18}$ cm$^{-3}$</td>
</tr>
<tr>
<td>$N_c$</td>
<td>Density of state in conduction band</td>
<td>$4.7 \times 10^{17}$ cm$^{-3}$</td>
</tr>
<tr>
<td>$\varepsilon_s$</td>
<td>Semiconductor Permittivity</td>
<td>12.9$ \times 8.85 \times 10^{-14}$ F/cm</td>
</tr>
<tr>
<td>$V_{p2}$</td>
<td>Pinch-off voltage in doped layer</td>
<td>1.75 V</td>
</tr>
<tr>
<td>$F$</td>
<td>Operating Frequency</td>
<td>1 GHz</td>
</tr>
<tr>
<td>$N$</td>
<td>No. of conduction Electrons</td>
<td>$10^{18}$</td>
</tr>
<tr>
<td>$\alpha_H$</td>
<td>Hooge’s Parameter</td>
<td>$2 \times 10^3$</td>
</tr>
</tbody>
</table>

Figure 2. Id vs. Vd characteristics for Lg = 1 um, Vd = (0 V to 5 V) with step size = 0.5 V for various Vg.
Figure 4 and Figure 5 show the I-V characteristics for Micro and Nano GaN HEMT. In this case we have kept Vd as constant value and for the variation of Vg. Even in this case if Lg = 100 nm in Figure 5, the drain current is hundred times greater than the device which has Lg = 1 um in Figure 4. From the figures (Figures 2-5), it is observed that the drain current is higher for nanometer GaN HEMT with increasing gate voltage and/or drain voltage compare to micrometer GaN HEMT.

**Figure 3.** Id vs. Vd characteristics for Lg = 100 nm, Vd = (0 V to 5 V) with step size = 0.5 V for various Vg.

**Figure 4.** Id vs. Vg characteristics for Lg = 1 um, Vg = (−2 V to 10 V) with step size = 1 V for various Vd.
The nano GaN HEMT drain current characteristics are compared with the simulation results are shown in Figure 6 and Figure 7 for Id versus Vd and Id versus Vg respectively.

The noise generated inside this device is due to drain current fluctuation caused due to material or crystal defects in GaN HEMTs. This noise power spectral density analysis stresses on the power spectral density of noise due to the drain currents are analyzed in

![Image](image1.png)

**Figure 5.** Id vs. Vg characteristics for Lg = 100 nm, Vg = (−2 V to 10 V) with step size = 1 V for various Vd.

![Image](image2.png)

**Figure 6.** Model and simulation Id vs. Vd characteristics for Lg = 100 nm for Vg = 0.5 V and −0.5 V.
Figure 8 and Figure 9.

The power spectral density (PSD) of the gate current noise shows a quadratic dependence on the gate current intensity. The noise PSD analysis on GaN HEMT comprises of plotting of noise PSD for different values of $V_g$ from $-1$ to $0.5$ V, over a range of $V_d$ from $0$ to $5$ V in Figure 8. Noise power spectrum density analysis of Nano HEMT using

![Figure 8. Vds-Sid characteristics of GaN nano HEMT with $L_g = 100$ nm over $V_g = -1$ V to 0.5 V.](image)

![Figure 7. Model and simulation Id vs. Vg characteristics for $L_g = 100$ nm for $V_d = 1.5$ V and 0.5 V.](image)
GaN material plotted with temperature is shown in Figure 9. Also the noise power spectral density model is compared with simulation results in Figure 10.

5. Conclusion

The model simulation for all the plots regarding Nano and Micro meter GaN HEMTs shows the comparative analysis for drain current characteristics. PSD analysis is done

![Figure 9](image9.png)

**Figure 9.** Temperature vs. Sid of GaN nano HEMT with Lg = 100 nm for various gate voltage.

![Figure 10](image10.png)

**Figure 10.** Model and simulation of Vds-Sid characteristics of GaN nano HEMT with Lg = 100 nm for Vg = 0.5 V and −0.5 V.
by considering reduced gate length in nanometer range of GaN HEMT. Also, it is observed that the amount of drain current generated by GaN Nano-HEMT is greater than normal GaN HEMT, but the former lacks in maintaining low noise PSD with higher frequency range, which is very poor as compared to GaN HEMT.

References


Solving of Klein-Gordon by Two Methods of Numerical Analysis

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Abstract

In this paper, the Decomposition Laplace-Adomian method and He-Laplace method are used to construct the solution of Klein-Gordon equation.

Keywords

Laplace-Adomian Method, He-Laplace Method, Klein-Gordon Equation

1. Introduction

In field theory, the description of the free particle for the wave function in quantum physics obeys to Klein-Gordon equation [1]. In addition, it also appears in nonlinear optics and plasma physics.

In sum, the Klein-Gordon equation rises in physics in linear and non linear forms. In this paper we examine the Klein-Gordon equation, using the Laplace-Adomian decomposition method and He-Laplace method to get the exact solution. The Klein-Gordon equation is described as:

\[ u_{\alpha} (x,t) = u_{\alpha} (x,t) + \alpha u(x,t) + \beta N(u(x,t)) + h(x,t) \]  

(1)

where \( \alpha, \beta \) are constants (spin zero) charged field, \( h(x,t) \) is a source term and \( N(u(x,t)) \) is a nonlinear function of \( u(x,t) \).

2. Describing of Both Method

2.1. The Laplace Transform [2]

Let’s note the laplace transform by

\[ U(x,s) = \mathcal{L}\{u(x,t)\} = \int_{0}^{\infty} u(x,t) e^{-st} dt \]  

(2)
From (1), we have:

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= \int_0^e e^{-\xi} \frac{\partial u}{\partial t} \, d\xi = s\left(u(x,t)\right) - u(x,0) \\
\frac{\partial^2 u(x,t)}{\partial t^2} &= s^2\left(u(x,t)\right) - su(x,0) - \frac{\partial u(x,0)}{\partial t} \\
\frac{\partial u(x,t)}{\partial x} &= \int_0^e e^{-\xi} \frac{\partial u}{\partial x} \, d\xi = \frac{dU}{dx} \\
\frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{d^2U}{dx^2}
\end{align*}
\]  

(3)

2.2. Laplace-Adomian Decomposition Method (LADM) [3]-[6]

Suppose that we need to solve the following equation:

\[
u_n(x,t) - u_{xx}(x,t) + \alpha u(x,t) + \beta N\left(u(x,t)\right) = h(x,t)
\]  

(4)

subject to initial conditions:

\[
u(x,0) = f(x); u_t(x,0) = g(x)
\]  

(5)

\(E\) is a Banach space, where \(F: E \rightarrow E\) is a linear or a nonlinear operator, \(h \in E\) and \(u\) is the unknown function.

Let’s suppose that operator \(F\) can be decomposed under the following form:

\[
F = L + R + N
\]  

(6)

where \(L + R\) is linear, \(N\) nonlinear. Let’s suppose that \(L\) is inversible to the sense of Adomian with \(L^{-1}\) as inverse.

From above, by applying the Laplace transform to both sides of Equation (4), we have:

\[
\left[u_n(x,t)\right] - \left[u_{xx}(x,t) - \alpha u(x,t)\right] + \beta \left[N\left(u(x,t)\right)\right] = \left(h(x,t)\right)
\]  

(7)

From the Equation (7), it follows:

\[
s^2\left(u(x,t)\right) - su(x,0) - u_t(x,0) - \left[u_{xx}(x,t) - \alpha u(x,t)\right] + \beta \left[N\left(u(x,t)\right)\right] = \left(h(x,t)\right)
\]  

(8)

and this equation gives

\[
s^2\left(u(x,t)\right) = su(x,0) + u_t(x,0) + \left(h(x,t)\right) + \left[u_{xx}(x,t) - \alpha u(x,t)\right] - \beta \left[N\left(u(x,t)\right)\right]
\]  

(9)

So, from the above Equation (9), we can write:

\[
\left(u(x,t)\right) = \frac{1}{s}u(x,0) + \frac{1}{s^2}u_t(x,0) + \frac{1}{s^2}\left(h(x,t)\right) + \frac{1}{s^2}\left[u_{xx}(x,t) - \alpha u(x,t)\right] - \frac{1}{s^2}\beta \left[N\left(u(x,t)\right)\right]
\]  

(10)

We have now \(u(x,t)\):

\[
u(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}f(x)\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}g(x)\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left(h(x,t)\right)\right]
\]  

\[
+ \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[u_{xx}(x,t) - \alpha u(x,t)\right]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2}\beta \mathcal{L}\left[N\left(u(x,t)\right)\right]\right]
\]  

(11)
We research solution of (4) in the following series expansion form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$  \hspace{1cm} (12)

and we consider

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n$$  \hspace{1cm} (13)

where \( A_n \) are the Adomian polynomials of \( u_0, u_1, \cdots, u_n \) and it can be calculated by formula given below.

$$A_n = \frac{1}{n!} \frac{d^n}{dp^n} \left[ N \sum_{i=0}^{\infty} \lambda_i u_i \right]_{|p=0}, \hspace{1cm} n = 0, 1, 2, 3, \cdots$$  \hspace{1cm} (14)

Using Equation (12) and Equation (13) in Equation (11) we have:

$$\sum_{n=0}^{\infty} u_n(x,t) = \mathcal{L}^{-1} \left( \frac{1}{s} f(x) \right) + \mathcal{L}^{-1} \left( \frac{1}{s} g(x) \right) + \mathcal{L}^{-1} \left( \frac{1}{s} \mathcal{L}(h(x,t)) \right) + \mathcal{L}^{-1} \left( \frac{1}{s^2} \mathcal{L} \left( \sum_{n=0}^{\infty} \frac{\partial^2}{\partial x^2} u_n(x,t) \right) \right) - \alpha \sum_{n=0}^{\infty} u_n(x,t) \right) - \mathcal{L}^{-1} \left( \frac{1}{s^2} \beta \mathcal{L} \left( \sum_{n=0}^{\infty} A_n \right) \right)$$  \hspace{1cm} (15)

From (15), we have the following Adomian algorithm:

$$\begin{cases}
u_0(x,t) = \mathcal{L}^{-1} \left( \frac{1}{s} f(x) \right) + \mathcal{L}^{-1} \left( \frac{1}{s} g(x) \right) + \mathcal{L}^{-1} \left( \frac{1}{s} \mathcal{L}(h(x,t)) \right) = K(x,t) \\
u_{n+1}(x,t) = \mathcal{L}^{-1} \left( \frac{1}{s^2} \mathcal{L} \left( \frac{\partial^2}{\partial x^2} u_n(x,t) \right) - \alpha u_n(x,t) \right) - \mathcal{L}^{-1} \left( \frac{1}{s^2} \beta \mathcal{L} \left( A_n \right) \right), n \geq 0
\end{cases}$$  \hspace{1cm} (16)

and we obtain the Adomian algorithm:

$$\begin{cases}
u_0(x,t) = K(x,t) \\
u_{n+1}(x,t) = \mathcal{L}^{-1} \left( \frac{1}{s^2} \mathcal{L} \left( \frac{\partial^2}{\partial x^2} u_n(x,t) \right) - \alpha u_n(x,t) \right) - \mathcal{L}^{-1} \left( \frac{1}{s^2} \beta \mathcal{L} \left( A_n \right) \right), n \geq 0
\end{cases}$$  \hspace{1cm} (17)

**Remark**

In order overcome the short coming, we assume that \( K(x,t) \) can be divided into the sum of two parts namely \( K_0(x,t) \) and \( K_1(x,t) \). Therefore,we get:

$$K(x,t) = K_0(x,t) + K_1(x,t)$$  \hspace{1cm} (18)

Instead of the iteration procedure Equation (17) we suggest the following modification

$$\begin{cases}
u_0(x,t) = K_0(x,t) \\
u_1(x,t) = K_1(x,t) + \mathcal{L}^{-1} \left( \frac{1}{s^2} \mathcal{L} \left( \frac{\partial^2}{\partial x^2} u_0(x,t) \right) - \alpha u_0(x,t) \right) - \mathcal{L}^{-1} \left( \frac{1}{s^2} \beta \mathcal{L} \left( A_0 \right) \right) \\
u_{n+1}(x,t) = \mathcal{L}^{-1} \left( \frac{1}{s^2} \mathcal{L} \left( \frac{\partial^2}{\partial x^2} u_n(x,t) \right) - \alpha u_n(x,t) \right) - \mathcal{L}^{-1} \left( \frac{1}{s^2} \beta \mathcal{L} \left( A_n \right) \right), n \geq 1
\end{cases}$$  \hspace{1cm} (19)
The solution through the modified Laplace decomposition method highly depends upon the choice of \( K_0(x,t) \) and \( K_1(x,t) \).

### 2.3. He-Laplace Method [7]

We consider a general nonlinear non homogeneous partial differential equation with initial conditions of the form

\[
\begin{align*}
&u_n(x,t) - u_n(x,t) + \alpha u(x,t) + \beta N\left(u(x,t)\right) = h(x,t) \\
u(x,0) = f(x) \\
u_1(x,0) = g(x)
\end{align*}
\]

(20)

\( N \) represents the general nonlinear differential operator and \( h(x,t) \) is the source term.

Taking the Laplace transform on both sides of (20), we obtain:

\[
\mathcal{L}\left[u_n(x,t)\right] - \mathcal{L}\left[u_n(x,t) - \alpha u(x,t)\right] + \beta \mathcal{L}\left[N\left(u(x,t)\right)\right] = \mathcal{L}(h(x,t))
\]

(21)

\[
\Leftrightarrow \\
s^2 \left(u(x,t)\right) = su(x,0) + u_1(x,0) + (h(x,t)) + \left[u_n(x,t) - \alpha u(x,t)\right] - \frac{1}{s^2} \beta \mathcal{L}\left[N\left(u(x,t)\right)\right]
\]

(22)

Applying the initial conditions given in (22), we have:

\[
\mathcal{L}\left(u(x,t)\right) = \frac{1}{s^2} f(x) + \frac{1}{s^2} g(x) + \frac{1}{s^2} \mathcal{L}(h(x,t)) + \frac{1}{s^2} \mathcal{L}\left[u_n(x,t) - \alpha u(x,t)\right] - \frac{1}{s^2} \beta \mathcal{L}\left[N\left(u(x,t)\right)\right]
\]

(23)

Operating the inverse Laplace transform on both sides of (23), we have

\[
u(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s} f(x)\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2} g(x)\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}(h(x,t))\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}\left[u_n(x,t) - \alpha u(x,t)\right]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2} \beta \mathcal{L}\left[N\left(u(x,t)\right)\right]\right]
\]

(24)

Now, we apply the homotopy perturbation method

\[
u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)
\]

(25)

and the non linear term can be decomposed as

\[
Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u)
\]

(26)

for some He's polynomials \( H_n(u) \) that are given by

\[
H_n(u) = \frac{1}{n!} \partial^n \left[ N \sum_{n=0}^{\infty} p^n u_n \right], n = 0, 1, 2, 3, \ldots
\]

(27)

Sustituting Equation (25) and Equation (26) in Equation (24), we get

\[
\sum_{n=0}^{\infty} p^n u_n(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s} f(x)\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2} g(x)\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}(h(x,t))\right] + p \left[ \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}\left[\partial^2 \left(p^n u_n(x,t)\right) - \alpha p^n u_n(x,t)\right]\right]\right] - \sum_{n=0}^{\infty} \beta \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}\left[p^n H_n(u)\right]\right]
\]

(28)
Comparing the coefficients of like powers of $p$, we have the following approximations:

\[
\begin{aligned}
 p^0 : u_0 (x,t) &= \mathcal{L}^{-1} \left( \frac{1}{s} f(x) \right) + \mathcal{L}^{-1} \left( \frac{1}{s^2} g(x) \right) + \mathcal{L}^{-1} \left( \frac{1}{s^2} \mathcal{L} \left( h(x,t) \right) \right) \\
p^1 : u_1 (x,t) &= \mathcal{L}^{-1} \left( \frac{1}{s^2} \mathcal{L} \left[ \frac{\partial^2 (u_0(x,t))}{\partial x^2} - \alpha u_0 (x,t) \right] \right) - \beta \mathcal{L}^{-1} \left( \frac{1}{s} \mathcal{L} \left[ H_0 (u) \right] \right) \\
p^2 : u_2 (x,t) &= \mathcal{L}^{-1} \left( \frac{1}{s^2} \mathcal{L} \left[ \frac{\partial^2 (u_1(x,t))}{\partial x^2} - \alpha u_1 (x,t) \right] \right) - \beta \mathcal{L}^{-1} \left( \frac{1}{s} \mathcal{L} \left[ H_1 (u) \right] \right) \\
& \vdots 
\end{aligned}
\]  

(29)

3. Illustrative Examples

To demonstrate the applicability of the above-presented method, we have applied it to two linear and two non-linear partial differential equations. These examples have been chosen because they have been widely discussed in literature.

3.1. Example 1

Consider the following linear Klein-Gordon equation

\[
\begin{aligned}
 u_{tt} (x,t) - u_{xx} (x,t) - u (x,t) &= -\cos x \sin t \\
u (x,0) &= 0 \\
u_1 (x,0) &= \cos x 
\end{aligned}
\]  

(30)

3.1.1. Application of the LADM

Applying the Laplace transform on both sides of Equation (30) with the initial conditions, we have:

\[
\mathcal{L} \left( u(x,t) \right) = \frac{1}{s^2} \cos x - \left( \frac{1}{s^2} \left( \frac{1}{s^2} + 1 \right) \right) \cos x + \frac{1}{s^2} \mathcal{L} \left[ u_{xx} (x,t) + u(x,t) \right] 
\]

(31)

The inverse Laplace transform gives us:

\[
\begin{aligned}
 u(x,t) &= \mathcal{L}^{-1} \left( \frac{1}{s^2} \cos x - \left( \frac{1}{s^2} \left( \frac{1}{s^2} + 1 \right) \right) \cos x + \frac{1}{s^2} \mathcal{L} \left( u_{xx} (x,t) + u(x,t) \right) \right) \\
& \Leftrightarrow \\
 u(x,t) &= \sin t \cos x + \mathcal{L}^{-1} \left( \frac{1}{s^2} \left( u_{xx} (x,t) + u(x,t) \right) \right) 
\end{aligned}
\]

(32)

(33)

We suppose that solution of (30) has the following form:

\[
\begin{aligned}
 u(x,t) &= \sum_{n=0}^{\infty} u_n (x,t) 
\end{aligned}
\]  

(34)

From (34) and (33), we have:

\[
\begin{aligned}
 \sum_{n=0}^{\infty} u_n (x,t) &= \sin t \cos x + \sum_{n=0}^{\infty} \mathcal{L}^{-1} \left( \frac{1}{s^2} \mathcal{L} \left[ \frac{\partial^2 (u_n(x,t))}{\partial x^2} + u_n (x,t) \right] \right) 
\end{aligned}
\]

(35)
This result guarantees that the following Adomian algorithm is:

\[
\begin{align*}
\left\{ \begin{array}{l}
u_0(x,t) = \sin t \cos x \\
u_{n+1}(x,t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \mathcal{L}\left( \frac{\partial^2}{\partial x^2} u_n(x,t) \right) + u_n(x,t) \right\}, \forall n \geq 0
\end{array} \right.
\end{align*}
\] (36)

Consequently, we obtain:

\[
\begin{align*}
\left\{ \begin{array}{l}
u_0(x,t) = \sin t \cos x \\
u_n(x,t) = 0, \forall n \geq 1
\end{array} \right.
\end{align*}
\] (37)

So that the solution of (30) is given by

\[
u(x,t) = \sum_{n=0}^{\infty} \nu_n(x,t) = \nu_0(x,t) = \sin t \cos x
\] (38)

which is the exact solution of the problem.

### 3.1.2. Application of the He-Laplace Method

Applying the Laplace transform on both sides of Equation (30) with the initial conditions, we obtain:

\[
\mathcal{L}\left( u(x,t) \right) = \frac{1}{s^2} \cos x - \left( \frac{1}{s^2 (s^2 + 1)} \right) \cos x + \frac{1}{s^2} \mathcal{L}\left[ u_\infty(x,t) + u(x,t) \right]
\] (39)

By applying inverse Laplace transform, we have:

\[
\begin{align*}
u(x,t) = \mathcal{L}^{-1}\left( \frac{1}{s^2} \cos x \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2 (s^2 + 1)} \right) \cos x + \mathcal{L}^{-1}\left( \frac{1}{s^2} \mathcal{L}\left( u_\infty(x,t) + u(x,t) \right) \right)
\end{align*}
\] (40)

\[
u(x,t) = \sin t \cos x + \mathcal{L}^{-1}\left( \frac{1}{s^2} (u_\infty(x,t) + u(x,t)) \right)
\] (41)

Now applying the homotopy perturbation method, we have:

\[
\sum_{n=0}^{\infty} p^n \nu_n(x,t) = \sin t \cos x + p \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left( \frac{1}{s^2} \mathcal{L}\left[ p^n \left( \frac{\partial^2}{\partial x^2} u_n(x,t) \right) + u_n(x,t) \right] \right)
\] (42)

Comparing the coefficient of like powers of \( p \), we have

\[
\begin{align*}
p^0 : \nu_0(x,t) &= \sin t \cos x \\
p^1 : \nu_1(x,t) &= \mathcal{L}^{-1}\left( \frac{1}{s^2} \mathcal{L}\left[ \frac{\partial^2}{\partial x^2} (\nu_0(x,t)) + \nu_0(x,t) \right] \right) \\
p^2 : \nu_2(x,t) &= \mathcal{L}^{-1}\left( \frac{1}{s^2} \mathcal{L}\left[ \frac{\partial^2}{\partial x^2} (\nu_1(x,t)) + \nu_1(x,t) \right] \right) \\
&\vdots \\
p^n : \nu_n(x,t) &= \mathcal{L}^{-1}\left( \frac{1}{s^2} \mathcal{L}\left[ \frac{\partial^2}{\partial x^2} (\nu_{n-1}(x,t)) + \nu_{n-1}(x,t) \right] \right)
\end{align*}
\] (43)

which gives us
\begin{align*}
\begin{cases}
p^0 : u_0(x,t) = \sin t \cos x \\
p^n : u_n(x,t) = 0, \forall n \geq 1
\end{cases}
\end{align*}
(44)

So that, the solution \( u(x,t) \) is given by:
\[
\lim_{p \to 1} \left( \sum_{n=0}^{\infty} p^n u_n(x,t) \right) = u_0(x,t) = \sin t \cos x
\]
(45)

### 3.2. Exemple 2

Consider the following nonlinear Klein-Gordon equation
\[
\begin{align*}
&u_n(x,t) - u_{xx}(x,t) + N(u(x,t)) = 2x + t^4 \\
u(x,0) = 0 \\
u_t(x,0) = 2
\end{align*}
\]
(46)

where \( N(u) = \left[ u_x(x,t) \right]^2 \).

#### 3.2.1. Laplace-Adomian Method

Using the Laplace transform, we have
\[
s^2 \mathcal{L}
\left(\begin{array}{c}
u(x,t) - u(x,0) \\
\partial_t u(x,t)
\end{array}\right) - s \mathcal{L}(u(x,0)) = \frac{2}{s} x + \frac{24}{s^2} + s^2 \left( \frac{\partial_t^2 u(x,t)}{\partial x^2} \right)
- \mathcal{L} \left( \left[ \frac{\partial u(x,t)}{\partial x} \right]^2 \right)
\]
(47)

\[
\Leftrightarrow
\]
\[
\mathcal{L}(u(x,t)) = \frac{2}{s^2} + \frac{2}{s} x + \frac{24}{s^2} + \frac{1}{s^2} \left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) - \frac{1}{s^2} \left( \left[ \frac{\partial u(x,t)}{\partial x} \right]^2 \right)
\]
(48)

by applying inverse Laplace transformation to Equation (48), we have
\[
u(x,t) = 2t + xt^2 + \frac{1}{30} t^6 + \mathcal{L}^{-1} \left( \frac{1}{s^2} \mathcal{L} \left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1} \left( \frac{1}{s^2} \mathcal{L} \left( \left[ \frac{\partial u(x,t)}{\partial x} \right]^2 \right) \right)
\]
(49)

Supposing that the solution of (46) has the following form:
\[
u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)
\]
(50)

and
\[
N \nu(x,t) = \sum_{n=0}^{\infty} A_n
\]
(51)

Taking (50) and (51) in to (49), we obtain:
\[
\sum_{n=0}^{\infty} u_n(x,t) = 2t + xt^2 + \frac{1}{30} t^6 + \sum_{n=0}^{\infty} \left( \mathcal{L}^{-1} \left( \frac{1}{s^2} \mathcal{L} \left( \frac{\partial^2 u_n(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1} \left( \frac{1}{s^2} \mathcal{L} (A_n) \right) \right)
\]
(52)

According to the standard Adomian algorithm (52), we need to chose
\[
u_0(x,t) = 2t + xt^2 + \frac{1}{30} t^6.
\]

Here, we choose by convenience \( \nu_0(x,t) = 2t + xt^2 \). So, we have the following Adomian algorithm
\[
\begin{align*}
\left\{ \begin{array}{l}
u_0(x,t) = xt^2 + 2t \\
u_1(x,t) = \frac{1}{30}t^6 + \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \mathcal{L}\left( \frac{\partial^2 u_0(x,t)}{\partial x^2} \right) \right\} - \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \mathcal{L}(A_0) \right\} \\
u_n(x,t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \mathcal{L}\left( \frac{\partial^2 u_n(x,t)}{\partial x^2} \right) \right\} - \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \mathcal{L}(A_n) \right\}
\end{array} \right.
\]
\end{align*}
\] (53)

then guarantee that:

\[
\begin{align*}
\left\{ \begin{array}{l}
u_0(x,t) = xt^2 + 2t \\
A_0 = N(u_n(x,t)) = t^4 \\
u_1(x,t) = \frac{1}{30}t^6 - \frac{1}{30}t^6 = 0 \\
A_n = 0, \forall n \geq 1 \Rightarrow u_n = 0, \forall n \geq 1
\end{array} \right.
\end{align*}
\] (54)

So the exact solution of (46) is

\[
u(x,t) = xt^2 + 2t
\] (55)

### 3.2.2. He-Laplace Method

Using the Laplace transform, we have:

\[
\mathcal{L}(u(x,t)) = \frac{2}{s^2} + \frac{3}{s} + \frac{2}{s} + \frac{1}{s} + \frac{1}{s^2} \mathcal{L}\left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) - \frac{1}{s} \mathcal{L}\left( \left( \frac{\partial u(x,t)}{\partial x} \right)^2 \right)
\] (56)

Now, we apply the inverse Laplace transformation to Equation (46), we have:

\[
u(x,t) = 2t + xt^2 + \frac{1}{30}t^6 + \mathcal{L}^{-1}\left( \frac{1}{s^2} \mathcal{L}\left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2} \mathcal{L}\left( \left( \frac{\partial u(x,t)}{\partial x} \right)^2 \right) \right)
\] (57)

Applying the homotopy perturbation method, we have:

\[
\sum_{n=0}^{\infty} p^n u_n(x,t) = 2t + xt^2 + \frac{1}{30}t^6 + p^0 \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left( \frac{1}{s^2} \mathcal{L}\left( \frac{\partial^2 u_n(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2} \mathcal{L}(H_n(u)) \right)
\] (58)

where \( H_n(u) \) are He’s polynomials. The first few components of He’s polynomials are given by

\[
\begin{align*}
H_0(u) &= \left( \frac{\partial u_0(x,t)}{\partial x} \right)^2 = t^4 \\
H_1(u) &= 2 \left( \frac{\partial u_0(x,t)}{\partial x} \right) \left( \frac{\partial u_1(x,t)}{\partial x} \right) = 0 \\
H_2(u) &= \left( \frac{\partial u_1(x,t)}{\partial x} \right)^2 + 2 \left( \frac{\partial u_0(x,t)}{\partial x} \right) \left( \frac{\partial u_2(x,t)}{\partial x} \right) = 0 \\
&\vdots
\end{align*}
\] (59)

Comparing the coefficients of the like powers of \( p \), we have:

\[
p^0 : u_0(x,t) = 2t + xt^2 + \frac{1}{30}t^6
\] (60)
\[ p^1 : u_1(x,t) = \mathcal{L}^{-1}\left( \frac{1}{s} \mathcal{L}\left( \frac{\partial^2 u_0(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2} \mathcal{L}\left( H_0(u) \right) \right) = -\frac{1}{30}t^6 \quad (61) \]

\[ p^n : u_n(x,t) = \mathcal{L}^{-1}\left( \frac{1}{s} \mathcal{L}\left( \frac{\partial^2 u_{n-1}(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2} \mathcal{L}\left( H_{n-1}(u) \right) \right) = 0, \forall n \geq 2 \quad (62) \]

So that, the exact solution \( u(x,t) \) is given by:

\[
\begin{align*}
  u(x,t) &= \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n(x,t) \\
  &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots \\
  &= 2t + xt^2 + \frac{1}{30}t^6 - \frac{1}{30}t^6 + 0 + 0 + \cdots \\
  &= 2t + xt^2 
\end{align*}
\]

### 4. Applications

#### 4.1. Problem 1

Consider the following linear Klein–Gordon equation

\[
\begin{align*}
  \frac{\partial^2 u(x,t)}{\partial t^2} &= \frac{\partial^2 u(x,t)}{\partial x^2} + 3u(x,t) - 3\sin x \sin t \\
  u(x,0) &= 0 \\
  u_t(x,0) &= \sin x
\end{align*}
\]

(64)

**Application of the LADM**

Using the Laplace transform, we have

\[
\begin{align*}
  s^2 \left( u(x,t) \right) - su(x,0) - \frac{\partial u(x,0)}{\partial t} &= \left( \frac{\partial^2 u(x,t)}{\partial x^2} + 3u(x,t) \right) - 3\sin x \sin t \\
\end{align*}
\]

\[
\implies
\]

\[
\begin{align*}
  \mathcal{L}\left( u(x,t) \right) &= \frac{1}{s} u(x,0) + \frac{1}{s^2} \frac{\partial u(x,0)}{\partial t} - 3\left( \frac{1}{s^2} \right) \sin x + \frac{1}{s^2} \mathcal{L}\left( \frac{\partial^2 u(x,t)}{\partial x^2} + 3u(x,t) \right)
\end{align*}
\]

(65)

\[
\begin{align*}
\mathcal{L}\left( u(x,t) \right) &= \frac{1}{s} u(x,0) + \frac{1}{s^2} \frac{\partial u(x,0)}{\partial t} - 3\left( \frac{1}{s^2} \right) \sin x + \frac{1}{s^2} \mathcal{L}\left( \frac{\partial^2 u(x,t)}{\partial x^2} + 3u(x,t) \right)
\end{align*}
\]

(66)

By applying the inverse Laplace transform, we have:

\[
\begin{align*}
  u(x,t) &= \mathcal{L}^{-1}\left( \frac{1}{s} u(x,0) \right) + \mathcal{L}^{-1}\left( \frac{1}{s^2} \frac{\partial u(x,0)}{\partial t} \right) - (3\sin x) \mathcal{L}^{-1}\left( \frac{1}{s^2} \left( \frac{1}{s^2} + 1 \right) \right) \\
  &+ \mathcal{L}^{-1}\left( \frac{1}{s^2} \mathcal{L}\left( \frac{\partial^2 u(x,t)}{\partial x^2} + 3u(x,t) \right) \right)
\end{align*}
\]

\[
\implies
\]

\[
\begin{align*}
  u(x,t) &= (\sin x) \mathcal{L}^{-1}\left( \frac{1}{s^2} \right) - (3\sin x) \mathcal{L}^{-1}\left( \frac{1}{s^2} \left( \frac{1}{s^2} + 1 \right) \right) + \mathcal{L}^{-1}\left( \frac{1}{s^2} \mathcal{L}\left( \frac{\partial^2 u(x,t)}{\partial x^2} + 3u(x,t) \right) \right)
\end{align*}
\]

\[
\implies
\]
\[ u(x,t) = 3 \sin t \sin x - 2t \sin x + \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \left( \frac{\partial^2 u(x,t)}{\partial x^2} + 3u(x,t) \right) \right\} \]  \hspace{1cm} (68)

From above equation, we have the following modified Adomian algorithm:

\[
\begin{aligned}
& u_0(x,t) = \sin t \sin x \\
& u_1(x,t) = 2 \sin t \sin x - 2t \sin x + \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \mathcal{L} \left( \frac{\partial^2 u_0(x,t)}{\partial x^2} + 3u_0(x,t) \right) \right\} \\
& u_{n+1}(x,t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \mathcal{L} \left( \frac{\partial^2 u_n(x,t)}{\partial x^2} + 3u_n(x,t) \right) \right\}, \forall n \geq 1 \\
\end{aligned}
\hspace{1cm} (69)

Equation (69) give us:

\[
\begin{aligned}
& u_1(x,t) = 2 \sin t \sin x - 2t \sin x + \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \mathcal{L} \left( 2 \sin t \sin x \right) \right\} \\
& u_1(x,t) = 2 \sin t \sin x - 2t \sin x + (2 \sin x) \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \mathcal{L} (\sin t) \right\} \\
& u_1(x,t) = 2 \sin t \sin x - 2t \sin x + 2t \sin x - 2 \sin x \sin t = 0 \\
\end{aligned}
\hspace{1cm} (70)

Thus

\[
\begin{aligned}
& u_0(x,t) = \sin x \sin t \\
& u_n(x,t) = 0, \forall n \geq 1 \\
\end{aligned}
\hspace{1cm} (71)

and the exact solution of Equation (64) is

\[ u(x,t) = \sin x \sin t \]  \hspace{1cm} (72)

### 4.2. Problem 2

Consider the following nonlinear Klein-Gordon equation

\[
\begin{aligned}
& \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) - u^2(x,t) + xt + x^2 t^2 \\
& u(x,0) = 1 \\
& \frac{\partial u(x,0)}{\partial x} = x \\
\end{aligned}
\hspace{1cm} (73)

Application of the LADM

Using the Laplace transform from (73), we have:

\[
\mathcal{L}\{u(x,t)\} = \frac{s}{s^2 - 1} \left( \frac{1}{s^2} - \frac{1}{s^2} \right) x + \frac{2}{s^3} \frac{x^2}{s^3} \\
+ \frac{1}{s^2 - 1} \mathcal{L} \left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) - \frac{1}{s^2 - 1} \mathcal{L} [u^2(x,t)] \\
\hspace{1cm} (74)
\]

Now, we apply the inverse Laplace transform, we have:
\[ u(x,t) = \mathcal{L}^{-1}\left( \frac{s}{s^2 - 1} \right) + \mathcal{L}^{-1}\left( \frac{1 + s^2}{s^2 (s^2 - 1)} \right)x + \mathcal{L}^{-1}\left( \frac{2}{s^2 (s^2 - 1)} \right)x^2 \]

\[ + \mathcal{L}^{-1}\left( \frac{1}{s^2 - 1}\mathcal{L}\left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2 - 1}\mathcal{L}\left[ u^2(x,t) \right] \right) \]  

(75)

Thus

\[ u(x,t) = 1 + xt + e^t x - 2xt - e^{-t}x + e^t x^2 + e^{-t}x^2 - t^2 x^2 - 2x^2 + \frac{e^t}{2} + \frac{e^{-t}}{2} - 1 \]

\[ + \mathcal{L}^{-1}\left( \frac{1}{s^2 - 1}\mathcal{L}\left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2 - 1}\mathcal{L}\left[ u^2(x,t) \right] \right) \]  

(76)

Denoting that the solution of (73) has the following form:

\[ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \]  

(77)

\[ u^2(x,t) = \sum_{n=0}^{\infty} A_n \]  

(78)

Taking (77) and (78) into (76), we have:

\[ \sum_{n=0}^{\infty} u_n(x,t) = 1 + xt + e^t x - 2xt - e^{-t}x + e^t x^2 + e^{-t}x^2 - t^2 x^2 - 2x^2 + \frac{e^t}{2} + \frac{e^{-t}}{2} - 1 \]

\[ + \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left( \frac{1}{s^2 - 1}\mathcal{L}\left( \frac{\partial^2 u_n(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2 - 1}\mathcal{L}\left[ u^2_n(x,t) \right] \right) \]  

(79)

and we obtain the following Adomian algorithm:

\[
\begin{cases}
    u_0(x,t) = 1 + xt \\
    u_1(x,t) = e^t x - 2xt - e^{-t}x + e^t x^2 + e^{-t}x^2 - t^2 x^2 - 2x^2 + \frac{e^t}{2} + \frac{e^{-t}}{2} - 1 \\
    + \mathcal{L}^{-1}\left( \frac{1}{s^2 - 1}\mathcal{L}\left( \frac{\partial^2 u_0(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2 - 1}\mathcal{L}\left( A_0 \right) \right)
\end{cases}
\]

(80)

Calculation \( u_l(x,t) \)

\[
\begin{cases}
    A_0 = N(u_0) = u_0^2(x,t) = 1 + 2xt + x^2 t^2 \\
    u_1(x,t) = e^t x - 2xt - e^{-t}x + e^t x^2 + e^{-t}x^2 - t^2 x^2 - 2x^2 + \frac{e^t}{2} + \frac{e^{-t}}{2} - 1 \\
    + \mathcal{L}^{-1}\left( \frac{1}{s^2 - 1}\mathcal{L}\left( \frac{\partial^2 u_0(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2 - 1}\mathcal{L}\left( 1 + 2xt + x^2 t^2 \right) \right)
\end{cases}
\]

(81)
Thus
\[
\begin{aligned}
    u_0(x,t) &= 1 + xt \\
    A_n &= 0, \quad \forall n \geq 1 \\
    u_n(x,t) &= 0, \forall n \geq 1
\end{aligned}
\] (82)

So that, the solution \( u(x,t) \) is given by:
\[
\begin{align*}
    u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\
    &= 1 + xt + 0 + 0 + \cdots = 1 + xt
\end{align*}
\] (83)

which is the exact solution of the problem.

### 4.3. Problem 3

Consider the following nonlinear Klein–Gordon equation
\[
\begin{aligned}
    \frac{\partial^2 u(x,t)}{\partial t^2} &= \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) - u^2(x,t) + x^2 \cos^2 t \\
    u(x,0) &= x \\
    \frac{\partial u(x,0)}{\partial t} &= 0
\end{aligned}
\] (84)

**Application of the LADM**

Using the Laplace transform, we have:
\[
\mathcal{L}(u(x,t)) = \frac{s}{s^2 + 1} x + \left( \frac{s^2 + 2}{s(s^2 + 1)(s^2 + 4)} \right) x^2 + \frac{1}{(s^2 + 1)} \mathcal{L}\left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) - \frac{1}{s^2 + 1} \mathcal{L}\left( u^2(x,t) \right)
\] (85)

The inverse Laplace transformation is applied to Equation (85) we get
\[
\begin{align*}
    u(x,t) &= x^{-1} \left( \frac{s}{s^2 + 1} \right) + x^2 \mathcal{L}^{-1}\left( \frac{s^2 + 2}{s(s^2 + 1)(s^2 + 4)} \right) \\
    &\quad + \mathcal{L}^{-1}\left( \frac{1}{(s^2 + 1)} \mathcal{L}\left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2 + 1} \mathcal{L}\left( u^2(x,t) \right) \right) \\
    &= x \cos t + \frac{1}{2} x^2 - \frac{1}{6} x^2 \cos 2t - \frac{1}{3} x^2 \cos t
\end{align*}
\] (86)

As before, we defines the solution \( u(x,t) \) by the series
\[
    u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)
\] (87)

and \( u_{xx} \) can be defined by an infinite series
\[
\frac{\partial^2 u(x,t)}{\partial x^2} = \sum_{n=0}^{\infty} \frac{\partial^2 u_n(x,t)}{\partial x^2}
\] (88)

The nonlinear term \( N(u) = u^2 \) is decomposed in term of Adomian polynomials.
\[ N\left(u(x,t)\right) = \sum_{n=0}^{\infty} A_n \] (89)

Substituting (87), (88) and (89) into both sides of Equation (86) we obtain

\[ \sum_{n=0}^{\infty} u_n(x,t) = x \cos t + \frac{1}{2} x^2 - \frac{1}{6} x^2 \cos 2t - \frac{1}{3} x^2 \cos t \]

\[ + \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left( \frac{1}{s^2+1} \mathcal{L}\left( \frac{\partial^2 u_0(x,t)}{\partial x^2} \right) \right) - \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left( \frac{1}{s^2+1} \mathcal{L}(A_n) \right) \] (90)

The recursive relation is defined by

\[
\begin{aligned}
    u_0(x,t) &= x \cos t \\
u_1(x,t) &= \frac{1}{2} x^2 - \frac{1}{6} x^2 \cos 2t - \frac{1}{3} x^2 \cos t + \mathcal{L}^{-1}\left( \frac{1}{s^2+1} \mathcal{L}\left( \frac{\partial^2 u_0(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2+1} \mathcal{L}(A_0) \right) \\
u_{n+1}(x,t) &= \mathcal{L}^{-1}\left( \frac{1}{s^2+1} \mathcal{L}\left( \frac{\partial^2 u_n(x,t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2+1} \mathcal{L}(A_n) \right); \forall n \geq 1
\end{aligned}
\] (91)

(91) give us

\[
\begin{aligned}
    A_0 &= u_0^2 = x^2 \cos^2 t \\
u_1(x,t) &= \frac{1}{2} x^2 - \frac{1}{6} x^2 \cos 2t - \frac{1}{3} x^2 \cos t \\
    &+ \mathcal{L}^{-1}\left( \frac{1}{s^2+1} \mathcal{L}\left( \frac{\partial^2 (x \cos t)}{\partial x^2} \right) \right) - \mathcal{L}^{-1}\left( \frac{1}{s^2+1} \mathcal{L}(A_0) \right) \\
    &= \frac{1}{2} x^2 - \frac{1}{2} x^2 - \frac{1}{6} x^2 \cos 2t + \frac{1}{6} x^2 \cos 2t - \frac{1}{3} x^2 \cos t + \frac{1}{3} x^2 \cos t \\
    &= 0
\end{aligned}
\] (92)

Thus

\[
\begin{aligned}
    u_0(x,t) &= x \cos t \\
    A_0 &= x^2 \cos^2 t \\
    u_n(x,t) &= 0, \forall n \geq 1 \\
    A_n &= 0, \forall n \geq 1
\end{aligned}
\] (93)

and the exact solution of Equation (84) is

\[ u(x,t) = x \cos t \] (94)

5. Conclusion

Through these examples, we showed again the usefulness of Laplace-Adomian Decomposition method and the He-Laplace method, in the search of an approximate solution of Klein-Gordon equation holds for the accepted forms of strong interaction of antiparticles in modern physics.

References


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The Pfaffian Technique: A (2 + 1)-Dimensional Korteweg de Vries Equation

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Abstract

The (2 + 1)-dimensional Korteweg de Vries (KdV) equation, which was first derived by Boiti et al., has been studied by various distinct methods. It is known that this (2 + 1)-dimensional KdV equation has rich solutions, such as multi-soliton solutions and dromion solutions. In the present article, a unified representation of its N-soliton solution is given by means of pfaffian. We'll show that this (2 + 1)-dimensional KdV equation is nothing but the Plücker identity when its $\tau$-function is given by pfaffian.

Keywords

The (2 + 1)-Dimensional Korteweg de Vries Equation, Hirota Bilinear Method, Pfaffian, Plücker Identity

1. Introduction

The solitary wave, so-called because it often occurs as a single entity and is localized, was first observed by J. Scott Russell on the Edinburgh-Glasgow Canal in 1834. It is known that many nonlinear evolution equations have soliton solutions, such as the Korteweg de Vries equation, the Sin-Gordon equation, the nonlinear Schrödinger equation, the Kadomtsev-Petviashvili equation, the Davey-Stewartson equation, etc. In order to study the property of nonlinear evolution equations, methods are developed to derive solitary wave solution or soliton solution to nonlinear evolution equations. Some of the most important methods are the inverse scattering transformation (IST) [1] method, the bilinear method [2]-[7], symmetry reduction method [8], the Bäcklund or Darboux transformation method [9] and so on. Having soliton solutions is one of the basic integrable properties of nonlinear evolution equations.

In this paper, we are interested in the general expression of $N$-soliton solution to the...
(2 + 1)-dimensional KdV equation,
\[
\frac{\partial u}{\partial t} + 3 \frac{\partial (uv)}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,
\]
\[
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0,
\]  
(1)

which was first derived by Boiti et al. by using the idea of the weak Lax pair [10]. This system can also be obtained from the inner parameter-dependent symmetry constraint of the KP equation [11]. Recently, the dromion solutions and some exact solutions are studied by Lou and Wazwaz respectively [12]-[14]. While as for the uniformed expression of its \(N\)-soliton solution is unknown yet.

In this article, we’ll study the \(N\)-soliton solution to the (2 + 1)-dimensional KdV system (1). A compact form of the \(N\)-soliton solution to Equation (1) is obtained by means of pfaffian technique, which is given in section 2. Conclusion and further discussions are given in section 3.

2. \(N\)-Soliton Solution to the (2 + 1)-Dimensional KdV Equation

Given a nonlinear evolution equation, if it has 3-soliton solution, then this equation is of great possibility of having \(N\)-soliton (\(3 \leq N\)) solution. Pfaffian technique is one of the methods that can help us to determine whether the evolution equation has multisoliton solutions or not. In this section, we first review some properties of pfaffian.

2.1. Pfaffian

Pfaffians are antisymmetric functions with respect to its independent variables
\[
\text{pf} (a, b) = -\text{pf} (b, a), \quad \text{for any } a \text{ and } b.
\]

An \(n\)-th order pfaffian \(\text{pf} (1, 2, \cdots, 2n)\) can be defined inductively by the expansion rule [3]
\[
\text{pf} (1, 2, \cdots, 2n) = \sum_{j=2}^{2n} (-1)^{j} \text{pf} (1, j) \text{pf} (2, 3, \cdots, \hat{j}, \cdots, 2n),
\]

where \(\hat{j}\) denotes the absence of letter \(j\). For example, when \(n = 2\), we have
\[
\text{pf} (1, 2, 3, 4) = \text{pf} (1, 2) \text{pf} (3, 4) - \text{pf} (1, 3) \text{pf} (2, 4) + \text{pf} (1, 4) \text{pf} (2, 3).
\]

There are various kinds of pfaffian identities. In this article, we just introduce the so-called Plücker relation for pfaffians [3]
\[
\begin{align*}
\text{pf} (\alpha_1, \alpha_2, \alpha_3, 1, 2, \cdots, 2n-1) \text{pf} (1, 2, \cdots, 2n) \\
= \text{pf} (\alpha_1, 1, 2, \cdots, 2n-1) \text{pf} (\alpha_2, \alpha_3, 1, 2, \cdots, 2n) \\
- \text{pf} (\alpha_2, 1, 2, \cdots, 2n-1) \text{pf} (\alpha_1, \alpha_3, 1, 2, \cdots, 2n) \\
+ \text{pf} (\alpha_3, 1, 2, \cdots, 2n-1) \text{pf} (\alpha_1, \alpha_2, 1, 2, \cdots, 2n),
\end{align*}
\]
(2)

which we are going to use. Hereafter, we let \((1, 2, \cdots, 2n)\) denote pfaffian \(\text{pf} (1, 2, \cdots, 2n)\) for simplicity.
2.2. N-Soliton Solutions

The Hirota form of the (2 + 1)-dimensional KdV system (1) is

\[ D_y \left[ D_z + D^2_x \right] f \cdot f = 0, \tag{3} \]

which is obtained by the dependent variable transformations

\[ u = 2 \frac{\partial^2 \ln f(x,y,t)}{\partial x \partial y}, \quad v = 2 \frac{\partial^2 \ln f(x,y,t)}{\partial x^2}. \tag{4} \]

Here the Hirota bilinear operator \( D^n_x D^m_y \) is defined by

\[ D^n_x D^m_y a \cdot b \equiv \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right)^n a(x,t) b(x',t) \bigg|_{x=x',t=t}, \]

with \( n \) and \( m \) are arbitrary nonnegative integers.

In [14], the 3-soliton solution to the (2 + 1)-dimensional KdV system (3) is obtained

\[ f = 1 + \sum_{i=1}^{3} \exp(\theta_i) + \sum_{1 \leq i < j < k \leq 3} c_{ij} \exp(\theta_i + \theta_j) + c_{123} \exp(\theta_1 + \theta_2 + \theta_3) \tag{5} \]

where

\[ \theta_i = k_i x + m_i y + q_i t + \theta_i^0, \quad q_i = -k_i^3, \quad k_i \neq 0, m_i \neq 0 \text{ and } \theta_i^0 \text{ are constants}; \]

\[ c_{ij} = \left( p_i - p_j \right) \left( m_i - m_j \right) \left( p_i + p_j \right) \left( m_i + m_j \right), \quad c_{123} = c_{12}c_{23}c_{13}, \]

via the perturbation method. It claims that the \( N \)-soliton solutions for \( 4 \leq N \) can also be obtained by using perturbation method, but the explicit expression of the multi-soliton solution is not given.

In this article, we'll study the the multi-soliton solution to Equation (3) using the pfaffian technique [15] [16]. A compact form of the \( N \)-soliton solution is given in terms of \( N \)-th order pfaffian.

**Proposition 1.** If the \( \tau \)-function \( f \) of the (2 + 1)-dimensional KdV system (3) is given by the pfaffian function

\[ f = \left( d_0, \beta_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N \right), \tag{6} \]

whose entries, for \( j, k = 1, 2, \ldots, N \), are defined by

\[ \left( d_0, a_j \right) = \exp(\eta_j), \quad \left( d_0, b_j \right) = -1, \quad \left( d_0, \beta_0 \right) = 1, \quad \left( a_j, a_k \right) = a_{jk} \exp(\eta_j + \eta_k), \]

\[ \left( b_j, \beta_0 \right) = 0, \quad \left( b_j, a_k \right) = b_{jk}, \quad \left( a_j, a_k \right) = \delta_{jk}, \]

with

\[ a_{jk} = \frac{p_j - p_k}{p_j + p_k}, \quad b_{jk} = \frac{m_j - m_k}{m_j + m_k}, \quad \delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \tag{7} \]

\[ \eta_j = p_j x + m_j y - p_j^0 t + \eta_j^0, \quad p_j \neq 0, m_j \neq 0 \text{ and } \eta_j^0 \text{ are constants}, \]

then this particular pfaffian function (6) gives an \( N \)-soliton solutions to the (2 + 1)-dimensional KdV system (3).

**Proof.** In the following, we will prove that the pfaffian function (6) satisfies the (2 + 1)-dimensional KdV Equation (3). By defining “differential operators” \( d_n (n = 1, 2, \ldots) \)
\( (d_x, a_j) = p_j \exp(\eta_j), \quad (d_x, b_j) = (d_x, \beta_0) = (d_x, d_0) = 0, \)  

we obtain the following differential formulae

\[
\frac{\partial f}{\partial x} = -(d_0, d_x, a_1, \cdots, a_N, b_1, \cdots, b_N), \quad \frac{\partial^2 f}{\partial x^2} = -(d_0, d_x, a_1, \cdots, a_N, b_1, \cdots, b_N), \\
\frac{\partial^3 f}{\partial x^3} = -(d_0, d_x, a_1, \cdots, a_N, b_1, \cdots, b_N) - (d_0, d_x, \beta_0, a_1, \cdots, a_N, b_1, \cdots, b_N), \\
\frac{\partial f}{\partial t} = (d_0, d_x, b_1, \cdots, b_N) - 2(d_0, d_x, \beta_0, a_1, \cdots, a_N, b_1, \cdots, b_N).
\]

(9)

In order to find the pfaffian expression for the differential functions with derivative of variable \( y \), we need to define another letter \( \beta \) [17]

\[
(\beta, b_j) = -m_j, \quad (\beta, a_j) = (\beta, \beta_0) = (\beta, d_0) = 0.
\]

(10)

Then we have

\[
\frac{\partial f}{\partial y} = (d_0, \beta, a_1, \cdots, a_N, b_1, \cdots, b_N), \\
\frac{\partial^2 f}{\partial x \partial y} = (d_0, d_x, \beta, a_1, \cdots, a_N, b_1, \cdots, b_N), \\
\frac{\partial^3 f}{\partial x^2 \partial y} = (d_0, d_x, \beta, a_1, \cdots, a_N, b_1, \cdots, b_N), \\
\frac{\partial^4 f}{\partial x^3 \partial y} = (d_0, d_x, \beta, \beta_0, a_1, \cdots, a_N, b_1, \cdots, b_N) - (d_0, d_x, \beta, a_1, \cdots, a_N, b_1, \cdots, b_N), \\
\frac{\partial^5 f}{\partial x^4 \partial y} = -(d_0, d_x, \beta, \beta_0, a_1, \cdots, a_N, b_1, \cdots, b_N) - 2(d_0, d_x, \beta, a_1, \cdots, a_N, b_1, \cdots, b_N).
\]

(11)

Substituting formulae (9) and (11) into the right hand side of Equation (3), we obtain nothing but the Plücker relation for pfaffians (2)

\[
D_1[D_1 + D_1^t] f \cdot f = f \left( \frac{\partial^2 f}{\partial y \partial t} + \frac{\partial^2 f}{\partial x \partial y} \right) - 3 \frac{\partial f}{\partial x} \frac{\partial^3 f}{\partial x^2 \partial y} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial f}{\partial t} \left( \frac{\partial f}{\partial x} + \frac{\partial^3 f}{\partial x^3} \right)
\]

\[
= 3 \left[ (d_1, \cdots)(d_2, \beta_3, \beta_0, \cdots) - (d_2, \cdots)(d_1, \beta_3, \beta_0, \cdots) \right] \\
+ \left( \beta_3, \cdots \right)(d_1, d_2, \beta_0, \cdots) - \left( \beta_0, \cdots \right)(d_1, d_2, \beta_3, \cdots) \right] \equiv 0,
\]

where \( (\cdots) \) denotes \( (d_0, a_1, \cdots, a_N, b_1, \cdots, b_N) \). Therefore the pfaffian function (6) solves the \((2 + 1)\)-dimensional KdV system (3).

Note that in order to derive the differential formulae of the pfaffian function (6), we have to define another extra letter \( \beta \) besides the "differential operators" \( d_x \). The multi-soliton solution to the nonlinear \((2 + 1)\)-dimensional KdV system (1) can be obtained by substituting pfaffian function (6) into the dependent variable transformation (4) directly.

3. Conclusion

In this article, a compact form of the multi-soliton solution to the \((2 + 1)\)-dimensional
KdV system is given via the pfaffian technique. As one can see, the key point of the proof is to derive suitable expressions of the differential formulae of pfaffian $\tau$-function $\tau$. It is worth pointing out that the method used in this article is different as the one for the proof of the BKP equation, which the differential formulae of the pfaffian $\tau$-function depend only on the “differential operators” $d_n$.

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References


Study of Temperature Behaviour on Thermally Induced Vibration of Non-Homogeneous Trapezoidal Plate with Bi-Linearly Varying Thickness

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Abstract

The main aim of the present work is to study the linear temperature behaviour of a non-homogeneous trapezoidal plate whose thickness varies linearly in both directions. The temperature behaviour considered linear along the length of the plate. Non-homogeneity in plate arises due to variation in density along the length of the plate. The two-term deflection function with clamped-simply supported-clamped-simply supported boundary condition is taken into consideration. The effect of structural parameters such as taper constants, thermal gradient, non-homogeneity constant and aspect ratio has been studied. Rayleigh-Ritz method is used to solve the governing differential equations and to obtain the fundamental frequencies for the first two modes of vibration. Results are presented in graphical form.

Keywords

Vibration, Trapezoidal Plate, Taper Constants, Thermal Gradient, Aspect Ratio, Non-Homogeneity, Thickness, Density, Frequencies

1. Introduction

Most of the machines and structures work under the control of high temperature. Due to this, system undergoes some vibrations. Vibrations affect the efficiency, strength and durability of the system. The purpose of vibration study is to reduce vibration through proper and accurate design of machines and structures. Therefore, it is necessary for researchers and design engineers to have pre-knowledge of vibrational characteristics of

After a careful study of literature, it is recognized that no work has been done on linear density variation with bilinear thickness variation on vibration of heated trape-
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In this paper, an analysis is presented to study the effect of thermally induced vibration of non-homogeneous trapezoidal plate with bi-linearly varying thickness. To acquire the natural frequencies for the first two modes of vibration, Rayleigh-Ritz’s method is used for a non-homogeneous trapezoidal plate whose two sides are clamped and two are simply-supported.

2. Thickness and Density

As depicted in Figure 1 a symmetric, non-homogeneous trapezoidal plate has been considered. Thickness varies bilinearly along length and width of the plate as

\[
h(\xi, \eta) = h_0 \left[ 1 - (1 - \beta_1) \left( \xi + \frac{1}{2} \right) \right] \left[ 1 - (1 - \beta_2) \left( \eta + \frac{1}{2} \right) \right]
\]

(1)

where \( h_0 = h \) at \( \xi = \eta = -1/2 \) and \( \beta_1, \beta_2 \) are taper constants.

The density is one of the most important aspects of any design. Due to variation in density, non-homogeneity occurs in plate’s material which varies linearly along the length of the plate. So, it can be considered as

\[
\rho = \rho_0 \left[ 1 - (1 - \beta) \left( \xi + \frac{1}{2} \right) \right]
\]

(2)

where \( \rho_0 = \rho \) is the mass density at \( \xi = -1/2 \) and \( \beta \) is non-homogeneity constant.

The temperature of the trapezoidal plate varies linearly along the length of the plate as

![Figure 1. Geometry of the trapezoidal plate.](image)
\[ \tau = \tau_0 \left(1 - \frac{x}{a}\right) \]  

(3)

where \( \tau \) denotes the excess above the reference temperature at a distance \( x/a \) and \( \tau_0 \) denotes the temperature excess above the reference temperature at the end \( x = -a \).

For most of the structural materials the temperature dependence of the modulus of elasticity is given by Nowacki [23] as

\[ E = E_0 \left(1 - \gamma \tau\right) \]  

(4)

where \( E_0 \) is Young’s modulus value at reference temperature \( \tau = 0 \) and \( \gamma \) is the slope of variation of \( E \) and \( \tau \).

By the use of Equation (3) in Equation (4), one obtains

\[ E = E_0 \left(1 - \alpha \left(1 - \frac{x}{a}\right)\right) \]  

(5)

where \( \alpha = \gamma \tau_0 \left(0 \leq \alpha \leq 1\right) \) known as thermal gradient.

3. Governing Differential Equations

The governing differential equations of kinetic energy \( T \) and strain energy \( V \) for a trapezoidal plate are given by [10] as

\[ T = \frac{ab}{2} \omega^2 \int h(\xi) \rho w^2 dA \]  

(6)

and

\[ V = \frac{ab}{2} \int D(\xi) \left\{ \frac{1}{a^2} \frac{\partial^2 w}{\partial \xi^2} + \frac{1}{b^2} \frac{\partial^2 w}{\partial \eta^2} \right\}^2 - 2(1-\nu) \frac{1}{a^2 b^2} \frac{\partial^2 w}{\partial \xi^2} \frac{\partial^2 w}{\partial \eta^2} \frac{1}{ab} \frac{\partial^2 w}{\partial \xi \partial \eta} \right\} dA \]  

(7)

where \( \nu \) is the Poisson’s ratio; \( \omega \) is the angular frequency of vibration and \( A \) is the area of the plate.

Flexural rigidity of the plate \( D(\xi) \) can be expressed as

\[ D(\xi) = D_0 \left[1 - (1-\beta_1)(\xi + \frac{1}{2})\right]^3 \]  

(8)

where \( \xi = \frac{x}{a}, \eta = \frac{y}{b} \) are non-dimensional variables. Here,

\[ D_0 = \frac{Eh_0^3}{12(1-\nu^2)} \]  

(9)

By using Equation (5) and Equation (9) in Equation (8), the flexural rigidity becomes

\[ D(\xi) = \frac{E(h_0^3)}{12(1-\nu^2)} \left[1 - (1-\beta_1)(\xi + \frac{1}{2})\right]^3 \left[1 - (1-\beta_2)(\eta + \frac{1}{2})\right]^3 \left(1 - \alpha \left(1 - \frac{x}{a}\right)\right) \]  

(10)

Using Equation (1) and Equation (2) in Equation (6), we get

\[ T = \frac{ab}{2} \rho_0 h_0 \omega^2 \int \left[1 - (1-\beta_1)(\xi + \frac{1}{2})\right]^3 \left[1 - (1-\beta_2)(\eta + \frac{1}{2})\right] x (1-\beta)(\xi + \frac{1}{2})] w^2 dA \]  

(11)
Using Equation (10) in Equation (7), we get
\[
V = \frac{ab}{2} \frac{E_0 h_0^3}{12(1-\nu^2)} \int \left[ 1 - (1 - \beta_1) \left( \frac{\xi + 1/2}{2} \right) \right] \left[ 1 - (1 - \beta_2) \left( \eta + 1/2 \right) \right] \left[ 1 - \alpha \left( \frac{1}{2} - \xi \right) \right] ^3 dA
\]
(12)
\[
\times \left( \frac{1}{a^2} \frac{\partial^2 w}{\partial \xi^2} + \frac{1}{b^2} \frac{\partial^2 w}{\partial \eta^2} \right) ^2 - 2(1-\nu) \left[ \frac{1}{a^2 b^2} \frac{\partial^2 w}{\partial \xi^2} \frac{\partial^2 w}{\partial \eta^2} - \left( \frac{1}{ab} \frac{\partial^2 w}{\partial \xi \partial \eta} \right) ^2 \right] \right) \right) dA.
\]

In the present study the two term deflection function which satisfies the boundary condition can be expressed as
\[
w = A_1 \left[ \left( \xi + \frac{1}{2} \right)^2 \eta \left( \beta \xi + \eta + \frac{1}{2} \right) \right] ^2 - \frac{b-c}{2} \left( \beta \xi \right) + \frac{b+c}{4} \left( \beta \xi \right) + \frac{b-c}{2} \left( \eta \right) - \frac{b+c}{4} \left( \eta \right) + \frac{b-c}{2} \left( \xi \right) - \frac{b+c}{4} \left( \xi \right) ^2
\]
(13)
\[
+ A_2 \left[ \left( \xi + \frac{1}{2} \right)^2 \eta \left( \beta \xi + \eta + \frac{1}{2} \right) \right] ^2 - \frac{b-c}{2} \left( \beta \xi \right) + \frac{b+c}{4} \left( \beta \xi \right) + \frac{b-c}{2} \left( \eta \right) - \frac{b+c}{4} \left( \eta \right) + \frac{b-c}{2} \left( \xi \right) - \frac{b+c}{4} \left( \xi \right) ^2,
\]
where \( A_1 \) and \( A_2 \) are two unknowns to be evaluated. For the solution of the problem the trapezoidal plate is considered whose two sides are clamped and two are simply supported. Therefore, the boundaries are defined by four straight lines
\[
\eta = \frac{c}{4b} - \frac{\xi}{2} + \frac{1}{4} + \frac{c^2}{4b},
\]
\[
\xi = \frac{1}{2},
\]
\[
\xi = \frac{1}{2}.
\]
(14)

4. Methodology

For the existing problem, Rayleigh-Ritz’s method has been employed. It requires the maximum strain energy must be equal to the maximum kinetic energy. Therefore, it is necessary that the consequent equation must be satisfied
\[
\delta (V - T) = 0.
\]
(15)

Using Equation (14) into Equation (11) and Equation (12), we obtain
\[
T = \frac{ab}{2} \rho \omega a^2 \int \frac{1}{2} \frac{c}{4b} \left[ \xi + \frac{1}{2} \right] \left[ 1 - (1 - \beta_1) \left( \xi + \frac{1}{2} \right) \right] \left[ 1 - (1 - \beta_2) \left( \eta + \frac{1}{2} \right) \right] \left[ 1 - \alpha \left( \frac{1}{2} - \xi \right) \right] ^3 d\eta \, d\xi
\]
(16)
\[
\times \left( \frac{1}{a^2} \frac{\partial^2 w}{\partial \xi^2} + \frac{1}{b^2} \frac{\partial^2 w}{\partial \eta^2} \right) ^2 - 2(1-\nu) \left[ \frac{1}{a^2 b^2} \frac{\partial^2 w}{\partial \xi^2} \frac{\partial^2 w}{\partial \eta^2} - \left( \frac{1}{ab} \frac{\partial^2 w}{\partial \xi \partial \eta} \right) ^2 \right] d\eta \, d\xi
\]
(17)
Using Equation (16) and Equation (17) into Equation (15), we get

$$\delta \left( V_i - \lambda^2 T_i \right) = 0. \quad (18)$$

where

$$T_i = \sum_{n=1}^{\infty} \int \left[ \frac{c}{4b} \frac{c}{z} \left( \frac{c}{z} \right) \left( \frac{c}{z} \right) \left( \frac{c}{z} \right) \right] \left[ 1 - (1 - \beta_1) \left( \frac{\xi + 1}{2} \right) \right] \left[ 1 - (1 - \beta_2) \left( \eta + \frac{1}{2} \right) \right] \left[ 1 - (1 - \beta) \left( \frac{\xi + 1}{2} \right) \right] w^2 d\eta d\xi, \quad (19)$$

$$V_i = \sum_{n=1}^{\infty} \int \left[ \frac{c}{4b} \frac{c}{z} \left( \frac{c}{z} \right) \left( \frac{c}{z} \right) \left( \frac{c}{z} \right) \right] \left[ 1 - (1 - \beta_2) \left( \xi + \frac{1}{2} \right) \right] \left[ 1 - (1 - \beta_2) \left( \eta + \frac{1}{2} \right) \right] \left[ 1 - (1 - \beta) \left( \frac{\xi + 1}{2} \right) \right] w^2 d\eta d\xi, \quad (20)$$

And

$$\lambda^2 = \frac{12 \omega^2 \rho_0 a^4 (1 - v^2)}{E_o h_0^2}. \quad (21)$$

is a frequency parameter.

The unknowns $A_1$ and $A_2$ in Equation (18) arises due to the substitution of the deflection function $w$ given by Equation (13). From Equation (18) these two constants can be determined, as follows

$$\frac{\partial}{\partial A_1} \left( V_i - \lambda^2 T_i \right) = 0,$$

$$\frac{\partial}{\partial A_2} \left( V_i - \lambda^2 T_i \right) = 0. \quad (22)$$

On simplifying (22), we get

$$b_{m1} A_1 + b_{m2} A_2 = 0, \quad m = 1, 2 \quad (23)$$

where $b_{m1}, b_{m2}$ $(m = 1, 2)$ involve parametric constants and the frequency parameter.

For a non-zero solution, the determinant of co-efficient of Equation (23) must be zero. Thus the frequency Equation for a (C-S-C-S) trapezoidal plate is given by

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = 0. \quad (24)$$

On simplifying Equation (24), a quadratic equation in $\lambda^2$ is obtained. Thus, it provides the two values of $\lambda^2$ corresponding to the first and second modes of vibration respectively.

5. Results and Discussions

Frequencies for the first two modes of vibration are calculated for non-homogeneous trapezoidal plate whose thickness varies linearly in both directions and density varies linearly in x-direction. Different values of taper constants $\beta_1$ & $\beta_2$, thermal gradient

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Kavita et al., 1942, aspect ratios a/b, c/b and non-homogeneity constant $\beta$ has been considered. The value of Poisson’s ratio $\nu$ is taken as 0.33. With the help of graphs all the results have been presented.

In Figure 2(a) and Figure 2(b) these figures show the variation of the frequency parameter $\lambda$ with the taper constant $\beta_i$ (0.0 to 1.0) for the first and second mode, respectively, for $\alpha = 0.0$ and $\alpha = 0.4$.

**Figure 2.** (a) Variation of frequency parameter $\lambda$ for different values of taper constant $\beta_i$ for the first mode. (b) Variation of frequency parameter $\lambda$ for different values of taper constant $\beta_i$ for the second mode.
1) \( a/b = 1.0, c/b = 0.5 \)
2) \( \beta = 0.4, 1.0 \)
3) \( \alpha = 0.0, 0.4 \)
4) \( \beta_2 = 0.6 \)

These figures demonstrate that as the taper constant \( \beta_1 \) increases, the frequency parameter \( \lambda \) also increases for both the modes of vibration.

In Figure 3(a) and Figure 3(b), these figures show the variation of the frequency parameter \( \lambda \) with the taper constant \( \beta_2 \) (0.0 to 1.0) for the first and second mode, respectively, for

![Graph](image1)

**Figure 3.** (a) Variation of frequency parameter \( \lambda \) for different values of taper constant \( \beta_1 \) for the first mode. (b) Variation of frequency parameter \( \lambda \) for different values of taper constant \( \beta_2 \) for the second mode.
1) \( a/b = 1.0, c/b = 0.5 \)
2) \( \beta = 0.4, 1.0 \)
3) \( \alpha = 0.0, 0.4 \)
4) \( \beta_i = 0.6 \)

These figures explain that as the taper constant \( \beta \) increases, and the frequency parameter \( \lambda \) also increases for both the modes of vibration.

In **Figure 4(a)** and **Figure 4(b)** these figures depict the behaviour of the frequency parameter \( \lambda \) with thermal gradient \( \alpha \) (varying from 0.0 to 1.0) for the first and second mode, respectively, for

**Figure 4.** (a) Variation of frequency parameter \( \lambda \) for different values of thermal gradient \( \alpha \) for the first mode. (b) Variation of frequency parameter \( \lambda \) for different values of thermal gradient \( \alpha \) for the second mode.
1) \( a/b = 1.0, \frac{c}{b} = 0.5 \)
2) \( \beta_1 = 0.2, \beta_2 = 0.6 \)
3) \( \beta_1 = \beta_2 = 0.0 \)
4) \( \beta = 0.4, 1.0 \)

It is clear from these figures that as the thermal gradient \( \alpha \) increases, the frequency parameter \( \lambda \) decreases for both the modes of vibration. In **Figure 5(a)** and **Figure 5(b)** these figures demonstrate the effect of aspect ratio \( c/b \) (varying from 0.25 to 1.0) on the frequency parameter \( \lambda \) for the first and second mode, respectively, for

**Figure 5.** (a) Variation of frequency parameter \( \lambda \) for different values of aspect ratio \( c/b \) for the first mode. (b) Variation of frequency parameter \( \lambda \) for different values of aspect ratio \( c/b \) for the second mode.
1) \( a/b = 0.75, 1.0 \)
2) \( \beta_1 = \beta_2 = 0.6 \)
   a) \( \alpha = 0.0, \beta = 0.0 \)
   b) \( \alpha = 0.4, \beta = 0.0 \)
   c) \( \alpha = 0.0, \beta = 0.4 \)
   d) \( \alpha = 0.4, \beta = 0.4 \)

It is evident from the figures that as aspect ratio \( c/b \) increases, the frequency parameter decreases for both the modes of vibration. From Figure 5(a) and Figure 5(b) it is observed that with increase in aspect ratio \( a/b \) the frequency increases for both the modes of vibration.

In Figure 6(a) and Figure 6(b) these figures show the effect of non-homogeneity constant \( \beta \) (varying from 0.0 to 1.0) on the frequency parameter \( \lambda \) for the first and second mode, respectively, for

**Figure 6.** (a) Variation of frequency parameter \( \lambda \) for different values of non-homogeneity constant \( \beta \) for the first mode. (b) Variation of frequency parameter \( \lambda \) for different values of non-homogeneity constant \( \beta \) for the second mode.
1) \( a/b = 1.0, c/b = 0.5, 1.0 \)
2) \( \beta_1 = \beta_2 = 0.0 \) and \( \beta_1 = 0.2, \beta_2 = 0.6 \)
3) \( \alpha = 0.0, 0.4 \)

These figures show that as the non-homogeneity constant \( \beta \) increases, the frequency parameter \( \lambda \) decreases for both the modes of vibration.

### 6. Conclusion

In the present paper, the effect of temperature on the vibration of symmetric, non-homogeneous trapezoidal plate of isotropic material with clamped-simply supported-clamped-simply supported-boundary condition has been studied by using the Rayleigh-Ritz method. Effect of other plate’s parameters such as non-homogeneity constant, aspect ratios, taper constants has also been considered. It is obvious from the graphs that by the increase of taper constants, aspect ratio \( a/b \) the frequency of both the modes of vibration increases. On the other hand, frequency decreases with increasing values of thermal gradient, aspect ratio \( c/b \) and non-homogeneity constant for both the modes of vibration. By the proper selection of various plate parameters such as taper constants, thermal gradient, aspect ratio and non-homogeneity constant, a desired frequency can be attained for the first two modes of vibration which would be helpful for the design engineers.

### References


The Third-Order Viscoelastic Acoustic Model Enables an Ice-Detection System for a Smart Deicing of Wind-Turbine Blade Shells

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Abstract

The present work is based on the third-order partial differential equation (PDE) of acoustics of viscoelastic solids for the quasi-equilibrium (QE) component of the average normal stress. This PDE includes the stress-relaxation time (SRT) for the material and is applicable at any value of the SRT. The notion of a smart deicing system (SDS) for blade shells (BSs) of a wind turbine is specified. The work considers the stress in a BS as the one caused by the operational load on the BS. The work develops key design issues of a prospective ice-detection system (IDS) able to supply an array of the heating elements of an SDS with the element-individual spatiotemporal data and procedures for identification of the material parameters of atmospheric-ice (AI) layer accreted on the outer surfaces of the BSs. Both the SDS and IDS flexibly allow for complex, curvilinear and space-time-varying shapes of BSs. The proposed IDS presumes monitoring of the QE components of the normal stresses in BSs. The IDS is supposed to include an array of pressure-sensing resistors, also known as force-sensing resistors (FSRs), and communication hardware, as well as the parameter-identification software package (PISP), which provides the identification on the basis of the aforementioned PDE and the data measured by the FSRs. The IDS does not have hardware components located outside the outer surfaces of, or implanted in, BSs. The FSR array and communication hardware are reliable, and both cost- and energy-efficient. The present work extends methods of structural-health/operational-load monitoring (SH/OL-M) with measurements of the operational-load-caused stress in closed solid shells and, if the prospective PISP is used, endows the methods with identification of material parameters of the shells. The identification algorithms that can underlie the PISP are computationally efficient and suitable for implementation in the real-time mode. The identification model and algorithms can deal with not only the single-layer systems such as the BS layer without the AI layer or two-layer systems but also multi-layer systems. The outcomes can be applied to not only
BSs of wind turbines but also non-QE closed single- or multi-layer deformable solid shells of various engineering systems (e.g., the shells of driver or passenger compartments of ships, cars, busses, airplanes, and other vehicles). The proposed monitoring of the normal-stress QE component in the mentioned shells extends the methods of SH/OL-M. The topic for the nearest research is a better adjustment of the settings for the FSR-based measurement of the mentioned components and a calibration of the parameter-identification model and algorithms, as well as the resulting improvement of the PISP.

**Keywords**


1. Introduction

In cold weather, a layer of atmospheric ice (AI) is accreted on the outer surfaces of the blade shells (BSs) of a wind turbine. As is well known, this layer can cause unexpected downtime and increased maintenance cost of the turbine, thereby resulting in reduced efficiency of the energy production by the turbines.

**Remark 1.1.** All of the cold-climate areas can be found within the Köppen-Geiger climate classification system (http://en.wikipedia.org/wiki/Köppen_climate_classification). The climates associated with the freezing temperatures comprise various weather conditions, from the hot summer continental climate (see also http://en.wikipedia.org/wiki/Humid_continental_climate) with the minimum temperatures in the coldest months down to −10°C to the tundra climate (see also http://en.wikipedia.org/wiki/Tundra) with the minimum temperatures in the coldest months down to −50°C.


Thus, the temperatures dealt with in connection with deicing of the BSs are in the interval from 0°C down to the aforementioned values depending on a specific climate.

*Note that all of the physical-quantity values are specified below in the SI without indication of the corresponding units, with the exception of the cases where the units are not the SI ones* (such as the temperature values in Remark 1.1).
The accreted AI is generally of different qualities. It can be continuous or porous, more specifically, hexagonal crystalline or low-density amorphous, clear, frozen dense snow, soft or hard rime, glaze, etc. In terms of continuum mechanics, these qualities are distinguished with values of the material parameters of the AI (e.g., [1], [2]).

According to [2], the AI (including porous and continuous ice) is regarded as porous medium where pores are filled with air. The corresponding (volumetric) density of the AI mass \( \rho \) is expressed as follows

\[
\rho = (1 - \varphi) \rho_i + \varphi \rho_a
\]

where \( \varphi \) is the AI porosity, i.e., the volume fraction occupied by air \((0 \leq \varphi < 1)\), \( \rho_i \) is the mass density of a continuous, non-porous ice, and \( \rho_a \) is the mass density of air. All of these parameters present the equilibrium values. Note that

\[
\rho_i = 916.7, \quad \text{at } 0^\circ \text{C},
\]

\[
\rho_a = 1.2 \quad \text{at sea level.}
\]

As follows from (1.1),

\[
\varphi = \left( \frac{\rho_i - \rho}{\rho_i - \rho_a} \right)
\]

The mass densities of the seasonal and dense snows are below and above 300, respectively.

Nowadays, BSs are usually fabricated of composite materials. As is well known (e.g., [3]), they are viscoelastic.

Viscoelasticity is also a property of AI. A viscosity of a medium presents the product of the corresponding elastic modulus and the relaxation time \( \theta \) of the quasi-equilibrium (QE) component of the Cauchy stress matrix to its equilibrium value. More specifically, if \( K \) and \( G \) are the bulk and shear moduli of an isotropic material, then parameters

\[
\eta = K\theta, \quad \mu = G\theta
\]

are the volume (or compressional) and shear viscosities. The stress relaxation exists in any material medium, in gases, liquids, and solids, no matter if the medium is spatially non-homogeneous or spatially homogeneous. Also note that, for the sake of simplicity, the stress-relaxation time (SRT) \( \theta \) is assumed to be a scalar rather than a matrix. Also, the same value of it is used in expressions (1.5) for both the viscosities. The physical picture for the viscosities of solids and the relevant theoretical-physics modeling approach are available (e.g., [4], §34). Many aspects of theoretical analysis and experimental study of viscoelastic materials are described in [5] and [6].

The fact that ice is viscoelastic rather than purely elastic (i.e., inviscid) was recognized more than a century ago [7]-[10]. This line of research was continued in [11]. It appeared that solid ice has high values of viscosity. For instance, the experimental data in ([11], the last column of Table 3) show that the ice shear viscosity values are in the interval \( 0.28 \times 10^{12} - 200 \times 10^{12} \). The shear-viscosity values for glaciers obtained for a number of the Swiss glaciers are \( 0.33 \times 10^{12} - 29.22 \times 10^{12} \) ([10], Table 1). This is confirmed with the fact that ([12], p. 305) values of the viscosity of polycrystalline ice vary in the range \( 0.1 \times 10^{12} - 100 \times 10^{12} \).
Basing on the pioneering work of J. C. McConnel published in 1891, the experimental data of ([10], Table 2) show that the ice viscosity rapidly increases from $0.0033 \times 10^{12}$ to $0.1342 \times 10^{12}$ when the ice temperature decreases from $-1.7 \degree C$ to $-15.3 \degree C$. These data can partly explain the spread of the numerical values in each of the aforementioned two intervals. “J. C. McConnel appears to have been the first to observe the exact conditions under which ice is capable of being deformed without fracture by stress. He showed that a crystal of ice can be sheared by very small stress in a direction at right angles to the optic axis, and that the rate of shear becomes greater as the stress is increased. His two main conclusions are 1) that the friction between the particles of ice along the shear planes becomes greater as the temperature falls, 2) that when the molecules of ice slide over each other the cube of the friction varies as the square of the velocity.” ([10], p. 253). Work [10] also notes that the shear viscosity of common ice at $0 \degree C$ is $2 \times 10^9$. Why the effective viscosities of glaciers are noticeably greater than the one of common ice remains unclear ([10], p. 259).

The following remark indicates a possible interval of the AI SRT.

**Remark 1.2.** The aforementioned experimental values for ice show that shear viscosity $\mu$ in (1.5) can be in the interval $2 \times 10^9 - 2.2 \times 10^{14}$. A typical value of the shear modulus for this ice is $3.07 \times 10^9$ (see [13], Table 1). Applying these data to the first equality in (1.5), one obtains that SRT for continuous ice varies in the range $0.65 - 7.17 \times 10^7$, i.e., in five orders.

In the case of polycrystalline ice (e.g., [12]), the range of the SRT values is even wider. Indeed, as already noted, values of the viscosity of polycrystalline ice vary from about $0.1 \times 10^9$ to $10^{14}$ ([12], p. 305). Combining a typical value $3.56 \times 10^9$ of the shear modulus for this ice (see [12], Table 4.1) with the second equality in (1.5), one obtains that the indicated interval corresponds to the SRT values in the six-order range $0.028 - 2.8 \times 10^7$.

According to Remark 1.2, the SRT values for continuous or polycrystalline ice can vary in five-six orders. The latter feature practically means that the corresponding viscoelastic acoustic model must be relevant at any value of the material SRT.

The AI layer accreted on it the outer surface of a BS is different at different locations on the surface and at different time points. More specifically, the AI landscape varies in both the space and time.

To prevent the losses mentioned at the beginning of this section, one usually applies deicing methods. The most common of them is heating. However, by now, all of the deicing techniques are based on a priori assumptions on the AI-layer parameters over the BS surface. In the case of heating, this often results in under-heating, i.e., reduced reliability of the deicing, or over-heating that damages the BS-material and, thus, reduce the cost-efficiency of the deicing.

The notion of a smart deicing system (SDS) presumes that the system is smart in the sense that is cost- and energy-efficient, safe to the BS material, and reliable. As noted above, the AI layer is space-time varying. The BS, which the AI landscape is accreted on, has a complex, curvilinear shape, which is generally space-time varying as well, due to
the action of the operational load (e.g., [14]). To be able to smartly remove the above
spatiotemporal landscape of the AI, an SDS must provide the heat levels, which are
individual to the landscape regions and specific time points. Consequently, these levels
should be determined on the basis of the spatiotemporal landscape of the AI-layer
material parameters including the layer thickness. To provide this landscape, each
heating element (e.g., [15]) should be accompanied by an appropriate sensor, which is
located near, but outside the area of, the heating element and measures a mechanical
variable that allows to identify the mentioned parameters of the AI layer. The heating
elements can be implanted in each of the three blades of a wind-turbine rotor in such a
way that they are distributed over the BS surface uniformly (say, one element per four
square meters), whereas the sensors can be attached to the inner surface of the BS at
their aforementioned locations. Thus, an SDS is supposed to consist of a sufficiently
large array of the identical heating elements (e.g., square- or disk-shaped) and com-
munication hardware that controls the array. Similarly, a related IDS is supposed to
consist of the corresponding sensor array and communication hardware that controls
this array. Both the controls can be implemented by a personal computer (PC).

To meet temporal variations of the AI landscape, both the SDS and IDS should
operate in the real-time mode. Both the SDS and IDS have components implanted in,
or attached to, the BSs. The BSs are rigidly attached to a wind-turbine rotor, which is in
general rotating, and, as follows from the published review works, both the SDS and
IDS must not have components located outside the body of a wind turbine. Therefore,
both the systems must be located inside the rotating rotor but accessible to a remote
operator at any time. For this reason, both the controls should be wireless. Moreover,
the controlling PC must also identify the parameters of the spatiotemporal AI layer
from the measurement data obtained by the sensors in the IDS, determine on the basis
of them the heat levels individual to the heating elements in the SDS, and send the level
data to the corresponding elements. The first of the three tasks should be implemented
in the parameter-identification software package (PISP) installed on the PC.

The main questions to be answered in the IDS design are the following:

• How can one design an IDS in order to enable it, firstly, to be relevant to an SDS for
  the BSs of a wind turbine, secondly, to be reliable, cost- and energy-efficient, and,
  thirdly, to supply the prospective PISP with the data measured by the ISD sensors?

• What are the parameter-identification procedures that can be implemented in the
  PISP?

The purpose of the present work is answering these questions by development of
design issues of an IDS relevant to an SDS for the wind-turbine BSs and procedures for
identification of the AI-layer parameters on the basis of acoustics of viscoelastic solids.
The work is underlain by the previous works of the authors [16] and [1]. Each of them
considers a thin curvilinear deformable solid layer associated with the BS and develops
a procedure for identification of the material parameters of the layer. Paper [16] deals
with a one-layer shell separating its interior and exterior, for instance, the non-QE
deformable solid shells of various engineering systems (e.g., the shells of driver or
passenger compartments of ships, cars, busses, airplanes, and other vehicles) or BS of a wind turbine. Paper [1] deals with the two-layer system of the BS layer and the AI layer accreted on the BS-layer outer surface.

Both works [16] and [1] consider the stress in a layer system as the one caused by the operational load on the system. The works describe the QE component of the average normal stress (ANS) with a third-order partial differential equation (PDE) of acoustic of viscoelastic solids. The deviatoric (or shear) stress is neglected for compactness of the model. The equation is introduced in [16], takes into account the stress-relaxation function (in the exponential approximation) in the integrand of the Boltzmann superposition integral, includes the stress-relaxation time (SRT) of the layer material, and is relevant at any value of the SRT. The latter property enables application to viscoelastic materials, in particular, AI (see Remark 1.2).

The parameter-identification procedures in papers [16] and [1] presume structural-health/operational-load monitoring (SH/OL-M) (e.g., [17]) with the help of the sensors located on the layer-system inner surface, i.e., the surface not affected by the environment. However, these papers consider a use of accelerometers as the sensors. This restricts application of the procedures to the QE systems only. In contrast to that, the present work regards a use of the sensors, which enable application of the parameter-identification procedures to general, non-QE layer systems. Also, papers [16] and [1] consider homogeneous conditions at the layer-system outer surface, i.e., the one affected by the environment, but do not include the operational-load-caused source term in the above equation. The present work overcomes these limitations as well.

The answer to the first question in the above bullet list is developed in terms of methods of SH/OL-M and with the emphasis on monitoring stress in BSs in Section 2. The answer to the second question in the mentioned list is developed in Appendixes A-C. Appendix A deals with the identification of the material parameters of a one-layer system. Appendix B shows how this identification approach is generalized for multilayer systems. Appendix C exemplifies this generalization in the case of the BS/AI-layer system, which comprises two layers. Section 3 summarizes the obtained results and presents the concluding remarks. The section "Notations", which also includes the list of the used abbreviations, completes the work.

2. Design Issues of an IDS, Which Is Relevant to an SDS for the BSs of a Wind Turbi

Wind turbines are driven by irregular wind under irregular weather conditions, more specifically, by the air flows at the outer surface of the BS/AI-layer. Practical techniques for measurement of the wind effect are still unknown, even if the outer surface of the BS is free from AI. The absence of the techniques deprives some of the most common acoustic methods. For example, structural dynamics (e.g., [18]) and lookup tables (e.g., [19]) cannot be applied because the data on the wind effect on the BS/AI-layer, which are necessary parts of the input data for these approaches, cannot be measured.

The only practical techniques are the ones, which do not use measurements of the
above wind effect. These techniques comprise methods of SH/OL-M. They are based on
the data measured by sensors and used in works [1] and [16]. Methods of SH/OL-M
apply various sensors, such as accelerometers or strain gauges (e.g., [20]).

In a deformable solid system, an acoustic signal, which is measured at a spatial point
of the system and can be regarded in SH/OL-M, is the QE component of one or another
mechanical variable at the point. Until now, the settings for measurements of the QE
components of the accelerations or strains in the non-QE systems are unknown.

The settings for measurements of the QE component of the stresses in the non-QE
systems are unknown either. However, the remark below proposes the corresponding
idea.

**Remark 2.1.** If a non-QE deformable solid system includes a cavity filled by air,
which remains at equilibrium and at the atmospheric pressure, then the QE component
of the stress in the wall of this cavity can be measured by an appropriate pressure/
normal-stress sensor as the difference between the values of the normal stresses in the
solid wall and air in the cavity.

An example of the non-QE systems noted in Remark 2.1 is the BS of an operating
wind turbine. Indeed, the BS is a hollow solid body where the cavity is isolated from the
external air. Other examples are the non-QE closed deformable solid hells and
interiors of driver or passenger compartments of ships, cars, busses, airplanes, and
other vehicles.

The sensors that can provide the measurements indicated in Remark 2.1 are the
pressure-sensing resistors also known as the force-sensing resistors (FSRs). They are
used in SH/OL-M. For example, in the wireless array of FSRs of the Honeywell Tech-
nology Center ([21], Sections 3.2.3 and 3.2.4), the FSRs are identical. Each of them is a
small-area thin planar resistor, which includes the film sensitive to the pressure or,
more precisely, normal-stress difference at the opposite planar surfaces of the film, or
to the related force.

The difference of the normal stresses at the opposite planar surfaces of an FSR,
\( \Pi(t) \), and the corresponding force, \( F(t) \), measured by the device, are coupled with
relation

\[
\Pi(t) = \frac{F(t)}{S_r}
\]  

(2.1)

where \( t \) is the time and \( S_r \) is the sensing area of the FSR. One of many examples of
FSRs is the FlexiForce A301 device [22].

**Remark 2.2.** The operating-temperature interval of the FlexiForce Standard Model
A301 FSR is from \(-40^\circ C\) up to \(60^\circ C\). The thickness of the device is \(0.203 \times 10^{-3}\).
The width and total length of the device is \(14 \times 10^{-3}\) and (approximately) \(32 \times 10^{-3}\),
respectively. This length includes 6-millimeter pins.

The sensing region of this device is a disk with the diameter \( 9.53 \times 10^{-3} \). The area of
this region, i.e., parameter \( S_r \) in (2.1), is \(0.7133 \times 10^{-4}\). The version "445 N" can
calculate force \( F(t) \) in the range from 0 to \(0.445 \times 10^3\). Taking into account that, the
mentioned area, and relation (2.1), one obtains that the interval of the measured
pressure is from 0 to \(6.24 \times 10^6\) (i.e., somewhat greater than 6 MPa). The force reading
change per degree of temperature change is 0.36. The output signal of the above sensor is presumed to be processed by the MCP6004 low-power operational amplifier of Microchip Technology. The operating-temperature range of this circuit includes the industrial temperature range, which is from −40°C to 85°C.

According to the sensor data sheet [23], the power consumed by one sensor together with the related electrical circuit, which includes one MCP6004 amplifier, does not exceed $20 \times 10^{-3}$.

The operating-temperature intervals noted in Remark 2.2 include a considerable part of the temperature values discussed in Remark 1.1. The interval of the measured pressure is also rather wide. Is it possible to measure the QE component of the normal stress in the BS/AI-layer system with FSRs? In order to answer this question, one can take a closer look at key aspects of the aerodynamics of the operation of a wind-turbine rotor.

The main aerodynamic effect, by which the operating rotor extracts the energy, is the pressure drop between the air domains directly in front of, and directly behind, the BS of a rotor. More specifically, the picture is the following.

If there is no wind, the BS as well as the air domains at the inner and outer surfaces of the BS are at equilibrium, and their pressures are the same and equal to the atmospheric pressure.

If there is a wind and the wind-turbine rotor operates, then:

• the air domain at the outer surface, which is directly in front of the BS, is at the above-atmospheric pressure,
• the air domain at the outer surface, which is directly behind the BS, is at the below-atmospheric pressure, and
• the BS is not at equilibrium,

whereas the pressure at the inner surface of the BS remains atmospheric. Consequently, in the course of the operation, there is always the pressure difference between the BS inner surface not affected by a wind and the BS outer surface affected by a wind. This difference manifests the presence of the stress distributed along the thickness of the BS.

This stress can be measured as follows. Assume that the working planar surface of an FSR is attached to the inner surface of the BS. Consequently, the opposite planar surface of the FSR contacts the equilibrium atmospheric-pressure air in the interior of the BS. Then, according to the well-known continuity of the QE component of the stress at, and normal to, the interface between two solids (e.g., [24], (1.48)), the normal stress at the FSR/BS interface is equal to the QE component of the stress at the interface. Similarly, according to the well-known continuity of the QE component of the stress at, and normal to, the interface between a solid and air, the QE component of the stress at, and normal to, the opposite planar surface of the FSR, i.e., the FSR/air interface, is equal to zero (e.g., [24], (1.49)). Since the FSR measures the normal-stress difference between its surfaces (see the text on (2.1)), it in fact measures the QE component of the mentioned stress in the BS at the FSR/BS interface.

The above part of the present section can be summarized as follows.
A sensor, which can measure the QE component of the normal stress in a non-QE closed deformable solid shell such as a BS of a wind turbine, is an FSR with its working planar surface attached to the BS inner surface and its opposite planar surface contacting the equilibrium air in the BS cavity. An array of the FSRs can be wirelessly connected to, and controlled by, a PC.

This picture is equally applicable to a more general case where the layer system measured by FSRs includes not only the BS layer but also the AI layer that can be accreted on the BS outer surface. The corresponding parameter-identification procedures are developed in Appendixes A-C.

The above settings can be implemented in a hardware configuration that includes an electronic communication subsystem for the sensor array. The components of it are standard manufactured (or off-the-shelf) products, very small and light, reliable, high-speed, inexpensive, and providing flexible scaling. Importantly, the sensor array and communication subsystem does not presume parts placed outside the BS.

To be specific, the present section considers a wind-turbine rotor that has three identical blades. The blade length is assumed to be of 50 (i.e., similar to the one of a fiberglass-reinforced epoxy blade of the Siemens SWT-2.3-101 wind turbine [25]). As one can estimate, the area of the BS layer of this blade is about 400. In order to sense the AI landscape discussed in Section 1, one should, near but outside the area of each heating element of an SDS (see Section 1), attach the FSRs, one per, say, four square meters. Thus, the area of 400 is covered with 100 sensors (see Row 1 of Table 1). As is noted above in this section, all of the sensors are attached to the inner surface of the BS. They can be connected by a flat cable of the Serial Peripheral Interface (SPI) bus (e.g., [26]) or the Inter-Integrated Circuit (I2C) bus (e.g., [27]). More specifically, the configuration for a rotor blade is the following.

**Table 1.** Key components of the electronic equipment for a three-blade rotor of a wind turbine according to the present approach. The communication hardware comprises the components in Rows 2 and 4 - 6.

<table>
<thead>
<tr>
<th>Rows</th>
<th>Quantity per blade</th>
<th>Quantity per rotor (of three blades)</th>
<th>Cost (excluding VAT) per rotor (kSEK)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>FlexiForce Standard Model A301 FSR (−40°C - 60°C)</td>
<td>100</td>
<td>300</td>
</tr>
<tr>
<td>2</td>
<td>Cable of the SPI or I²C bus (−40°C - 105°C)</td>
<td>1 (if the length is about 210)</td>
<td>3 (if the total length is about 630)</td>
</tr>
<tr>
<td>3</td>
<td>MCP6004 operational amplifier (−40°C - 85°C)</td>
<td>100</td>
<td>300</td>
</tr>
<tr>
<td>4</td>
<td>ATMEGA48 micro-controller (−40°C - 85°C)</td>
<td>100</td>
<td>300</td>
</tr>
<tr>
<td>5</td>
<td>GSM/GPRS communication module (−40°C - 85°C)</td>
<td>Not applicable</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>Rechargable, long-life battery (−40°C - 40°C)</td>
<td>Not applicable</td>
<td>1</td>
</tr>
</tbody>
</table>

* if one orders at the quantity of 300.
First, the bus cable of the length of about 210 is attached to the inner surface of the BS (see Row 2 of Table 1) in the form of a meander.

Next, 100 sensors are rigidly attached (stuck) to the same surface along, and in a close proximity to, the cable at the distances of about 2 between any two neighboring sensors. Note that the cable and sensors are placed in such a way that each sensor corresponds to the surface area of about four square meters. Then each of the sensors is connected to the cable via an amplifying circuit (e.g., see [23] for the recommended circuit, which includes the amplifier) and the micro-controller (see Rows 3 and 4 of Table 1).

Finally, each of the three cables is carried out from a respective blade into the rotor hub and connected to a micro-controller, a GSM/GPRS communication module, and a rechargeable long-life battery that are rigidly attached to the solid body of the hub (see Rows 5 and 6 of Table 1). The macro-controller can be of the same type as the one indicated in Row 4 of Table 1. The battery is recharged from the energy produced by the turbine.

The GSM/GPRS module wirelessly transmits the measured data to a remote PC, which is also equipped with a GSM/GPRS module, processes the data for identification of the material parameters of the AI that can be accreted on the BS, and controls the sensor array. The corresponding estimated cost for the described hardware does not exceed 31 kSEK per one wind turbine with three 50-meter blades (see the last row and column of Table 1). As follows from the last sentence in Remark 2.2, the power consumed by the above electronic system necessary for the operation of 300 sensors does not exceed 6.

The outcomes of the present section is the answer to the question in the first bullet in Section 1. The answer to the question in the second bullet is developed in Appendixes A-C.

3. Obtained Results and Concluding Remarks

Summing up the present work, one can note the following.

The present work is based on the third-order partial differential equation (PDE) of acoustics of viscoelastic solids for the quasi-equilibrium (QE) component of the average normal stress derived and used in the previous papers of the authors. This PDE includes the stress-relaxation time (SRT) for the material and is applicable at any value of the SRT.

The work specifies the notion of a smart deicing system (SDS) for blade shells (BSs) of a wind turbine. The stress in a BS is considered as the one caused by the operational load on the BS. The work developed key design issue of a prospective ice-detection system (IDS) able to supply an array of the heating elements of an SDS with the element-individual spatiotemporal data (see Section 2) and procedures for identification of the material parameters of atmospheric ice (AI) layer accreted on the outer surfaces of the BSs (see Appendixes A-C). Both the SDS and IDS flexibly allow for complex, curvilinear and space-time-varying shapes of BSs.
The proposed IDS presumes monitoring of the QE components of the normal stresses in BSs. The IDS is supposed to include an array of force-sensing resistors (FSRs) and communication hardware, as well as the parameter-identification software package (PISP), which provides the identification on the basis of the aforementioned PDE and the data measured by the FSRs. The IDS does not have hardware components located outside the outer surfaces of, or implanted in, the BSs. The FSR array and communication hardware are:

- Reliable because they comprise standard manufactured (or off-the-shelf) products only, which can, moreover, operate at the temperatures between −40°C and +40°C;
- Cost efficient because their estimated cost is 30 - 35 kSEK for a rotor with three 50-meter blades;
- Energy efficient because their estimated power consumption is within 10 watts in the case of the above rotor.

The present work extends methods of structural-health/operational-load monitoring (SH/OL-M) with measurements of the operational-load-caused stress in closed solid shells and, if the prospective PISP is used, endows the methods with identification of material parameters of the shells. The identification algorithms that can underlie the PISP are computationally efficient and suitable for implementation in the real-time mode.

The identification model and algorithms can deal with not only the single-layer systems such as the BS layer without the AI layer (see Appendix A as well as Table A1 for the input data and the parameters that can be identified) or two-layer systems such as the BS with the AI layer accreted on it (see Appendix C as well as Table C1 for the input data and the parameters that can be identified) but also multi-layer systems (see Appendix B). The outcomes are applicable to not only the BSs of wind turbines but also the non-QE closed single- or multi-layer deformable solid shells of various engineering systems (e.g., the shells of driver or passenger compartments of ships, cars, busses, airplanes, and other vehicles). The proposed monitoring of the normal-stress QE component in the mentioned shells extends methods of SH/OL-M.

The outcomes of the present work complement and further develop the results of the previous works of the authors, more specifically, papers [1], [16], and [32]. The topic for the nearest research is a better adjustment of the settings for the FSR-based measurement of the normal-stress QE components in BSs and a calibration (e.g., see Remark A.2) of the parameter-identification model and algorithms, as well as the resulting improvement of the PISP.

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sensing resistors and suggestion on the hardware configuration described in Section 2.

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Appendixes

This section comprises Appendixes A-C. They consider the stress in a layer system as the one caused by the operational load on the system and present necessary details on the models and methods for identification of the material parameters of the BS and AI layers.

A. Identification of the Material Parameters of the BS of an Operating Wind Turbine

The present section considers the case where the AI layer is not present at the outer surface of the BS. The QE component of ANS, $P_s$, can be described with linear PDE

$$\theta_s \frac{\partial^2 P_s}{\partial t^2} + \frac{\partial^2 P_s}{\partial t^2} = s_s^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)(1 + 2\theta_s \frac{\partial}{\partial t})P_s + L_s, \quad (A.1)$$

which is the non-homogeneous generalization of partial differential equation (PDE) (16), (2.11)). In (A.1), $x$, $y$, and $z$ are the spatial coordinates and term $L_s$ is due to the operational load on the BS (in particular, associated with the position and wind-induced rotation of the distributed mass of the BS). Consequently, $L_s$ in general depends on $x$, $y$, $z$, and $t$. Note that, at equilibrium, $L_s$ need not be zero (e.g., [14]).

The meanings of two other terms in (A.1) are the following: $s_s$ is the speed of the bulk acoustic waves (i.e., the ones that correspond to compressions/rarefactions in the shear-free case) in the BS and $\theta_s$ is the stress-relaxation time in the BS. Note that inequalities

$$s_s > 0, \quad (A.2)$$
$$\theta_s \geq 0 \quad (A.3)$$

hold due to the physical meanings of the two parameters. Speed $s_s$ is determined as follows (e.g., [16], (2.7))

$$s_s = \sqrt{\frac{K_s}{\rho_s}} \quad (A.4)$$

where $K_s$ is the bulk modulus and $\rho_s$ is the volumetric mass density. If both $\theta_s$ and $K_s$ are available, then the volume viscosity of the BS, $\eta_s$, can be determined as follows $\eta_s = K_s \theta_s$ (cp., the first equality in (1.5)).

There are other contributions of the operational load. For example,

- the QE component of the outer-air (wind) pressure acts at, and perpendicularly to, the outer surface of the BS,

$$\text{the gradient of the QE component of the outer-air (wind) pressure creates the air acceleration that acts at, and perpendicularly to the outer surface of the BS} \quad (A.5)$$

Remark A.1. Parameters $s_s$ and $\theta_s$ are the material parameters of the BS to be identified. If an analytical or tabular dependence of the speed of the bulk acoustic waves in the BS on the volumetric mass density of it, say, $s_s(\rho)$ is available, then $\rho_s$ is determined as the solution of equation

$$s_s = s_s(\rho_s) \quad (A.7)$$
This enables one to evaluate $K_s$ from (A.4). Thus, $\rho_s$ and $K_s$ can also be available.

The material parameters of the BS to be identified, $s$, and $\theta$, are parameters of Equation (A.1). Usually, parameters of an equation can be identified if a solution of the equation is available. However, Equation (A.1) includes term $L_s$, which is unknown. Moreover, the pressure and acceleration noted in features (A.5) and (A.6) are also unknown. Consequently, description (A.1), (A.5), (A.6) seems useless for the parameter identification.

However, before rejecting it, one can consider particular, more specific cases of this description in hope that a better specificity will change the role of the unknown terms in such a way that it will be possible to include the “successors” of them in the identification procedure. For the reason explained in Section 2, one can focus on the distribution of $P_s$ along the thickness of the BS, say, the x-axis. Note that

$$y = 0 \text{ and } z = 0, \quad \text{in the x-axis.} \quad (A.8)$$

In order to pass to the x-version of Equation (A.1), we first rewrite it in the following equivalent form

$$\theta_s \frac{\partial^3 P_s}{\partial t^3} + \frac{\partial^2 P_s}{\partial t^2} = s_s \frac{\partial^2}{\partial x^2} \left(1 + 2\theta_s \frac{\partial}{\partial t}\right) P_s + L_s$$

where

$$L_s = s_s \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(1 + 2\theta_s \frac{\partial}{\partial t}\right) P_s + L_s, \quad (A.10)$$

and then substitute equalities (A.8) into (A.9). This results in

$$\theta_s \frac{\partial^3 P_s(x,t)}{\partial t^3} + \frac{\partial^2 P_s(x,t)}{\partial t^2} = s_s \frac{\partial^2}{\partial x^2} \left(1 + 2\theta_s \frac{\partial}{\partial t}\right) P_s(x,t) + L_s(x,t), \quad (A.11)$$

where

$$P_s(x,t) = P_s \big|_{y=0, z=0}, \quad (A.12)$$

$$L_s(x,t) = L_s \big|_{y=0, z=0}. \quad (A.13)$$

Without a loss of generality, one can assume that the x-axis is normal to the BS layer and point $x = 0$ corresponds to the inner surface of the BS. Common relation for the QE component of the ANS (A.12) at the solid-air interface $x = 0$ is (e.g., [24], (1.49))

$$P_s(0,t) \equiv 0,$$

(A.14)

The inner air is located to the left from this interface. As noted in Section 2, it remains at the atmospheric pressure.

Let $h$ be the thickness of the BS along the x-axis. Then point $x = h$ corresponds to the outer surface of the BS. One can also assume that the BS layer is spatially regular sufficiently in order to allow to regard the x-axis as normal to the outer BS surface at point $x = h$.

For the sake of simplicity, we consider the air acceleration indicated in (A.6) to be identically zero. Then, the effects of the phenomena (A.5) and (A.6) at point $x = h$
can be formulated as follows

\[ P_x(h_x,t) = \Pi(t), \quad (A.15) \]
\[ \partial P_x(h_x,t) / \partial x = 0 \quad (A.16) \]

where \( \Pi(t) \) is the outer-air pressure along the \( x \)-axis at the BS/outer-air interface \( x = h_x \). The obtained description for the QE component of the ANS (A.12) comprises (A.11) and (A.13)-(A.16).

It is in one spatial coordinate and is formally simpler than description (A.1), (A.5), (A.6) in three spatial coordinates. Still, the new settings do not enable one to obtain solutions of equation (A.11) because term \( L_x(x,t) \) (see (A.13) and (A.10)) in it and term \( \Pi(t) \) in (A.15) are unknown. However, data of a sensor can help to specify at least some terms in Equation (A.11) or its particular version.

Indeed, if one, at point \( x = 0 \), separates the BS from the inner air with a body of an FSR (see Section 2), then the interface at point \( x = 0 \) is the one between two solids (rather than a solid and air) that motivates to change relation (A.14) to (e.g., see [24], (1.48))

\[ P_x(0,t) = \Pi(t) \quad (A.17) \]

which expresses the continuity of the stresses at the mentioned interface. The passing from (A.14) to (A.17) make it possible to obtain a prototype for an ordinary differential equation (ODE), which can be used for the identification. Indeed, applying value \( x = 0 \) to Equation (A.11), one obtains

\[ \theta_x \partial^3 P_x(0,t) / \partial t^3 + \partial^2 P_x(0,t) / \partial t^2 = s_x \partial^2 / \partial x^2 \left( 1 + 2 \theta_x \partial / \partial t \right) P_x(0,t) + L_x(0,t) \quad (A.18) \]

that, after substitution of (A.17), takes the following form

\[ \theta_x \partial^2 \Pi(t) / \partial t^2 + \partial \Pi(t) / \partial t = s_x \partial^2 / \partial x^2 \left( 1 + 2 \theta_x \partial / \partial t \right) P_x(0,t) + L_x(0,t) \quad (A.19) \]

where the derivatives on the left-hand side are known because values of function \( \Pi(t) \) are available as the FSR data. Relation (A.19) is an ODE with the known solution \( \Pi(t) \). This ODE can be specified further if the spatial derivative on the right-hand side is expressed in terms of the solution.

The simplest way to this expression is based on an approximation for \( P_x(x,t) \), as function of \( x \), in the form of a polynomial in \( x \) with the \( t \)-dependent coefficients that can be determined from the information on \( P_x(x,t) \) (cp., the method of ([16], Section 4]). This information comprises three identities, (A.15)-(A.17). Consequently, the polynomial has three coefficients. We consider its following specific form

\[ P_x(x,t) = g_x(t) + \left[ h_x(t) / 2 \right] x^2 + \left[ c_x(t) / n! \right] x^n \quad (A.20) \]

where the \( t \)-independent integer number

\[ n > 2 \quad (A.21) \]

is the degree of the polynomial. Representation (A.20) provides the estimation for the aforementioned spatial derivative, namely
\[
\frac{\partial^2}{\partial x^2}(1 + 2 \theta_j \partial/\partial t) P_j(0, t) = b_j(t) + 2 \theta_j \frac{db_j(t)}{dt}.
\]  
(A.22)

Substitution of (A.20) into (A.15)-(A.17) results in

\[
g_j(t) = \Pi(t),
\]
(A.23)

\[
b_j(t) = \frac{1}{h^2_j} \frac{2n}{n-2} \left[-\Pi(t) + \bar{\Pi}(t)\right],
\]
(A.24)

\[
c_j(t) = \frac{(n-1)!}{h^2_j} \frac{2n}{n-2} \left[\Pi(t) - \bar{\Pi}(t)\right].
\]
(A.25)

Combination of (A.22) and (A.24) transforms (A.19) into ODE

\[
\theta_j d^3\Pi(t)/dt^3 + d^2\Pi(t)/dt^2 + 2q_j \theta_j d\Pi(t)/dt + q_j \Pi(t) = Q_j(t)
\]
(A.26)

for function \(\Pi(t)\) where

\[
Q_j(t) = q_j \left(1 + 2 \theta_j \frac{d}{dt} \Pi(t) + L_j(0, t)\right),
\]
(A.27)

\[
q_j = s^2_j \frac{1}{h^2_j} \frac{2n}{n-2}.
\]
(A.28)

Note that, by virtue of (A.2) and (A.21), inequality

\[
q_j > 0
\]
(A.29)

holds. We also note that, in view of the role, which parameter

\[
\kappa_j = \sqrt{\frac{2n}{n-2} \frac{1}{h^2_j}}
\]
(A.30)

plays in ODE (A.26) for value (A.17) of a solution of PDE (A.11), this parameter can be interpreted as the characteristic wave number for the mentioned value.

Equation (A.26) presents an ODE with:

- Solution \(\Pi(t)\) available as the measurement results provided by the FSR,
- Unknown parameters \(\theta_j\) and \(q_j\), and
- Function \(Q_j(t)\), which is unknown due to the fact that term \(\Pi(t)\) in (A.27) and term \(L_j\) in (A.10), and, thus, term \(L_j\) in (A.27) are unknown.

The latter feature indicates that function \(Q_j(t)\) should, in one or another way, be included in the identification of parameters \(\theta_j\) and \(s_j\) or, by virtue of (A.28), \(q_j\). This can be implemented by the procedure, which is described below and starts with the following three auxiliary steps. The corresponding input data and parameters that can be identified are listed in Table A1.

One considers ODE (A.26) at any three successive time points, say, \(t_1\), \(t_2\), and \(t_3\) measured by the FSR (see the upper half of Table A1)

\[
\theta_j \Pi''_j + \Pi''_j + 2q_j \theta_j \Pi'_j + q_j \Pi_j = Q_j(t_k), \quad k = 1, 2, 3.
\]
(A.31)

where \(\Pi'_j\), \(\Pi''_j\), and \(\Pi''_j\) are the finite-difference (FD) approximations for the first, second, and third time derivatives of function \(\Pi(t)\) at the mentioned points calculated on the basis on of the data measured by the FSR (see the upper half of Table A1). We assume that the length of the time interval between the left and right time points, \(i.e., t_3 - t_1\), is much smaller than the characteristic time of function \(Q_j(t)\) in
Table A1. Identification method for one-layer systems: The input data and the parameters that can be identified.

<table>
<thead>
<tr>
<th>Input data</th>
</tr>
</thead>
<tbody>
<tr>
<td>* t&lt;sub&gt;1&lt;/sub&gt;,···,t&lt;sub&gt;N&lt;/sub&gt;, the successive time points, at which the FSR measures the normal stress at the inner surface of the BS, ( N \geq 4 )</td>
</tr>
<tr>
<td>( \Pi_1, \ldots, \Pi_N ), the values of the stress ( \Pi(t) ) measured by the FSR at the above time points</td>
</tr>
<tr>
<td>( n ), the parameter determined from the calibrating identification as is described in Remark A.2</td>
</tr>
<tr>
<td>( h ), the thickness of the BS</td>
</tr>
<tr>
<td>Optional: an analytical or tabular dependence of the speed of the bulk acoustic waves in the BS on the volumetric mass density of the BS (i.e., dependence ( s_s(\cdot) ) in Remark A.1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters that can be identified</th>
</tr>
</thead>
<tbody>
<tr>
<td>At any three successive time points measured by the FSR (see the upper half of this table), one can identify the following parameters:</td>
</tr>
<tr>
<td>( s_s ), the speed of the bulk acoustic waves in the BS</td>
</tr>
<tr>
<td>( \theta ), the stress-relaxation time of the BS</td>
</tr>
<tr>
<td>( Q_\theta ), the source term in the acoustic equation (see the text on (A.31) and (A.32))</td>
</tr>
<tr>
<td>( \rho ), the volumetric mass density of the BS (if the above option is available)</td>
</tr>
<tr>
<td>( K ), the bulk modulus of the BS (if the above option is available)</td>
</tr>
</tbody>
</table>

this interval. This feature allows to assume that \( Q_s(t_1) = Q_s(t_2) = Q_s(t_3) \). Denoting this value with \( Q_\star \), and substituting it into (A.31), one gets

\[
q_i \Pi_{i+1} + \theta_i \Pi_{i+2} - Q_\star = -\Pi_{i+2} - 2q_i \theta_i \Pi_i, \quad k = 1, 2, 3. \tag{A.32}
\]

Equations (A.32) present a system of bilinear equations with constant coefficients for unknown numbers \( q_s \), \( \theta_s \), and \( Q_\star \). The left-hand sides of equalities (A.32) present the linear part of this system. Solution of the system can be simplified if one expresses \( q_s \), \( \theta_s \), and \( Q_\star \) on the left-hand sides in terms of the right-hand sides of the system. Assuming that matrix of the linear part is nonsingular, i.e.,

\[
\begin{vmatrix}
\Pi_1 & \Pi_2^\prime & \Pi_3^\prime & -1 \\
\Pi_2 & \Pi_3^\prime & -1 & 0 \\
\Pi_3 & -1 & 0 & 0 \\
\end{vmatrix} = (\Pi_1^\prime \Pi_2 - \Pi_2^\prime \Pi_1^\prime + (\Pi_3^\prime \Pi_1 - \Pi_1^\prime \Pi_3^\prime) + (\Pi_2^\prime \Pi_3 - \Pi_3^\prime \Pi_2^\prime) \neq 0, \tag{A.33}
\]

one easily calculates the \( \alpha \) and \( \beta \) coefficients in the resulting expressions

\[
q_s = \alpha_q + 2\beta_q q_s \theta_s, \tag{A.34}
\]
\[
\theta_s = \alpha_\theta + 2\beta_\theta q_s \theta_s, \tag{A.35}
\]
\[
Q_\star = \alpha_Q + 2\beta_Q q_s \theta_s. \tag{A.36}
\]

If the inequality in relation (A.33) is not valid, then the rank of the matrix in this relation is less than three. Consideration of this special case is not difficult. However, it is beyond the scope of the present work.

Solution of system (A.34)-(A.36) with respect to \( q_s \), \( \theta_s \), and \( Q_\star \) is rather simple in the following three cases.
If $\beta_q = 0$ and $\beta_\theta = 0$, then $q_s = \alpha_q$, $\theta_s = \alpha_\theta$, and $Q_{s*} = \alpha_0 + 2\beta_q \alpha_q \alpha_\theta$ are the identified values of the three parameters provided that inequalities (A.29) and (A.3) hold.

If $\beta_q = 0$ and $\beta_\theta \neq 0$, then $q_s = \alpha_q$, $\theta_s = \alpha_\theta$, and

\[ Q_{s*} = \alpha_0 + 2\beta_\theta \alpha_q \alpha_\theta (1 - 2\beta_q \alpha_q) \]

are the identified values of the three parameters provided that inequalities (A.29) and (A.3) hold.

If $\beta_q \neq 0$ and $\beta_\theta = 0$, then

\[ q_s = \alpha_q (1 - 2\beta_q \alpha_q), \quad \theta_s = \alpha_\theta, \quad \text{and} \]

\[ Q_{s*} = \alpha_0 + 2\beta_q \alpha_q \alpha_\theta (1 - 2\beta_q \alpha_q) \]

are the identified values of the three parameters provided that inequalities (A.29) and (A.3) hold.

The case where $\beta_q \neq 0$ and $\beta_\theta \neq 0$ is more complex. In this case, we multiply (A.34) and (A.35) by $\beta_\theta$ and $\beta_q$, respectively, and subtract the second of the resulting equalities from the first one. This results in relation $\beta_q q_s - \beta_\theta \theta_s = \beta_\theta \alpha_q - \beta_q \alpha_\theta$ or, equivalently,

\[ \beta_\theta q_s = \beta_q \theta_s + (\beta_\theta \alpha_q - \beta_q \alpha_\theta). \]  \hspace{1cm} (A.37)

Substitution of (A.37) into (A.35) transforms the latter into quadratic equation

\[ 2\beta_q \theta_s^2 + [2\beta_\theta \alpha_q + \beta_q \alpha_\theta - 1] \theta_s + \alpha_\theta = 0 \]  \hspace{1cm} (A.38)

If this equation has exactly one root, which meets condition (A.3), and the corresponding value of $q_s$ indicated by (A.37) meets condition (A.29), then the determined values are the identified parameters. They determine the identified value of $Q_{s*}$ by means of (A.36).

**Remark A.2.** In each of the above cases, parameter $s_i$ is determined from (A.28) by means of the identified $q_s$ and parameter $n$ (see (A.21)). This means that $n$ must be a part of the input data and, thus, be available before the identification. It can be determined from the calibrating identification, i.e., the one at an already available $s_i$ (see (A.4)). In this case, $n$ is calculated from (A.28) by means of the already known $s_i$ and identified $q_s$.

Thus, the input data cannot be obtained without measured data for stress $\Pi(t)$. □

**Remark A.3.** Importantly, the above identification of parameters $q_s$ and $\theta_s$ also provides identification of parameter $Q_{s*}$ (see the text above (A.32)). It is the corresponding value of unknown function (A.27), which includes unknown, wind-related terms $\Pi(t)$ and (A.10). Due to this, the identified source term $Q_{s*}$ in (A.32) presents valuable information.

The above also shows that the proposed method is applicable in spite of the presence of the unknown terms in the model. This is an important practical advantage of the method. □

As soon as $q_s$ is identified, parameter $s_i$ is identified according to (A.28). The rest of the identification procedure is described in the last sentence of Remark A.1.

**Remark A.4** (cp., the discussion in [16], Section 4). The proposed method identifies the parameters in a time interval comprising three successive time points, thereby preserving that the parameters are independent of time in this interval. The method
can be applied to each of the three successive-point intervals that can be considered in the time-point sequence indicated as a part of the input data for the method (see the upper half of Table A1). As a result, one obtains each of the identified parameters in the form of a piecewise constant function of the time. This function need is not single-valued because the three-point intervals are mutually intersecting and, at the intersections, the function can have two values. The time dependences of this type are rather irregular and need special techniques for smoothing or other processing. These techniques are beyond the scope of the present work.

The next section outlines how the proposed approach can be generalized for multi-layer systems.

B. Generalization of the Parameter Identification from One-Layer Systems to Multi-Layer Systems

The previous section proposes a method for the identification of the material parameters of the BS layer of an operating wind-turbine rotor. The present section explains how this approach is generalized for the identification of the parameters of multi-layer system.

One can consider the $l$-layer system where $l \geq 2$, the $x$-axis is directed along the normal to all of the layers, point $x = 0$, as before, corresponds to the inner surface of the system, and $h_j$, $j = 1, \cdots, l$, is the thickness of the $j$th layer. Then $x_1 = h_1$ and $x_j = x_{j-1} + h_j$, $j = 2, \cdots, l$, are the coordinates of the boundaries of the layers. Note that $x_l$ is the outer surface of the layer system. Thus, the $l$ layers correspond to $l$ intervals: $0 < x < x_i$ and $x_{j-1} < x < x_j$, $j = 2, \cdots, l$.

Equations (A.18) or (A.19) corresponds to the left boundary of the $x$-interval considered in the one-layer treatment. The counterparts of (A.19), i.e., the equations corresponding to the left boundaries of the above $l$ intervals are

$$
\begin{align*}
\theta_j \frac{d^3 P_j(x_{j-1}, t)}{dx^3} + d^2 P_j(x_{j-1}, t)/dx^2 = s_j^2 \frac{\partial^2}{\partial x^2} (1 + 2\theta_j \frac{\partial}{\partial t}) P_j(x_{j-1}, t) + L_j(x_{j-1}, t), \\
0 < x < x_j,
\end{align*}
$$

Equation (B.1)

$$
\begin{align*}
\theta_j \frac{d^3 P_j(x_{j-1}, t)}{dx^3} + d^2 P_j(x_{j-1}, t)/dx^2 = s_j^2 \frac{\partial^2}{\partial x^2} (1 + 2\theta_j \frac{\partial}{\partial t}) P_j(x_{j-1}, t) + L_j(x_{j-1}, t), \\
x_{j-1} < x < x_j, \quad j = 2, \cdots, l.
\end{align*}
$$

Equation (B.2)

The boundary data at the layer surfaces are

$$
\begin{align*}
P_j(0, t) &\equiv \Pi(t), \\
P_{j-1}(x_{j-1}, t) &\equiv P_j(x_{j-1}, t), \\
-\rho_{j-1} \frac{\partial P_{j-1}(x_{j-1}, t)}{\partial x} &\equiv -\rho_j \frac{\partial P_j(x_{j-1}, t)}{\partial x}, \\
P_j(x_j, t) &\equiv \Pi(t), \\
\frac{\partial P_j(x_j, t)}{\partial x} &\equiv 0.
\end{align*}
$$

Equation (B.3)

Relation (B.3) is similar to (A.17) because it is set at the inner surface of the layer system. Relations (B.6) and (B.7) are similar to (A.15) and (A.16) because they are set at
the outer surface of the layer system. Relations (B.4) and (B.5) express the continuity of the normal stress and normal acceleration at the interfaces between the layers.

Remark B.1. The number of the boundary equalities (B.3)-(B.7) is $2l + 1$. □

Terms $P_{1}(0,t)$ and $P_{j-1}(x_{j-1},t)$, $j = 2,\cdots, l$, are the boundary values included in (B.3) and (B.4). Relation (B.3) determines $P_{1}(0,t)$ in terms of normal stress $\Pi(t)$ measured by the FSR. This enables one to rewrite equation (B.1) as

$$\theta_{1} d^{3}\Pi(t)/dt^{3} + d^{2}\Pi(t)/dt^{2} = \frac{s_{1}^{2}}{s_{1}} \frac{\partial^2 \Pi}{\partial x^2} (1 + 2\theta_{1} \partial/\partial t) P_{1}(0,t) + L_{1}(0,t),$$  \hspace{1cm} (B.8)

which is similar to (A.19). One can also consider a possibility to express boundary values $P_{j-1}(x_{j-1},t)$, $j = 2,\cdots, l$, in terms of $\Pi(t)$. Moreover, following the idea of the method presented in the previous section (see the text below (A.19)), one needs to express the spatial derivatives on the right-hand sides of (B.8) and (B.2) in terms of $\Pi(t)$.

Proceeding in these directions, one can involve a polynomial similar to (A.20) for each layer. This results in $l$ polynomials

$$P_{1}(x,t) = g_{1}(t) + [h_{1}(t)/2]x^2 + [c_{1}(t)/n!]x^n, \hspace{1cm} 0 < x < x_{1},$$  \hspace{1cm} (B.9)

$$P_{j}(x,t) = g_{j}(t) + [h_{j}(t)/2](x-x_{j-1})^2 + [c_{j}(t)/n!] (x-x_{j-1})^n, \hspace{1cm} x_{j-1} < x < x_{j}, \hspace{1cm} j = 2,\cdots, l,$$  \hspace{1cm} (B.10)

where $n$ and $c_{j}(t)$ are independent of $j$, and $n$ is described as before (see the text on (A.21)). They in particular provide estimations (cp., (A.22))

$$\frac{\partial^2 \Pi}{\partial x^2} (1 + 2\theta_{1} \partial/\partial t) P_{1}(0,t) = b_{1}(t) + 2\theta_{1} \frac{db_{1}(t)}{dt},$$  \hspace{1cm} (B.11)

$$\frac{\partial^2 \Pi}{\partial x^2} (1 + 2\theta_{j} \partial/\partial t) P_{j}(x_{j-1},t) = b_{j}(t) + 2\theta_{j} \frac{db_{j}(t)}{dt}, \hspace{1cm} j = 2,\cdots, l.$$  \hspace{1cm} (B.12)

The number of the coefficients of polynomials (B.9) and (B.10) is $2l + 1$, i.e., the same as the number of the boundary equalities (see Remark B.1). Consequently, these coefficients can be determined. The resulting expressions enable one to express $P_{1}(0,t)$, $P_{j-1}(x_{j-1},t)$, $j = 2,\cdots, l$, and spatial derivatives (B.12), (B.13) in terms of $\Pi(t)$. This, in turn, allows obtaining the corresponding versions of ODE (A.26) and implementing the analysis analogous to the one described in the previous section.

The above generalization for multi-layer systems is exemplified with application to a two-layer system, which comprises the BS and AI layers, in the next section.

C. Identification of the Parameters of the AI Layer Accreted on the BS of an Operating Wind Turbine

Section A considers the case where the $x$-interval for the BS is $0 < x < h_{1}$. The present section deals with a more general case where the AI layer of the thickness $h$ is accreted on the outer surface of the BS $x = h_{1}$. Consequently, the $x$-interval for the AI layer is $h_{1} < x < h_{1} + h$. Thus, the layer system under consideration comprises two layers.

We specify the two-layer version of equations (B.8), (B.2), boundary data (B.3)-(B.7), polynomials (B.9), (B.10), and terms (B.11), (B.12) in the following forms

$$\theta_{2} d^{3}\Pi(t)/dt^{3} + d^{2}\Pi(t)/dt^{2} = \frac{s_{2}^{2}}{s_{2}} \frac{\partial^2 \Pi}{\partial x^2} (1 + 2\theta_{2} \partial/\partial t) P_{2}(0,t) + L_{2}(0,t),$$  \hspace{1cm} (C.1)
\[
\theta d^3 P(h, t)/dt^3 + d^2 P(h, t)/dt^2 = s^2 \partial^2/\partial x^2 (1 + 2\theta \partial/\partial t) P(h, t) + L(h, t), \quad (C.2)
\]
\[
P_r(0, t) \equiv \Pi(t), \quad (C.3)
\]
\[
P_r(h, t) \equiv P(h, t), \quad (C.4)
\]
\[
-\rho^{-1} \partial P_r(h, t)/\partial x \equiv -\rho^{-1} \partial P(h, t)/\partial x, \quad (C.5)
\]
\[
P(h + h, t) \equiv \Pi(t), \quad (C.6)
\]
\[
\partial P(h + h, t)/\partial x \equiv 0, \quad (C.7)
\]
\[
P_r(x, t) = g_r(t) + \left[ b_r(t)/2 \right] x^2 + \left[ c_r(t)/n! \right] x^n, \quad 0 < x < h_r, \quad (C.8)
\]
\[
P(x, t) = g(t) + \left[ b(t)/2 \right] (x - h)^2 + \left[ c(t)/n! \right] (x - h)^n, \quad h < x < h + h, \quad (C.9)
\]
\[
\partial^2/\partial x^2 (1 + 2\theta \partial/\partial t) P_r(0, t) = b_r(t) + 2\theta \partial_1 b_r(t)/\partial t, \quad (C.10)
\]
\[
\partial^2/\partial x^2 (1 + 2\theta \partial/\partial t) P_r(h, t) = b(t) + 2\theta \partial b(t)/\partial t \quad (C.11)
\]

where \( P \) is the QE component of the ANS in the AI and the coefficients without the subscript “s” corresponds to the AI. Note that, in view of (C.9),
\[
g(t) = P(h, t). \quad (C.12)
\]

One can determine the five coefficients of polynomials (C.8) and (C.9) from the five boundary relations (C.3)-(C.7). The results are:
\[
g_r(t) = \Pi(t), \quad (C.13)
\]
\[
b_r(t) = \frac{1}{h^2} \frac{2n}{n - 2} \frac{1}{1 + u} \left[ -\Pi(t) + \Pi(t) \right], \quad (C.14)
\]
\[
g(t) = \frac{1 + u}{1 + u} \Pi(t) + \frac{u}{1 + u} \Pi(t), \quad (C.15)
\]
\[
b(t) = \frac{1}{h^2} \frac{2n}{n - 2} \frac{u}{1 + u} \left[ -\Pi(t) + \Pi(t) \right], \quad (C.16)
\]
\[
c_r(t) = \frac{(n-1)!}{h^2} \frac{2n}{n - 2} \frac{1}{1 + u} \left[ \Pi(t) - \Pi(t) \right] \quad (C.17)
\]

where
\[
u = (h/h)^n. \quad (C.18)
\]

**Remark C.1.** In the limit case as \( h \downarrow 0 \), i.e., in the limit where the AI layer is not present, relation \( u \downarrow 0 \) also holds (see (C.18)), and values (C.13), (C.14), and (C.17) tend to (A.23)-(A.25), respectively, as must be. Moreover, in this limit case, \( P(x, t) \) tends to \( \Pi(t) \) due to (C.9), (C.18) and (C.15).

Application of (C.14) and (C.16) to (C.10) and (C.11), respectively, leads to
\[
\frac{\partial^2}{\partial x^2} (1 + 2\theta \partial/\partial t) P_r(0, t)
\]
\[
= -\frac{1}{h^2} \frac{2n}{n - 2} \frac{1}{1 + u} \left[ \Pi(t) + 2\theta \partial_1 \Pi(t)/\partial t \right] + \frac{1}{h^2} \frac{2n}{n - 2} \frac{1}{1 + u} \left[ \Pi(t) + 2\theta \partial_1 \Pi(t)/\partial t \right], \quad (C.19)
\]
\[
\frac{\partial^2}{\partial \xi^2} \left(1 + 2\theta \frac{\partial}{\partial \xi}\right) P(h, t)
= -\frac{1}{h^2} \frac{2n}{n-2} \frac{u}{1 + u} \left[\Pi(t) + 2\theta \frac{\partial \Pi(t)}{\partial t}\right] + \frac{1}{h^2} \frac{2n}{n-2} \frac{u}{1 + u} \left[\widetilde{\Pi}(t) + 2\theta \frac{\partial \widetilde{\Pi}(t)}{\partial t}\right]. \tag{C.20}
\]

Substituting (C.19) and (C.20) into the right-hand sides of (C.1) and (C.2), respectively, one obtains
\[
\theta^2 \frac{d^3 \Pi(t)}{d t^3} + d^2 \Pi(t) + \left[2q, \theta \frac{d \Pi(t)}{dt} + q \Pi(t)\right] \nu = Q_1(t), \tag{C.21}
\]
\[
\theta^2 \frac{u}{1 + u} \frac{d^3 \Pi(t)}{d t^3} + \frac{u}{1 + u} \frac{d^2 \Pi(t)}{d t^2} + 2q, \theta \frac{u}{1 + u} \frac{d \Pi(t)}{dt} + q \Pi(t) = Q(t). \tag{C.22}
\]

where \( q \) is described with (A.28), parameters \( s \) and \( \theta \) in (C.21) are a part of the input data (see the upper half of \textbf{Table C1}), the role of \( n \) is the same as the one described in Remark A.2, and
\[
\nu = \frac{1}{1 + u}, \tag{C.23}
\]
\[
q = s^2 \frac{2n}{h^2 n - 2}, \tag{C.24}
\]
\[
Q_1(t) = \frac{1}{1 + u} q_s \left(1 + 2\theta_s \frac{d}{dt}\right) \Pi(t) + L_s(0, t), \tag{C.25}
\]

\textbf{Table C1.} Identification method for two-layer systems: The input data and the parameters that can be identified.

<table>
<thead>
<tr>
<th>Input data</th>
<th>Parameters that can be identified</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1, \cdots, t_n ), the successive time points, at which the FSR measures the normal stress at the inner surface of the BS, ( N \geq 4 )</td>
<td>( Q_s ), the source term in the acoustic equation for the BS (see the text on (C.29) and (C.31))</td>
</tr>
<tr>
<td>( \Pi_1, \cdots, \Pi_n ), the values of the QE component of the stress ( \Pi(t) ) measured by the FSR at the above time points</td>
<td>( h ), the thickness of the AI</td>
</tr>
<tr>
<td>( n ), the parameter determined from the calibrating identification for the BS without the AI layer as is described in Remark A.2</td>
<td>( s ), the speed of the bulk acoustic waves in the AI</td>
</tr>
<tr>
<td>( h ), the thickness of the BS</td>
<td>( \theta ), the stress-relaxation time of the BS</td>
</tr>
<tr>
<td>( s ), the speed of the bulk acoustic waves in the BS</td>
<td>( Q ), the source term in the acoustic equation for the AI layer (see the text on (C.29) and (C.31))</td>
</tr>
<tr>
<td>( \theta ), the stress-relaxation time of the BS</td>
<td>( \rho ), the volumetric mass density of the AI</td>
</tr>
<tr>
<td>an analytical or tabular dependence of the speed of the bulk acoustic waves in the AI on the volumetric mass density of the AI, ( \rho ), dependence ( s(\cdot) ) in (C.36) (e.g., see (2.1), Figure 1)</td>
<td>( K ), the bulk modulus of the AI</td>
</tr>
<tr>
<td>( \rho ), the volumetric mass density of air (e.g., see (1.2))</td>
<td>( \varphi ), the porosity of the AI</td>
</tr>
<tr>
<td>( \varphi ), the porosity of the AI</td>
<td></td>
</tr>
</tbody>
</table>
\begin{align*}
Q(t) &= -\frac{1}{1+u}(1+\theta \frac{d}{dt})d^{2}\Pi(t)/dt^{2} + \frac{1}{1+u}q(1+2\theta \frac{d}{dt})\Pi(t) + L(h_{i},t). \tag{C.26}
\end{align*}

In view of (C.23) and (C.18), the acceptable values of \( v \) are such that
\[ 0 < v \leq 1. \tag{C.27} \]

Equalities (C.21) and (C.22) present two different ODEs for function \( \Pi(t) \). This function is a part of the input data (see the upper half of Table C1). The source terms (C.25) and (C.26) are unknown functions because they include unknown terms: \( L_{s}(0,t) \), \( L(h_{i},t) \), and \( \Pi(t) \). Thus, ODEs (C.21), (C.22) and source terms (C.25), (C.26) are the two-layer-system counterparts of the one-layer-system (A.26) and source term (A.27).

The ODEs can be used for identification of the AI layer parameters \( q \) and \( \theta \) provided that \( s_{s} \) (and, thus (see (A.28)), \( q_{s} \), and \( \theta_{s} \) are available (see the upper half of Table C1). This can be implemented with the help of equation (C.22). In order to enable that, the procedure should include identification of the corresponding source term. The value of \( u \) in (C.22) can be determined from (C.18) by means of identification of the two parameters, \( v \) and the corresponding source term in (C.21). Thus, the three parameters, \( q \), \( \theta \), and the source term in (C.22), can be determined after the above two parameters are determined.

The corresponding procedure can follow the already familiar way (see the text on (A.31) and (A.32)). More specifically, one considers ODEs (C.21), (C.22) in the interval comprising any three successive time points, say, \( t_{1}, t_{2}, \) and \( t_{3} \) measured by the FSR (see the upper half of Table C1) where any two of the three points, say, \( t_{1} \) and \( t_{3} \), are used for ODE (C.21) to identify the related two parameters, and all of the three points are used for ODE (C.22) to identify the other three parameters. This leads to the following versions of the ODEs
\begin{align*}
2q_{s}\theta_{s}\Pi_{k} + q_{i}\Pi_{k}v - Q_{s}(t_{k}) &= -\left(\theta_{s}\Pi^{s}_{k} + \Pi^{s}_{k}\right), \quad k = 1, 3, \tag{C.28}
\end{align*}
\begin{align*}
q\frac{u}{1+u}\Pi_{k} + \theta\frac{u}{1+u}\Pi^{s}_{k} - Q_{s}(t_{k}) &= -\frac{u}{1+u}\Pi^{s}_{k} - 2q\theta\frac{u}{1+u}\Pi^{s}_{k}, \quad k = 1, 2, 3, \tag{C.29}
\end{align*}
where \( \Pi_{k}, \Pi^{s}_{k}, \) and \( \Pi^{w}_{k} \) are the FD approximations for the first, second, and third time derivatives of function \( \Pi(t) \) at the mentioned points calculated on the basis on of the data measured by the FSR (see the upper half of Table C1). We assume that the length of the time interval between the left and right time points, i.e., \( t_{3} - t_{1} \) is much smaller than the characteristic times of functions \( Q_{s}(t) \) and \( Q(t) \) in this interval. This feature allows to assume that \( \Pi_{s}(t_{1}) = \Pi_{s}(t_{2}) = \Pi_{s}(t_{3}) \) and \( Q(t_{1}) = Q(t_{2}) = Q(t_{3}) \). Denoting these values with \( \Pi^{s}_{s} \) and \( Q_{s} \), respectively, and substituting them into (C.28) and (C.29), one gets
\begin{align*}
(2q_{s}\theta_{s}\Pi_{k} + q_{i}\Pi_{k})v - Q_{s} &= -\left(\theta_{s}\Pi^{s}_{k} + \Pi^{s}_{k}\right), \quad k = 1, 3, \tag{C.30}
\end{align*}
\begin{align*}
q\frac{u}{1+u}\Pi_{k} + \theta\frac{u}{1+u}\Pi^{s}_{k} - Q_{s} &= -\frac{u}{1+u}\Pi^{s}_{k} - 2q\theta\frac{u}{1+u}\Pi^{s}_{k}, \quad k = 1, 2, 3. \tag{C.31}
\end{align*}
By solving system (C.30) of two linear algebraic equations, one determines \( v \) and
where \( \nu \) should meet requirement (C.27). Then, \( u \) and \( h \) are evaluated according to (C.23) and (C.18),

\[
\begin{align*}
    u &= \frac{1}{\nu - 1}, \\
    h &= h_s u^{\nu_n}.
\end{align*}
\]  

If \( u = 0 \), then (see (C.33)) \( h = 0 \) as well that means that the AI-layer thickness is zero, \( i.e. \), this layer is not present. In this case, there are no material parameters of the layer to be identified.

If \( u \neq 0 \), then (see (C.32) and (C.27)) \( u > 0 \). In this case, one substitutes \( u \) into equation system (C.31) and identifies parameters \( q \), \( \theta \), and \( Q_1 \) from it. Since system (C.31) is exactly of the same form as the one of system (A.32), the parameters can be identified with the same method, \( i.e. \), the one described in the text on (A.33)-(A.38). The first two of the parameters should meet conditions (cp., (A.29), (A.3))

\[
\begin{align*}
    q &> 0, \\
    \theta &\geq 0.
\end{align*}
\]

As soon as \( q \) is available, parameter \( s \) is evaluated from (C.24) (where \( h \) is determined with (C.33)). Then (cp., Remark A.1), \( \rho \) is calculated as the unique positive solution of equation

\[
    s = s_\rho(\rho)
\]

(see the upper half of Table C1 for function \( s_\rho(\cdot) \)). The obtained value of \( \rho \) provides the bulk modulus and porosity of the AI, \( K \) and \( \varphi \), by means of (cp., (A.4)) \( K = \rho s^2 \) and (1.4), respectively.

The proposed identification method is a generalization of the one of Appendix A for two-layer systems. However, the discussion on the one-layer method, in particular, Remarks A.3 and A.4 (along with Remark A.2 already mention in the present section) are equally applicable to the present, two-layer case.
Notations

Abbreviations

AI—atmospheric ice
ANS—average normal stress
BS—blade shell
FD—finite difference
FSR—force-sensing resistor
IDS—ice-detection system
ODE—ordinary differential equation
PC—personal computer
PDE—partial differential equation
PISP—parameter-identification software package
QE—quasi-equilibrium
SDS—smart deicing system
SH/OL-M—structural-health/operational-load monitoring
SRT—stress-relaxation time

Roman Uppercase Letters

\( F(t) \)—force corresponding to \( \Pi(t) \) (see (2.1))
\( G \)—shear modulus of AI
\( K \)—bulk modulus of AI
\( K_s \)—bulk modulus of a BS
\( L \)—counterpart of term \( L_s \) in the case of the AI layer in the BS/AI-layer system
\( L_s \)—term for a BS determined with (A.10)
\( L_j \)—version of term \( L_s \) in the case of the \( j \)th layer of a multi-layer system
\( L^*_s \)—term in (A.1) due to the position and wind-induced rotation of the distributed mass of a BS
\( N \)—number of time points \( t_1, \ldots, t_N \)
\( P \)—version of quantity \( P_s \) in the case of the AI layer in the BS/AI-layer system
\( P_j \)—version of quantity \( P_s \) in the case of the \( j \)th layer in a multi-layer system
\( P_s \)—QE component of the ANS in a BS
\( Q(t) \)—term for the AI layer in the BS/AI-layer system determined with (C.26)
\( Q_j(t) \)—term for the BS determined with (A.27) in Appendix A and with (C.25) in Appendix C
\( Q* \)—approximate value of function \( Q_s(t) \) determined as is described in the text between (A.31) and (A.32)
\( S \)—sensing area of an FSR

Roman Lowercase Letters

\( b(t) \)—version of coefficient \( b_s(t) \) in the case of the AI in the BS/AI-layer system
\( b_j(t) \)—version of coefficient \( b_s(t) \) in the case of the \( j \)th layer in a multi-layer system
\( b_s(t) \)—coefficient of polynomial (A.20) for the BS layer
\( c_i(t) \) — coefficient of polynomial (A.20) for the BS layer
\( c_i(t) \) — version of coefficient \( c_i(t) \) in the case of a multi-layer system
\( g_i(t) \) — version of coefficient \( g_i(t) \) in the case of the AI layer in the BS/AI-layer system
\( g_i(t) \) — version of coefficient \( g_i(t) \) in the case of the \( j \)th layer in a multi-layer system
\( g_i(t) \) — coefficient of polynomial (A.20) for a BS

\( h \) — thickness of the AI layer
\( h_i \) — thickness of the BS layer

\( j \) — integer index, \( j = 1, \ldots, l \)

\( k \) — integer index, \( k = 1, \ldots, N \)

\( l \) — number of the layers in a multi-layer system

\( n \) — degree of polynomial (A.20) or polynomials (B.9), (B.10)

\( q \) — parameter determined with (C.24)

\( q_i \) — parameter determined with (A.28)

\( s \) — speed of the bulk acoustic waves in AI

\( s_i \) — version of parameter \( s_i \) in the case of the \( j \)th layer of a multi-layer system

\( s_i \) — speed of the bulk acoustic waves in a BS

\( s_i(\cdot) \) — analytical or tabular dependence of the speed of the bulk acoustic waves in AI on \( \rho \)

\( s_i(\cdot) \) — analytical or tabular dependence of the speed of the bulk acoustic waves in a BS on \( \rho_i \)

\( t \) — time

\( t_1, \ldots, t_N \) — successive time points, at which an FSR measures the normal stress at the inner surface of a BS

\( u \) — parameter determined with (C.18)

\( x, y, z \) — three scalar spatial coordinates

\( x_j \) — thickness of the first \( j \) layers in a multi-layer system (see the second paragraph of Section B)

**Greek Uppercase Letters**

\( \Pi(t) \) — difference of the normal stresses at the opposite planar surfaces of an FSR; stress in a BS at the location of an FSR on the inner surface of the BS

\( \Pi_1, \ldots, \Pi_N \) — values of the stress \( \Pi(t) \) measured by an FSR at time points \( t_1, \ldots, t_N \)

\( \Pi'_1, \Pi'_2, \Pi'_3 \) — FD approximations for the first, second, and third time derivatives of function \( \Pi(t) \) at time point \( t_k \) calculated on the basis on of values \( t_1, \ldots, t_N \) and \( \Pi_1, \ldots, \Pi_N \)

\( \Pi(t) \) — outer-air pressure along the \( x \)-axis at the interface \( x = h_i \) between a BS and the outer air or at the interface \( x = x_j \) between the last layer in a multi-layer system (e.g., the AI layer in the BS/AI-layer) and the outer air (in the AI-layer case, this interface is at \( x = h_i + h \))

**Greek Lowercase Letters**

\( \alpha_a, \alpha_b, \alpha_c \) — coefficients described in the text on (A.34)-(A.36)

\( \beta_a, \beta_b, \beta_c \) — coefficients described in the text on (A.34)-(A.36)
φ — porosity of AI
η — volume (or compressional) viscosity of AI
η_0 — volume (or compressional) viscosity of a BS
θ — stress-relaxation time in AI
θ_j — version of parameter θ_j in the case of the jth layer of a multi-layer system
θ_s — stress-relaxation time in a BS
κ — parameter determined with (A.30)
μ — shear viscosity of AI
ν — parameter determined with (C.23)
ρ — volumetric mass density of AI
ρ_0 — volumetric mass density of air
ρ_i — volumetric mass density of a continuous, non-porous ice
ρ_j — version of parameters ρ_j in the case of the jth layer of a multi-layer system
ρ_s — volumetric mass density of a BS
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