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The Inertial Manifold for a Class of Nonlinear Higher-Order Coupled Kirchhoff Equations with Strong Linear Damping

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Abstract

This paper considers the long-time behavior for a system of coupled wave equations of higher-order Kirchhoff type with strong damping terms. Using the Hadamard graph transformation method, we obtain the existence of the inertial manifold while such equations satisfy the spectral interval condition.

Keywords

Higher-Order Kirchhoff System, Hadamard Graph Transformation, Spectral Gap Condition, Inertial Manifold

1. Introduction

The concept of inertial manifold proposed by C. Foias, G. R. Sell and R. Temam [1] in 1985 is a very convenient tool to describe the long-time behavior of solutions of evolutionary equations, these inertial manifolds are smooth finite dimensional invariant Lipschitz manifolds which contain the global attractor and attract all orbits of the underlying solutions exponentially. It is closely related to infinite and finite dimensional dynamic systems, that is, the existence of inertial manifold in infinite-dimensional dynamical system is reduced to the existence of inertial manifold in finite-dimensional dynamical system. Furthermore, when the system demonstrated by restriction to the inertial manifold, it reduces to finite-dimensional ordinary differential equation, at this point, the system is called the *inertial system*. As in this following, the existence of such manifold relies on a spectral gap condition that turns out to be very restrictive for the applications.

It is well known that early researches on inertial manifold have yielded considerable results. In 1988, the concept of spectral barriers was utilized in the Hilbert space to attempt to refine spectral separation condition by Constantin *et al.* [2], after that, the inertial manifold was constructed with using an elliptic regularization method by Fabes, Luskin, Sell in [3] (See [4] for other research results). Among then, the two well-known methods used to show the existence of inertial manifold are the Lyapunov-Perron method and the Hadamard graph transformation method.

In recent years, there have been many works which focus on using the latter method to study it. Wu Jingzhu and Lin Guoguang introduced the graph transformation method in [5] to obtain the existence of inertial manifold for a two-dimensional damped Boussinesq equation with $\alpha > 2$,

$$u_{tt} - \alpha\Delta u_t - \Delta u + u^{2k+1} = f(x, y).$$

Subsequently, Xu Guigui, Wang Libo, and Lin Guoguang dealt with the existence of inertial manifold for second-order nonlinear wave equation with delays in the literature [6] under the assumption that the time lag is sufficiently small,

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial u}{\partial t} - \beta\Delta\frac{\partial u}{\partial t} - \Delta u + g(u) = f(x) + h(t, u_t).$$

In addition, Guo Yamei and Li Huahui obtained the existence of inertial manifold for a class of strongly dissipative nonlinear wave equation in [7]:

$$u_{tt} - \alpha\Delta u_t + \Delta^2 u_t - \Delta u + \Delta^2 u + \Delta g(u) = f(x).$$

Chen Ling, Wang Wei and Lin Guoguang discussed the situation of higher-order Kirchhoff equation in [8]:

$$u_{tt} + (-\Delta)^m u_t + \phi\left(\left\|\nabla^m u\right\|^2\right)(-\Delta)^m u + g(u) = f(x).$$

In this paper, basing on previous studies, the existence of the inertial manifold for nonlinear Kirchhoff type equations with higher-order strong damping is considered by using the Hadamard graph transformation method. The paper is arranged as follows. In Section 2, some notations, definitions and lemmas are given. In Section 3, in order to acquire the result of the existence of the inertial manifold, we show spectral gap condition.

$$\begin{aligned} &u_{tt} + M\left(\left\|D^m u\right\|^2 + \left\|D^m v\right\|^2\right)(-\Delta)^m u + \beta(-\Delta)^m u_t + g_1(u, v) \\ &= f_1(x), \text{ in } \Omega\times[0, +\infty), \end{aligned} \tag{1.1}$$

$$\begin{aligned} &v_{tt} + M\left(\left\|D^m u\right\|^2 + \left\|D^m v\right\|^2\right)(-\Delta)^m v + \beta(-\Delta)^m v_t + g_2(u, v) \\ &= f_2(x), \text{ in } \Omega\times[0, +\infty), \end{aligned} \tag{1.2}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \tag{1.3}$$

$$v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in \Omega, \tag{1.4}$$

$$\frac{\partial^i u}{\partial n^i} = 0, \quad \frac{\partial^i v}{\partial n^i} = 0, \quad i = 0, 1, 2, \cdots, m-1, \ x \in \partial\Omega, \ t \geq 0, \tag{1.5}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $\beta > 0$ is real number and $m \geq 1$ is positive integer, $M(s)$ is a nonnegative C^1 function and satisfies $0 < m_0 \leq M(s) \leq m_1 \leq \frac{\beta^2 \mu_k}{4}$, $g_j(u,v)$ and $f_j(x)(j=1,2)$ are nonlinear terms and external force terms respectively.

2. Preliminaries

For convenience, we need the following notations in subsequent article. Considering a family of Hilbert spaces $V_a = D\left(A^{a/2}\right), a \in R$, whose inner product and norm are given by $(\cdot,\cdot)_{V_a} = \left(A^{a/2}, A^{a/2}\right)$ and $\|\cdot\|_{V_a} = \left\|A^{a/2} \cdot\right\|$.

Apparently

$$V_0 = L^2(\Omega),\ V_m = H^m(\Omega) \cap H_0^1(\Omega),\ V_{2m} = H^{2m}(\Omega) \cap H_0^1(\Omega).$$

Definition 2.1 [9] Let $S = \left(S(t)\right)_{t \geq 0}$ be a solution semigroup on a Banach space *X*, a subset $\mu \subset X$ is said to be an inertial manifold if it satisfies the following three properties:

1) μ is a finite-dimensional Lipschitz manifold;

2) μ is positively invariant, *i.e.*, $S(t)\mu \subset \mu$, for all $t \geq 0$;

3) μ attracts exponentially all orbits of solution , that is, there are constants $\eta > 0, c > 0$ such that

$$dist\left(S(t)x, \mu\right) \leq c\mathrm{e}^{-\eta t},\ t \geq 0, \tag{2.1}$$

for every $x \in X$, and the rate of decay in (2.1) is exponential, uniformly for x in bounded sets in *X*. property 3) implies that the inertial manifold must contain the universal attractor.

In order to describe the spectral interval condition, we firstly consider that the nonlinear term $F : X \to X$ is globally bounded and Lipschitz continuous, and has a positive Lipschitz constant l_F; its operator *A* has several positive real part eigenvalues, and the eigenfunctions expand to the corresponding orthogonal spaces in *X*.

Lemma 2.1 Let the operator $A : X \to X$ have countable positive real part eigenvalues whose eigenfunctions expand to the corresponding orthogonal spaces in *X*, and $F \in C_b(X,X)$ satisfies the Lipschitz condition:

$$\left\|F(u) - F(v)\right\|_X \leq l_F \left\|u - v\right\|_X,\ u, v \in X, \tag{2.2}$$

and operator *A* satisfies spectral interval condition related to *F*, if the point spectrum of the operator *A* can be divided into the following two parts σ_1 and σ_2, where σ_1 is finite,

$$\begin{aligned} \wedge_1 &= \sup\left\{\operatorname{Re}\lambda \middle| \lambda \in \sigma_1\right\}, \\ \wedge_2 &= \inf\left\{\operatorname{Re}\lambda \middle| \lambda \in \sigma_2\right\}, \end{aligned} \tag{2.3}$$

$$X_i = span\left\{w_j \middle| \lambda_j \in \sigma_i\right\},\ (i = 1,2). \tag{2.4}$$

Then

$$\wedge_2 - \wedge_1 > 4l_F, \tag{2.5}$$

$$X = X_1 \oplus X_2, \tag{2.6}$$

hold with continuous orthogonal projection $P_1 : X \to X_1, P_2 : X \to X_2$.

Lemma 2.2 $g_i : V_m \times V_m \to V_m \times V_m \ (i = 1,2)$ is uniformly bounded and global Lipschitz continuous functions.

Proof. $\forall (\tilde{u},\tilde{v}),(u,v) \in V_m \times V_m$,

$$\begin{aligned}
&\left\| g_1(\tilde{u},\tilde{v}) - g_1(u,v) \right\|_{V_m \times V_m} \\
&= \left\| g_{1u}\left(u + \theta(\tilde{u}-u), v + \theta(\tilde{v}-v)\right)(\tilde{u}-u) + g_{1v}\left(u + \theta(\tilde{u}-u), v + \theta(\tilde{v}-v)\right)(\tilde{v}-v) \right\| \\
&\le \left\| g_{1u}\left(u + \theta(\tilde{u}-u), v + \theta(\tilde{v}-v)\right)(\tilde{u}-u) \right\|_{V_m \times V_m} \\
&\quad + \left\| g_{1v}\left(u + \theta(\tilde{u}-u), v + \theta(\tilde{v}-v)\right)(\tilde{v}-v) \right\|_{V_m \times V_m} \\
&\le l\left(\left\| \tilde{u}-u \right\|_{V_m} + \left\| \tilde{v}-v \right\|_{V_m} \right).
\end{aligned}$$

$$\left\| g_2(\tilde{u},\tilde{v}) - g_2(u,v) \right\|_{V_m \times V_m} \le l\left(\left\| \tilde{u}-u \right\|_{V_m} + \left\| \tilde{v}-v \right\|_{V_m} \right).$$

Lemma 2.3 Let eigenvalues $\lambda_k^{\pm} \ (k \ge 1)$ be arranged in non-decreasing order, then for $m \in N$, there is $N \ge m$ such that λ_N^- and λ_{N+1}^- are adjacent values.

3. Inertial Manifold

Equations (1.1)-(1.2) are equivalent to the following first-order evolution equation

$$U_t + A^* U = F(U), \tag{3.1}$$

with

$$A^* = \begin{pmatrix} 0 & -I & 0 & 0 \\ M(s)(-\Delta)^m & \beta(-\Delta)^m & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & M(s)(-\Delta)^m & \beta(-\Delta)^m \end{pmatrix}, \tag{3.2}$$

$$F(U) = \begin{pmatrix} 0 \\ f_1(x) - g_1(u,v) \\ 0 \\ f_2(x) - g_2(u,v) \end{pmatrix}. \tag{3.3}$$

$$D(A^*) = \left\{ (u,v) \in V_m \times V_m \mid (u,v) \in V_0 \times V_0, \left((-\Delta)^m u, (-\Delta)^m v \right) \in V_0 \times V_0 \right\} \times V_0 \times V_0,$$

$$X = V_m \times V_0 \times V_m \times V_0.$$

To determine characteristic values of operator A^*, we consider the graph norm on X, which induced by the scale product

$$(U,V)_X = \left(M(s) D^m u, D^m \bar{x} \right) + (\bar{y}, p) + \left(M(s) D^m v, D^m \bar{z} \right) + (\bar{w}, q), \tag{3.4}$$

where $U = (u,p,v,q), V = (x,y,z,w), \bar{x},\bar{y},\bar{z},\bar{w}$ represent the conjugation of x,y,z,w respectively. Moreover, the operator A^* defined in (3.2) is monotone. Indeed, for $U \in D(A^*)$,

$$
\begin{aligned}
\left(A^{*}U,U\right)_X &= -\left(M(s)D^m p, D^m\overline{u}\right)+\left(\overline{p}, M(s)(-\Delta)^m u+\beta(-\Delta)^m p\right)\\
&\quad -\left(M(s)D^m q, D^m\overline{v}\right)+\left(\overline{q}, M(s)(-\Delta)^m v+\beta(-\Delta)^m q\right)\\
&= -\left(M(s)D^m p, D^m\overline{u}\right)+\left(\overline{p}, M(s)(-\Delta)^m u\right)+\left(\overline{p}, \beta(-\Delta)^m p\right)\\
&\quad -\left(M(s)D^m q, D^m\overline{v}\right)+\left(\overline{q}, M(s)(-\Delta)^m v\right)+\left(\overline{q}, \beta(-\Delta)^m q\right)\\
&= \beta\left(\left\|D^m p\right\|^2+\left\|D^m q\right\|^2\right)\ge 0,
\end{aligned}
\tag{3.5}
$$

therefore, $\left(A^{*}U,U\right)_X$ is a non-negative real number.

To further determine the eigenvalues of A^{*}, we consider the following characteristic equation

$$
A^{*}U=\lambda U,\ U=(u,p,v,q)\in X. \tag{3.6}
$$

That is

$$
\begin{cases}
-p=\lambda u,\\
M(s)(-\Delta)^m u+\beta(-\Delta)^m p=\lambda p,\\
-q=\lambda v,\\
M(s)(-\Delta)^m v+\beta(-\Delta)^m q=\lambda q.
\end{cases}
\tag{3.7}
$$

Substituting the first and third equations of (3.7) into the second and fourth equations, thus u,v satisfy the problem of eigenvalues

$$
\begin{cases}
\lambda^2 u-\lambda\beta(-\Delta)^m u+M(s)(-\Delta)^m u=0,\\
\lambda^2 v-\lambda\beta(-\Delta)^m v+M(s)(-\Delta)^m v=0,\\
\left.\dfrac{\partial^i u}{\partial n^i}\right|_{\partial\Omega}=\left.\dfrac{\partial^i v}{\partial n^i}\right|_{\partial\Omega}=0,\ i=0,1,2,\cdots,m-1,
\end{cases}
\tag{3.8}
$$

taking the inner product of u,v on both sides of the first and second equations of (3.8) respectively, we acquire

$$
\begin{cases}
\lambda^2\|u\|^2-\lambda\beta\left\|D^m u\right\|^2+M(s)\left\|D^m u\right\|^2=0,\\
\lambda^2\|v\|^2-\lambda\beta\left\|D^m v\right\|^2+M(s)\left\|D^m v\right\|^2=0,
\end{cases}
\tag{3.9}
$$

that is to say

$$
\lambda^2\left(\|u\|^2+\|v\|^2\right)-\lambda\beta\left(\left\|D^m u\right\|^2+\left\|D^m v\right\|^2\right)+M(s)\left(\left\|D^m u\right\|^2+\left\|D^m v\right\|^2\right)=0. \tag{3.10}
$$

(3.10) is a quadratic equation about λ, bringing u_k,v_k to the position of u,v, for any positive integer *k*, the equation (3.6) has paired eigenvalues

$$
\lambda_k^{\pm}=\frac{\beta\mu_k\pm\sqrt{\left(\beta\mu_k\right)^2-4M\left(\mu_k\right)\mu_k}}{2}, \tag{3.11}
$$

where μ_k is the characteristic value of $(-\Delta)^m$ in $V_m\times V_m$, then $\mu_k=\lambda_1 k^{\frac{m}{n}}$.

If

$$
\left(\beta\mu_k\right)^2\ge 4M\left(\mu_k\right)\mu_k,
$$

that is

$$\beta \ge 2\sqrt{\frac{M(\mu_k)}{\mu_k}} \quad \text{or} \quad \mu_k \ge \frac{4M(\mu_k)}{\beta^2},$$

then the eigenvalues of the operator A^* are all real numbers, and the corresponding characteristic functions are

$$U_k^{\pm} = \left(u_k, -\lambda_k^{\pm} u_k, v_k, -\lambda_k^{\pm} v_k\right).$$

For convenience, we note that for any $k \ge 1$,

$$\left\|D^m u_k\right\|^2 + \left\|D^m v_k\right\|^2 = \mu_k,\ \left\|u_k\right\|^2 + \left\|v_k\right\|^2 = 1,\ \left\|D^{-m} u_k\right\|^2 + \left\|D^{-m} v_k\right\|^2 = \frac{1}{\mu_k}.$$

Theorem 3.1 Suppose $0 < \beta < 2\sqrt{\frac{M(\mu_k)}{\mu_k}}$, and l be the Lipschitz constant of $g_i(u,v)(i=1,2)$ in (3.1), set $N_1 \in N$ be so large such that if $N > N_1$,

$$\beta\left(\mu_{N+1} - \mu_N\right) \ge 16l. \tag{3.12}$$

Then the operator A^* satisfies the spectral interval condition of Definition 1.2.

Proof. We firstly estimate the Lipschitz property of *F*, from (3.1) and (3.4), we have

$$\begin{aligned} \left\|F(U) - F(V)\right\|_X &= \left\|g_1(u,v) - g_1(\tilde{u},\tilde{v})\right\| + \left\|g_2(u,v) - g_2(\tilde{u},\tilde{v})\right\| \\ &\le 2l\left(\left\|\tilde{u} - u\right\|_{V_m} + \left\|\tilde{v} - v\right\|_{V_m}\right). \end{aligned} \tag{3.13}$$

That is $l_F \le 2l$. Next it can be known from (3.11) that $\lambda_k^{\pm}$ to be real numbers if and only if $\beta \ge 2\sqrt{\frac{M(\mu_k)}{\mu_k}}$. By assumption $M(s) > 0$, A^* has at most number N_0 for finite real eigenvalues, and when $N_0 = 0$, $\beta < 2\sqrt{\frac{M(\mu_k)}{\mu_k}}$, $\wedge_0 = \max\left\{\lambda_k^{\pm} \middle| k \le N_0\right\}$. The eigenvalues are complex, and

$$\operatorname{Re}\lambda_k^{\pm} = \frac{\beta\mu_k}{2}, \tag{3.14}$$

therefore, there exists $N_1 \ge N_0 + 1$ making $\operatorname{Re}\lambda_k^{\pm} > \wedge_0, k > N_1$.

Let $N \ge N_1$ be such that (3.12) holds, decomposing the point spectrum of A^*

$$\sigma_1 = \left\{\lambda_k^{\pm} \middle| k \le N\right\},\ \sigma_2 = \left\{\lambda_k^{\pm} \middle| k \ge N+1\right\}, \tag{3.15}$$

meanwhile, define the corresponding subspaces of X

$$X_1 = span\left\{U_k^{\pm} \middle| k \le N\right\},\ X_2 = span\left\{U_k^{\pm} \middle| k \ge N+1\right\}, \tag{3.16}$$

there is no k such that $\lambda_k^- \in \sigma_1$ and $\lambda_k^+ \in \sigma_2$, *i.e.*, it is impossible to have $U_k^- \in X_1$ and $U_k^+ \in X_2$, vice versa, so X_1 and X_2 are orthogonal subspaces of *X*. From (2.3) and (3.14), we have $\wedge_1 = \operatorname{Re}\lambda_N^+, \wedge_2 = \operatorname{Re}\lambda_{N+1}^-$,

$$\wedge_2 - \wedge_1 = \mathrm{Re}\left(\lambda_{N+1}^- - \lambda_N^+\right) = \frac{\beta\left(\mu_{N+1} - \mu_N\right)}{2}. \tag{3.17}$$

Thus, (3.12) implies that A^* satisfies the spectral interval inequality (2.5), in conclusion, A^* satisfies the spectral interval condition.

The proof of Theorem 3.1 is completed.

Theorem 3.2 Suppose l be the Lipschitz constant of $g_i(u,v)(i=1,2)$ in (3.1), assume $\mu_k \ge \frac{4m_1}{\beta^2} \ge \frac{4M(\mu_k)}{\beta^2}$, $N_1 \in N$ be large enough, when $N \ge N_1$, the following inequalities hold, either

$$\frac{\left(\mu_{N+1} - \mu_N\right)\left(\beta - \sqrt{\beta^2\mu_1 - 4M(s)}\right)}{2} \ge \frac{8\sqrt{2}l}{\sqrt{\beta^2\mu_1 - 4M(s)}} + 1, \tag{3.18}$$

$$\left|\sqrt{R(N)} - \sqrt{R(N+1)} + \left(\mu_{N+1} - \mu_N\right)\sqrt{\beta^2\mu_1 - 4M(s)}\right| < 2. \tag{3.19}$$

Or

$$\left(\mu_{N+1} - \mu_N\right)\frac{\beta+1}{2} \ge \frac{8\sqrt{2}l}{\sqrt{\beta^2\mu_1 - 4M(s)}} + 1, \tag{3.20}$$

$$\left|\sqrt{R(N)} - \sqrt{R(N+1)} + \mu_N - \mu_{N+1}\right| < 2, \tag{3.21}$$

where

$$R(N) = \beta^2\mu_N^2 - 4M(\mu_N)\mu_N. \tag{3.22}$$

Then in either case, the operator A^* satisfies the spectral interval condition (2.5).

Proof. Due to $\mu_k \ge \frac{4M(\mu_k)}{\beta^2}$, the eigenvalues of A^* are all real numbers, and we know that both $\left\{\lambda_k^-\right\}_{k\ge1}$ and $\left\{\lambda_k^+\right\}_{k\ge1}$ are monotonically increasing sequences.

The three steps to prove Theorem 3.2 are as follows:

Step1 Setting

$$Z_0 = \left\{1 \le k \le N \middle| \beta \ge 2\sqrt{\frac{M(\mu_k)}{\mu_k}}\right\}, Z_1 = \left\{k \in N \middle| 0 < \beta < 2\sqrt{\frac{M(\mu_k)}{\mu_k}}\right\}. \tag{3.23}$$

If $k \in Z_0$, then $\lambda_k^\pm \in R$; and if $k \in Z_1$, then $\lambda_k^\pm \in C$.

In addition, if $k \in Z_0$,

$$0 < \lambda_1^- < \cdots < \lambda_{N_0+1}^- < \beta\mu_k/2 < \lambda_{N_0+1}^+ < \cdots < \lambda_1^+, \tag{3.24}$$

where $N_0 = \sup Z_0, \mathrm{Re}\lambda_k^\pm = \frac{\beta\mu_k}{2}, \forall k > N_0$.

If $N_0 \ge N$, let

$$\sigma_1 = \left\{\lambda_j^- \middle| 1 \le j \le N\right\},\ \sigma_2 = \left\{\lambda_j^+, \lambda_k^\pm \middle| 1 \le j \le N \le k\right\}. \tag{3.25}$$

Step 2 Consider the corresponding decomposition of X

$$X_1 = span\left\{U_j^- \middle| 1 \le j \le N\right\},\\ X_2 = span\left\{U_j^+, U_k^\pm \middle| 1 \le j \le N \le k\right\}, \tag{3.26}$$

the equivalent inner product $\left(\left(U,V\right)\right)_X$ on X will be given below so that X_1, X_2 are orthogonal. Given

$$\begin{cases} X_2 = X_C \oplus X_R, \\ X_C = span\left\{U_1^+, \cdots, U_N^+\right\}, \\ X_R = span\left\{U_k^\pm \middle| k \ge N+1\right\}, \end{cases} \tag{3.27}$$

and set $X_N = X_1 \oplus X_C$.

Now we introduce two functions $\Phi : X_N \to R, \Psi : X_R \to R$, defined as

$$\begin{aligned} \Phi\left(U,V\right) = & -4M\left(s\right)\left(u,\overline{x}\right) + 2\beta^2\left(D^m u, D^m \overline{x}\right) + 2\beta\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{y}, \left(-\Delta\right)^{\frac{m}{2}} u\right) \\ & + 2\beta\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{p}, \left(-\Delta\right)^{\frac{m}{2}} x\right) + 4\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{y}, \left(-\Delta\right)^{-\frac{m}{2}} p\right) \\ & - 4M\left(s\right)\left(v,\overline{z}\right) + 2\beta^2\left(D^m v, D^m \overline{z}\right) + 2\beta\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{w}, \left(-\Delta\right)^{\frac{m}{2}} v\right) \\ & + 2\beta\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{q}, \left(-\Delta\right)^{\frac{m}{2}} z\right) + 4\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{w}, \left(-\Delta\right)^{-\frac{m}{2}} q\right). \\ \Psi\left(U,V\right) = & \ 2\beta^2\left(D^m u, D^m \overline{x}\right) + \beta\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{y}, \left(-\Delta\right)^{\frac{m}{2}} u\right) \\ & + \beta\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{p}, \left(-\Delta\right)^{\frac{m}{2}} x\right) + 4\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{y}, \left(-\Delta\right)^{-\frac{m}{2}} p\right) \\ & + 2\beta^2\left(D^m v, D^m \overline{z}\right) + \beta\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{w}, \left(-\Delta\right)^{\frac{m}{2}} v\right) \\ & + \beta\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{q}, \left(-\Delta\right)^{\frac{m}{2}} z\right) + 4\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{w}, \left(-\Delta\right)^{-\frac{m}{2}} q\right). \end{aligned} \tag{3.28}$$

With $U = \left(u, p, v, q\right), V = \left(x, y, z, w\right) \in X_N$ or X_R.

For $U = \left(u, p, v, q\right) \in X_N$, then

$$\begin{aligned} \Phi\left(U,U\right) = & -4M\left(s\right)\left(u,\overline{u}\right) + 2\beta^2\left(D^m u, D^m \overline{u}\right) + 2\beta\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{p}, \left(-\Delta\right)^{\frac{m}{2}} u\right) \\ & + 2\beta\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{p}, \left(-\Delta\right)^{\frac{m}{2}} u\right) + 4\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{p}, \left(-\Delta\right)^{-\frac{m}{2}} p\right) \\ & - 4M\left(s\right)\left(v,\overline{v}\right) + 2\beta^2\left(D^m v, D^m \overline{v}\right) + 2\beta\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{q}, \left(-\Delta\right)^{\frac{m}{2}} v\right) \\ & + 2\beta\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{q}, \left(-\Delta\right)^{\frac{m}{2}} v\right) + 4\left(\left(-\Delta\right)^{-\frac{m}{2}}\overline{q}, \left(-\Delta\right)^{-\frac{m}{2}} q\right) \\ \ge & -4M\left(s\right)\left(\left\|u\right\|^2 + \left\|v\right\|^2\right) + 2\beta^2\left(\left\|D^m u\right\|^2 + \left\|D^m v\right\|^2\right) \\ & - 4\beta\left(\left\|D^{-m} p\right\|\left\|D^m u\right\| + \left\|D^{-m} q\right\|\left\|D^m v\right\|\right) + 4\left(\left\|D^{-m} p\right\|^2 + \left\|D^{-m} q\right\|^2\right) \end{aligned}$$

$$\begin{aligned}&\geq -4M(s)\left(\|u\|^2+\|v\|^2\right)+2\beta^2\left(\left\|D^m u\right\|^2+\left\|D^m v\right\|^2\right)+4\left(\left\|D^{-m}p\right\|^2+\left\|D^{-m}q\right\|^2\right)\\&\quad -4\left(\left\|D^{-m}p\right\|^2+\left\|D^{-m}q\right\|^2\right)-\beta^2\left(\left\|D^m u\right\|^2+\left\|D^m v\right\|^2\right)\\&=\beta^2\left(\left\|D^m u\right\|^2+\left\|D^m v\right\|^2\right)-4M(s)\left(\|u\|^2+\|v\|^2\right)\\&\geq\left(\beta^2\mu_1-4M(s)\right)\left(\|u\|^2+\|v\|^2\right).\end{aligned} \tag{3.29}$$

For any *k*, there is $\beta^2\mu_k \geq 4M(\mu_k)$, and according to the initial hypothesis $0<m_0\leq M(s)\leq m_1\leq \dfrac{\beta^2\mu_k}{4}$, that is $\Phi(U,U)\geq 0$, Φ is positive definite.

Analogously, for $U=(u,p,v,q)\in X_R$, then

$$\begin{aligned}\Psi(U,V)&=2\beta^2\left(D^m u,D^m\bar{u}\right)+\beta\left((-\Delta)^{-\frac{m}{2}}\bar{p},(-\Delta)^{\frac{m}{2}}u\right)\\&\quad+\beta\left((-\Delta)^{-\frac{m}{2}}\bar{p},(-\Delta)^{\frac{m}{2}}u\right)+4\left((-\Delta)^{-\frac{m}{2}}\bar{p},(-\Delta)^{-\frac{m}{2}}p\right)\\&\quad 2\beta^2\left(D^m v,D^m\bar{v}\right)+\beta\left((-\Delta)^{-\frac{m}{2}}\bar{q},(-\Delta)^{\frac{m}{2}}v\right)\\&\quad+\beta\left((-\Delta)^{-\frac{m}{2}}\bar{q},(-\Delta)^{\frac{m}{2}}v\right)+4\left((-\Delta)^{-\frac{m}{2}}\bar{q},(-\Delta)^{-\frac{m}{2}}q\right)\\&\geq 2\beta^2\left(\left\|D^m u\right\|^2+\left\|D^m v\right\|^2\right)-4\left(\left\|D^{-m}p\right\|^2+\left\|D^{-m}q\right\|^2\right)\\&\quad-\beta^2\left(\left\|D^m u\right\|^2+\left\|D^m v\right\|^2\right)+4\left(\left\|D^{-m}p\right\|^2+\left\|D^{-m}q\right\|^2\right)\\&\geq\beta^2\mu_1\left(\|u\|^2+\|v\|^2\right).\end{aligned} \tag{3.30}$$

that is $\Psi(U,U)\geq 0$, Ψ is positive definite.

Specify the inner product of *X*:

$$\left((U,V)\right)_X=\Phi\left(P_N U,P_N V\right)+\Psi\left(P_R U,P_R V\right), \tag{3.31}$$

where P_N and P_R are projections of X to X_N and X_R respectively, for briefly, (3.31) can be abbreviated as the following

$$\left((U,V)\right)_X=\Phi(U,V)+\Psi(U,V).$$

In the inner product of *X*, to prove that X_1 and X_2 are orthogonal, as long as X_1 and X_C are proved to be orthogonal, *i.e.*,

$$\left(\left(U_j^-,U_j^+\right)\right)_X=\Phi\left(U_j^-,U_j^+\right)=0\left(U_j^-\in X_1,U_j^+\in X_C\right). \tag{3.32}$$

Recalling (3.28)

$$\begin{aligned}\Phi\left(U_j^-,U_j^+\right)&=-4M(\mu_j)\left(u_j,\bar{u}_j\right)+2\beta^2\left(D^m u_j,D^m\bar{u}_j\right)-2\beta\lambda_j^+\left(D^{-m}\bar{u}_j,D^m u_j\right)\\&\quad-2\beta\lambda_j^-\left(D^{-m}\bar{u}_j,D^m u_j\right)+4\lambda_j^-\lambda_j^+\left(D^{-m}\bar{u}_j,D^{-m}u_j\right)\\&\quad-4M(\mu_j)\left(v_j,\bar{v}_j\right)+2\beta^2\left(D^m v_j,D^m\bar{v}_j\right)-2\beta\lambda_j^+\left(D^{-m}\bar{v}_j,D^m v_j\right)\\&\quad-2\beta\lambda_j^-\left(D^{-m}\bar{v}_j,D^m v_j\right)+4\lambda_j^-\lambda_j^+\left(D^{-m}\bar{v}_j,D^{-m}v_j\right)\end{aligned}$$

$$
\begin{aligned}
&= -4M(\mu_j)\left(\|u_j\|^2+\|v_j\|^2\right)+2\beta^2\left(\|D^m u_j\|^2+\|D^m v_j\|^2\right)\\
&\quad -2\beta\left(\lambda_j^+ + \lambda_j^-\right)\left(\|u_j\|^2+\|v_j\|^2\right)+4\lambda_j^-\lambda_j^+\left(\|D^{-m}u_j\|^2+\|D^{-m}v_j\|^2\right)\\
&= -4M(\mu_j)+2\beta^2\mu_j-2\beta\left(\lambda_j^+ + \lambda_j^-\right)+4\lambda_j^-\lambda_j^+\cdot\frac{1}{\mu_j},
\end{aligned}
\tag{3.33}
$$

according to (3.10)

$$
\lambda_j^+ + \lambda_j^- = \beta\mu_j,\ \lambda_j^+\lambda_j^- = M(\mu_j)\mu_j,
$$

thus, (3.33) is equivalent to

$$
\Phi\left(U_j^-,U_j^+\right) = -4M(\mu_j)+2\beta^2\mu_j-2\beta\left(\lambda_j^+ + \lambda_j^-\right)+4\lambda_j^-\lambda_j^+\cdot\frac{1}{\mu_j}=0.
$$

Step3 The orthogonal decomposition (2.6) has now been established. Let us prove that A^* satisfies the spectral interval condition (2.5) and its equivalent norm on X is shown in (3.31), for this, we must estimate Lipschitz constant l_F of F in (2.2).

Recalling that

$$
F(U) = \left(0, f_1(x)-g_1(u,v), 0, f_2(x)-g_2(u,v)\right)^{\mathrm{T}},
$$

$g_i(u,v): V_m\times V_m \to V_m\times V_m$ is Lipschitz continuous. Assume P_1, P_2 be the orthogonal maps of $X\to X_1, X\to X_2$ respectively, P_1, P_2 are their corresponding mappings on $V_m\times V_m$ and $V_0\times V_0$, from (3.29) and (3.30), for

$$
U=(u,p,v,q)\in X, U_1=(u_1,p_1,v_1,q_1)\in P_1U, U_2=(u_2,p_2,v_2,q_2)\in P_2U,
$$

then

$$
P_1u=u_1, P_1v=v_1, P_2u=u_2, P_2v=v_2.
$$

$$
\begin{aligned}
\|U\|_X^2 &= \Phi^*\left(P_1U,P_1U\right)+\Psi^*\left(P_2U,P_2U\right)\\
&\ge \left(\beta^2\mu_1-4M(s)\right)\left(\|P_1u\|^2+\|P_1v\|^2\right)+\beta^2\mu_1\left(\|P_2u\|^2+\|P_2v\|^2\right)\\
&\ge \left(\beta^2\mu_1-4M(s)\right)\left(\|u\|^2+\|v\|^2\right).
\end{aligned}
\tag{3.34}
$$

Given $U=(u,p,v,q), V=(\tilde{u},\tilde{p},\tilde{v},\tilde{q})\in X$, we get

$$
\begin{aligned}
\|F(U)-F(V)\|_X &= \|g_1(u,v)-g_1(\tilde{u},\tilde{v})\|_{V_m\times V_m}+\|g_2(u,v)-g_2(\tilde{u},\tilde{v})\|_{V_m\times V_m}\\
&\le 2l\left(\|u-\tilde{u}\|_{V_m}+\|v-\tilde{v}\|_{V_m}\right)\\
&\le \frac{2\sqrt{2}l}{\sqrt{\beta^2\mu_1-4M(s)}}\|U-V\|_X,
\end{aligned}
$$

thus

$$
l_F \le \frac{2\sqrt{2}l}{\sqrt{\beta^2\mu_1-4M(s)}},
\tag{3.35}
$$

by (3.35), then the spectral interval condition (2.5) holds if

$$\wedge_2-\wedge_1=\lambda_{N+1}^- -\lambda_N^- > \frac{8\sqrt{2}l}{\sqrt{\beta^2\mu_1-4M(s)}}. \tag{3.36}$$

Recalling (3.22), we have

$$\lambda_{N+1}^- -\lambda_N^- = \frac{\sqrt{R(N)}-\sqrt{R(N+1)}}{2}+\frac{\beta(\mu_{N+1}-\mu_N)}{2}, \tag{3.37}$$

and

$$\lim_{N\to+\infty}\left(\sqrt{R(N)}-\sqrt{R(N+1)}+\sqrt{\beta^2\mu_1-4M(s)}(\mu_{N+1}-\mu_N)\right)=0. \tag{3.38}$$

For formula (3.38), in fact, setting

$$R'(N)=\frac{\beta^2\mu_N-4M(\mu_N)}{\mu_N(\beta^2\mu_1-4M(s))},$$

we compute

$$\begin{aligned}&\sqrt{R(N)}-\sqrt{R(N+1)}+\sqrt{\beta^2\mu_1-4M(s)}(\mu_{N+1}-\mu_N)\\&=\sqrt{\beta^2\mu_1-4M(s)}\left(\mu_{N+1}\left(1-\sqrt{R'(N+1)}\right)-\mu_N\left(1-\sqrt{R'(N)}\right)\right),\end{aligned} \tag{3.39}$$

$$\lim_{N\to+\infty}\mu_N\left(1-\sqrt{R'(N)}\right)=0.$$

Consequently, (3.38) is obtained.

From the condition (3.19), it can be determined that $N_1>0$ such that for all $N\ge N_1$, and with (3.37)

$$\wedge_2-\wedge_1=\lambda_{N+1}^- -\lambda_N^- \ge \frac{\mu_{N+1}-\mu_N}{2}\left(\beta-\sqrt{\beta^2\mu_1-4M(s)}\right)-1, \tag{3.40}$$

this shows that (3.36) is established by the conditions (3.18), (3.37), and (3.40), that the Theorem 3.2 is certified completely under the previous hypothesis.

At this point, we continue to use the latter hypothesis to prove, setting

$$R_1(N)=-\sqrt{\frac{R(N)}{\mu_N^4}},$$

then (3.37) is equivalent to

$$\wedge_2-\wedge_1=\lambda_{N+1}^- -\lambda_N^- = \frac{\beta(\mu_{N+1}-\mu_N)}{2}+\frac{1}{2}\left(\mu_{N+1}^2\sqrt{R_1(N+1)}-\mu_N^2\sqrt{R_1(N)}\right), \tag{3.41}$$

arranging

$$R_2(N)=\mu_N^2R_1(N),$$

then (3.41) means

$$\wedge_2-\wedge_1=\lambda_{N+1}^- -\lambda_N^- = \frac{\beta(\mu_{N+1}-\mu_N)}{2}+\frac{R_2(N+1)-R_2(N)}{2}, \tag{3.42}$$

making

$$R_3(N)=1-\frac{R_2(N)}{\mu_N},$$

this implies

$$R_2(N+1)-R_2(N)+\mu_N-\mu_{N+1}=\mu_N R_3(N)-\mu_{N+1}R_3(N+1), \tag{3.43}$$

from (3.43), we easily get

$$\lim_{N\to+\infty} R_2(N+1)-R_2(N)+\mu_N-\mu_{N+1}=0.$$

Namely

$$\left|\sqrt{R(N)}-\sqrt{R(N+1)}+\mu_N-\mu_{N+1}\right|<2,$$

to be specific

$$-2<\sqrt{R(N)}-\sqrt{R(N+1)}+\mu_N-\mu_{N+1}<2,$$

$$-2-(\mu_N-\mu_{N+1})<\sqrt{R(N)}-\sqrt{R(N+1)}<2-(\mu_N-\mu_{N+1}),$$

therefore

$$\begin{aligned}\wedge_2-\wedge_1=\lambda_{N+1}^{-}-\lambda_N^{-}&=\frac{\beta(\mu_{N+1}-\mu_N)}{2}+\frac{\sqrt{R(N)}-\sqrt{R(N+1)}}{2}\\&>\frac{\beta}{2}(\mu_{N+1}-\mu_N)-\frac{1}{2}(\mu_N-\mu_{N+1})-1\\&>\frac{\beta+1}{2}(\mu_{N+1}-\mu_N)-1\\&\geq\frac{8\sqrt{2}l}{\sqrt{\beta^2\mu_1-4M(s)}}\geq 4l_F.\end{aligned} \tag{3.44}$$

Under the latter assumption, Theorem 3.2 is proved completely.

Theorem 3.3 In the conclusions of Theorem 3.1 and Theorem 3.2, initial boundary value problems (1.1)-(1.5) admits an inertial manifold μ in X of the form

$$\mu=graph(\Gamma)=\{\xi+\Gamma(\xi):\xi\in X_1\}, \tag{3.45}$$

where $\Gamma:X_1\to X_2$ is Lipschitz continuous with the Lipschitz constant l_F, and $graph(\Gamma)$ represents the diagram of Γ.

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