

# Random Attractors for the Kirchhoff-Type Suspension Bridge Equations with Strong Damping and White Noises

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**How to cite this paper:** Qin, C.L., Du, J.J. and Lin, G.G. (2017) Random Attractors for the Kirchhoff-Type Suspension Bridge Equations with Strong Damping and White Noises. *International Journal of Modern Nonlinear Theory and Application*, 6, 134-147.

<https://doi.org/10.4236/ijmnta.2017.64012>

**Received:** October 28, 2017

**Accepted:** December 18, 2017

**Published:** December 21, 2017

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## Abstract

In this paper, we investigate the existence of random attractor for the random dynamical system generated by the Kirchhoff-type suspension bridge equations with strong damping and white noises. We first prove the existence and uniqueness of solutions to the initial boundary value conditions, and then we study the existence of the global attractors of the equation.

## Keywords

Kirchhoff-Type Suspension Bridge Equations, Random Attractors, Random Dynamical System

## 1. Introduction

In this paper, we consider the following stochastic Kirchhoff-type suspension bridge equations

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta^2 u_t + (p - |\nabla u|^2) \Delta u + bu^+ + f(u) = q(x) \dot{W}, & \Omega \times [\tau, +\infty), \tau \in \mathbb{R}, \\ u(x, t) = \nabla u(x, t) = 0, & x \in \partial\Omega, t \geq \tau, \\ u(x, \tau) = u_0(x), u_t(x, \tau) = u_1(x) & x \in \Omega, \end{cases} \quad (1.1)$$

where  $u(x, t)$  is an unknown function, which represents the downward deflection of the road bed in the vertical plane,  $u^+ = u$  for  $u \geq 0$  and  $u^+ = 0$  for  $u < 0$ .  $b > 0$  denotes the spring constant of the ties, the real constant  $p$  represents the axial force acting at the end of the road bed of the bridge in the reference configuration. Namely,  $p$  is negative when the bridge is stretched, positive when compressed.  $\Omega$  is an open bounded subset of  $\mathbb{R}^2$  with

sufficiently smooth boundary  $\partial\Omega$ .  $q(x) \in H^3(\Omega)$  is not identically equal to zero,  $f$  is a nonlinear function satisfying certain conditions.  $\dot{W}$  is the derivative of a one-dimensional two-valued Wiener process  $W(t)$  and  $q(x)\dot{W}$  formally describes white noise.

We assume that the nonlinear function  $f \in C^2(\mathbb{R}, \mathbb{R})$  with  $f(0) = 0$ , which satisfies the following assumptions:

(a) Growth conditions:

$$|f(s)| \leq C_0(1 + |s|^k), \quad k \geq 1, \forall s \in \mathbb{R}, \quad (1.2)$$

where  $C_0$  is a positive constant. For example, obviously,  $f(s) = |s|^{k-1}s$  satisfies (1.2).

(b) Dissipation conditions:

$$F(s) := \int_0^s f(r) dr \geq C_1(|s|^{k+1} - 1), \quad \forall s \in \mathbb{R}, \quad (1.3)$$

and

$$sf(s) \geq C_2(F(s) - 1), \quad \forall s \in \mathbb{R}, \quad (1.4)$$

where  $C_1, C_2$  are positive constants.

When  $f(u) = 0$  and  $q(x) = 0$ , Equation (1.1) is regarded as a model of naval structures, which is originally in [1] introduced by Lazer and McKenna. To the best of our knowledge, Qin [2] [3] proved random attractor for stochastic Kirchhoff equation with white noise, Ma [4] investigated the asymptotic behavior of the solution for the floating beam, that is, the “noise” is absent in (1.1). No one else has studied the long-time behavior of the solutions about these problems, it is just our interest in this paper. As far as the other related problems are concerned, we refer the reader to [2]-[7] and the references therein.

It is well known that Crauel and Flandoli originally introduced the random attractor for the infinite-dimensional RDS [8] [9]. A random attractor of RDS is a measurable and compact invariant random set attracting all orbits. It is the appropriate generalization of the now classical attractor exists, it is the smallest attracting compact set and the largest invariant set [10]. Zhou *et al.* [11] studied random attractor for damped nonlinear wave equation with white noise. Fan [12] proved random attractor for a damped stochastic wave equation with multiplicative noise. These abstract results have been successfully applied to many stochastic dissipative partial differential equations. The existence of a random attractors for the wave equations has been investigated by several authors [8] [9] [10].

The outline of this paper is as follows: In Section 2, we recall many basic concepts related to a random attractor for general random dynamical system. In Section 3, We prove the existence and uniqueness of the solution corresponding to system (1.1) which determines RDS. In Section 4, we prove the existence of random attractor of the random dynamical system.

## 2. Random Dynamical System

In this section, we recall some basic concepts related to RDS and a random

attractor for RDS in [8] [9] [10], which are important for getting our main results.

Let  $(X, \|\cdot\|_X)$  be a separable Hilbert space with Borel  $\sigma$ -algebra  $\mathcal{B}(X)$ , and let  $(\Omega, \mathcal{F}, \mathcal{P})$  be a probability space.  $\theta_t : \Omega \rightarrow \Omega, t \in \mathbb{R}$  is a family of measure preserving transformations such that  $(t, \omega) \mapsto \theta_t \omega$  is measurable,  $\theta_0 = id$  and  $\theta_{t+s} = \theta_t \theta_s$  for all  $t, s \in \mathbb{R}$ . The flow  $\theta_t$  together with the probability space  $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t \in \mathbb{R}})$  is called a metric dynamical system.

**Definition 2.1.** Let  $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t \in \mathbb{R}})$  be a metric dynamical system. Suppose that the mapping  $\phi : \mathbb{R}^+ \times \Omega \times X \rightarrow X$  is  $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X))$ -measurable and satisfies the following properties:

- 1)  $\phi(0, \omega)x = x, x \in X$  and  $\omega \in \Omega$ ;
- 2)  $\phi(t+s, \omega) = \phi(t, \theta_s \omega) \circ \phi(s, \omega)$ , for all  $t, s \in \mathbb{R}^+, x \in X$  and  $\omega \in \Omega$ .

Then  $\phi$  is called a random dynamical system (RDS). Moreover,  $\phi$  is called a continuous RDS if  $\phi$  is continuous with respect to  $x$  for  $t \geq 0$  and  $\omega \in \Omega$ .

**Definition 2.2.** A set-valued map  $D : \Omega \rightarrow 2^X$  is said to be a closed (compact) random set if  $D(\omega)$  is closed (compact) for  $P$ -a.s.  $\omega \in \Omega$ , and  $\omega \mapsto d(x, D(\omega))$  is  $P$ -a.s. measurable for all  $x \in X$ .

**Definition 2.3.** If  $K$  and  $B$  are random sets such that for  $P$ -a.s.  $\omega$  there exists a time  $t_B(\omega)$  such that for all  $t \geq t_B(\omega)$ ,

$$\phi(t, \theta_{-t} \omega) B(\theta_{-t} \omega) \subset K(\omega),$$

then  $K$  is said to absorb  $B$ , and  $t_B(\omega)$  is called the absorption time.

**Definition 2.4.** A random set  $\mathcal{A} = A(\omega)_{\omega \in \Omega} \subset X$  is called a random attractor associated to the RDS  $\phi$  if  $P$ -a.s.:

- 1)  $\mathcal{A}$  is a random compact set, i.e.,  $A(\omega)$  is compact for  $P$ -a.s.  $\omega \in \Omega$ , and the map  $\omega \mapsto d(x, A(\omega))$  is measurable for every  $x \in X$ ;
- 2)  $\mathcal{A}$  is  $\phi$ -invariant, i.e.,  $\phi(t, \omega) A(\omega) = A(\theta_t \omega)$  for all  $t \geq 0$  and  $P$ -a.s.  $\omega \in \Omega$ ;
- 3)  $\mathcal{A}$  attracts every set  $B$  in  $X$ , i.e., for all bounded (and non-random)  $B \subset X$ ,

$$\lim_{t \rightarrow \infty} d(\phi(t, \theta_{-t} \omega) B(\theta_{-t} \omega), A(\omega)) = 0,$$

where  $d(\cdot, \cdot)$  denotes the Hausdorff semi-distance:

$$d(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y), A, B \in X.$$

Note that  $\phi(t, \theta_{-t} \omega)x$  can be interpreted as the position of the trajectory which was in  $x$  at time  $-t$ . Thus, the attraction property holds from  $t = -\infty$ .

**Theorem 2.1.** [8] (Existence of a random attractor) Let  $\phi$  be a continuous random dynamical system on  $X$  over  $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t \in \mathbb{R}})$ . Suppose that there exists a random compact set  $K(\omega)$  absorbing every bounded non-random set  $B \subset X$ . Then the set

$$\mathcal{A} = A(\omega)_{\omega \in \Omega} = \overline{\bigcup_{B \subset X} \wedge_B(\omega)}$$

is a global random attractor for  $\phi$ , where the union is taken over all bounded  $B \subset X$ , and  $\wedge_B(\omega)$  is the  $\omega$ -limitsset of  $B$  given by

$$\wedge_B(\omega) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \phi(t, \theta_{-t}\omega) B(\theta_{-t}\omega)}$$

### 3. Existence and Uniqueness of Solutions

With the usual notation, we denote

$$H = L^2(\Omega), \quad V = H^2(\Omega)$$

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad D(A^2) = \{u \in H^4(\Omega) : A^2u \in L^2(\Omega)\},$$

where  $A = -\Delta, A^2 = \Delta^2$ . We denote  $H, V$  with the following inner products and norms, respectively:

$$(u, v) = \int_{\Omega} uv dx, \quad \|u\|^2 = (u, u), \quad \forall u, v \in H,$$

$$((u, v)) = \int_{\Omega} \Delta u \Delta v dx, \quad \|u\|_2^2 = ((u, u)), \quad \forall u, v \in V.$$

And we introduce the space  $E = D(A) \times H$ , which is used throughout the paper and endow the space  $E$  with the following usual scalar product and norm:

$$(y_1, y_2)_E = ((u_1, u_2)) + (v_1, v_2), \quad \forall y_i = (u_i, v_i)^T \in E, i = 1, 2,$$

$$\|y\|_E^2 = (y, y)_E, \quad \forall y = (u, v)^T \in E$$

where  $T$  denotes the transposition.

More generally, define  $\mathcal{H}^r = D\left(A^{\frac{r}{2}}\right)$  for  $r \in \mathbb{R}$ , which turns out to be a Hilbert space with the inner product  $(u, v)_r = \left(A^{\frac{r}{2}}u, A^{\frac{r}{2}}v\right)$ , we denote by

$$\|\cdot\|_r = \left\|A^{\frac{r}{2}}\right\| \text{ the norm on } \mathcal{H}^r \text{ induced by the above inner product. Let } \lambda > 0$$

be the first eigenvalue of  $A^2u = \lambda u, u(x, t) = \nabla u(x, t) = 0, x \in \partial\Omega$ , by the compact embeddings  $\mathcal{H}^{r+1} \rightarrow \mathcal{H}^r$  along with the generalized Poincaré inequality, we have

$$\|u\|_{r+1}^2 \geq \lambda \|u\|_r^2. \tag{3.1}$$

It is convenient to reduce (1.1) to an evolution of the first order in time

$$\begin{cases} u_t = v, \\ v_t = -A^2u - A^2v + (p - |\nabla u|^2)Au - bu^+ - f(u) + q(x)\dot{W}, \\ u(x, \tau) = u_0(x), u_t(x, \tau) = u_1(x), x \in \Omega, \end{cases} \tag{3.2}$$

whose equivalent Itó equation is

$$\begin{cases} du = v dt, \\ dv = -A^2u dt - A^2v dt + (p - |\nabla u|^2)A u dt - bu^+ dt - f(u) dt + q(x) dW, \\ u(x, \tau) = u_0(x), u_t(x, \tau) = u_1(x), x \in \Omega, \end{cases} \tag{3.3}$$

where  $W(t)$  is a one-dimensional two-sided real-valued Wiener process on  $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t \in \mathbb{R}})$ . Without loss of generality, we can assume that

$$\Omega = \{\omega(t) = W(t) \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

where  $P$  is a Wiener measure. We can define a family of measure preserving and ergodic transformations  $(\theta_t)_{t \in \mathbb{R}}$  by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) \omega(t), t \in \mathbb{R}, \omega \in \Omega.$$

Let  $z = v - q(x)W$ , we consider the random partial differential equation equivalent to (3.3)

$$\begin{cases} \frac{du}{dt} = z + q(x)W, \\ \frac{dz}{dt} = -A^2u - A^2z + (p - |\nabla u|^2)Au - bu^+ - f(u) - A^2q(x)W, \\ u(x, \tau) = u_0(x), z(\tau, \omega) = z(x, \tau, \omega) = u_1(x) - q(x)W(\tau), x \in \Omega. \end{cases} \tag{3.4}$$

Apparently, there is no stochastic differential in (3.4) by comparing with stochastic differential Equation (3.3). Let

$$\varphi = \begin{pmatrix} u \\ z \end{pmatrix}, L = \begin{pmatrix} 0 & I \\ -A^2 & -A^2 \end{pmatrix}, F(\varphi, \omega) = \begin{pmatrix} q(x)W \\ (p - |\nabla u|^2)Au - bu^+ - f(u) - A^2q(x)W \end{pmatrix},$$

then (3.4) can be written as

$$\dot{\varphi} = L\varphi + F(\varphi, \omega), \quad \varphi(\tau, \omega) = (u_0, z(\tau, \omega))^T. \tag{3.5}$$

From [13] we know that  $L$  is the infinitesimal generators of  $C_0$ -semigroup  $e^{Lt}$  on  $E$ . It is not difficult to check that the functions  $F(\varphi, \omega): E \rightarrow E$  is locally Lipschitz continuous with respect to  $\varphi$  and bounded for every  $\omega \in \Omega$ . By the classical semigroup theory of existence and uniqueness of solutions of evolution differential equations [13], so we have the following theorem:

**Theorem 3.1.** Consider (3.5). For each  $\omega \in \Omega$  and initial value  $\varphi(\tau, \omega) = (u_0, z(\tau, \omega))^T = (u_0, u_1(x) - q(x)W(\tau))^T \in E$ , there exists a unique function  $\varphi(t, \omega)$  such that satisfies the integral equation

$$\varphi(t, \omega) = e^{L(t-\tau)}\varphi(\tau, \omega) + \int_{\tau}^t F(\varphi(s), \omega) ds, \quad \forall \varphi(\tau, \omega) \in E.$$

By theorem 3.1, we can prove that for  $P$ -a.s. every  $\omega \in \Omega$  the following statements hold for all  $T > 0$ :

- 1) If  $\varphi(\tau, \omega) \in E$ , then  $\varphi(t, \omega) \in C([\tau, \tau + T]; D(A)) \times C([\tau, \tau + T]; H)$ .
- 2)  $\varphi(t, \varphi(\tau, \omega))$  is continuous in  $t$  and  $\varphi(\tau, \omega)$ .
- 3) The solution mapping of (3.5) satisfies the properties of RDS.

Equation (3.5) has a unique solution for every  $\omega \in \Omega$ . Hence the solution mapping

$$\tilde{S}(t, \omega): \varphi(\tau, \omega) \mapsto \varphi(t, \omega) \tag{3.6}$$

generates a random dynamical system, so the transformation

$$S(t, \omega): \varphi(\tau, \omega) + (0, q(x)W(\tau))^T \mapsto \varphi(t, \omega) + (0, q(x)W(t))^T \quad (3.7)$$

also determines a random dynamical system corresponding to Equation (3.2).

#### 4. Existence of a Random Attractor

In this section, we prove the existence of a random attractor for RDS (3.7) in  $E$ .

Let  $\tilde{z} = z + \varepsilon u, \psi = (u, \tilde{z})^T$ , where

$$\varepsilon = \frac{\lambda^2}{4\lambda^2 + 3\lambda + 4}. \quad (4.1)$$

So Equation (3.4) can be written as

$$\dot{\psi} + Q\psi = \tilde{F}(\psi, \omega), \quad \psi(\tau, \omega) = (u_0, z(\tau, \omega) + \varepsilon u_0)^T, \quad t \geq \tau. \quad (4.2)$$

where

$$Q = \begin{pmatrix} \varepsilon I & -I \\ (1-\varepsilon)A^2 + \varepsilon^2 I & A^2 - \varepsilon I \end{pmatrix},$$

$$\tilde{F}(\psi, \omega) = \begin{pmatrix} q(x)W \\ (p - |\nabla u|^2)Au - bu^+ - f(u) + (\varepsilon - A^2)q(x)W \end{pmatrix}.$$

The mapping

$$\tilde{S}_\varepsilon(t, \omega): (u_0, z(\tau, \omega) + \varepsilon u_0)^T \mapsto (u(t), z(t) + \varepsilon u(t))^T, \quad E \rightarrow E, t \geq \tau$$

is defined by (4.2).

To show the conjugation of the solution of the stochastic partial differential Equation (1) and the random partial differential Equation (4.2), we introduce the homeomorphism

$$R_\varepsilon: (u, z)^T \mapsto (u, z + \varepsilon u)^T$$

with the inverse homeomorphism  $R_{-\varepsilon}$ . Then the transformation

$$\tilde{S}_\varepsilon(t, \omega) = R_\varepsilon S(t, \omega) R_{-\varepsilon} \quad (4.3)$$

also determines RDS corresponding to Equation (1). Therefore, for RDS (7) we only need consider the equivalent random dynamical system

$S_\varepsilon(t, \omega) = R_\varepsilon S(t, \omega) R_{-\varepsilon}$ , where  $S_\varepsilon(t, \omega)$  is decided by

$$\dot{\xi} + Q\xi = G(\xi, \omega), \quad \xi(\tau, \omega) = (u_0, u_1 + \varepsilon u_0)^T, \quad t \geq \tau, \quad (4.4)$$

where

$$\xi = \begin{pmatrix} u(t) \\ u_t(t) + \varepsilon u(t) \end{pmatrix}, \quad G(\xi, \omega) = \begin{pmatrix} 0 \\ (p - |\nabla u|^2)Au - bu^+ - f(u) + q(x)W \end{pmatrix}.$$

Next, we prove a positivity property of the operator  $Q$  in  $E$  that plays a vital role throughout the paper.

**Lemma 4.1.** For any  $\phi = (u, z)^T \in E$ , there holds

$$(Q\phi, \phi)_E \geq \frac{\varepsilon}{2} \|\phi\|_E^2 + \frac{\varepsilon}{4} \|u\|_2^2 + \frac{\lambda}{2} \|z\|^2.$$

**Proof.** Since  $Q\phi = (\varepsilon u - z, (1 - \varepsilon)A^2u + \varepsilon^2u + A^2z - \varepsilon z)^T$ , by using the Poincaré inequality and the Young inequality, we conclude that

$$\begin{aligned} (Q\phi, \phi)_E &= \varepsilon \|u\|_2^2 - \varepsilon(Au, Az) + \varepsilon^2(u, z) + \|Az\|^2 - \varepsilon \|z\|^2 \\ &\geq \varepsilon \|u\|_2^2 - \frac{\varepsilon}{8} \|u\|_2^2 - 2\varepsilon \|Az\|^2 - \frac{\varepsilon}{8} \|u\|_2^2 - \frac{2\varepsilon^3}{\lambda} \|z\|^2 + \|Az\|^2 - \varepsilon \|z\|^2 \\ &\geq \frac{\varepsilon}{2} \|\phi\|_E^2 + \frac{\varepsilon}{4} \|u\|_2^2 + (1 - 2\varepsilon)\lambda \|z\|^2 - \left(\frac{2\varepsilon}{\lambda} + \frac{3\varepsilon}{2}\right) \|z\|^2 \\ &= \frac{\varepsilon}{2} \|\phi\|_E^2 + \frac{\varepsilon}{4} \|u\|_2^2 + \frac{\lambda}{2} \|z\|^2. \end{aligned}$$

where  $\varepsilon = \frac{\lambda^2}{4\lambda^2 + 3\lambda + 4}$ .  $\square$

**Lemma 4.2.** Let (1.2)-(1.4) hold, there exist a random variable  $r_1(\omega) > 0$ , and a bounded ball  $B_0$  of  $E$  centered at 0 with random radius  $r_0(\omega) > 0$  such that for any bounded non-random set  $B$  of  $E$ , there exists a deterministic  $T(B) \leq -1$  such that the solution  $\psi(t, \omega, \psi(\tau, \omega)) = (u(t, \omega), \tilde{z}(t, \omega))^T$  of (4.2) with initial value  $(u_0, u_1 + \varepsilon u_0)^T \in B$  satisfies for  $P$ -a.s.  $\omega \in \Omega$ ,

$$\|\psi(-1, \omega, \psi(\tau, \omega))\|_E \leq r_0(\omega), \quad \tau \leq T(B),$$

and for all  $\tau \leq t \leq 0$

$$\|\psi(t, \omega, \psi(\tau, \omega))\|_E^2 \leq R(\tau, \omega), \tag{4.5}$$

where  $\tilde{z}(t\omega) = u_t(t) + \varepsilon u(t) - q(x)W(t)$ , and  $R(\tau, \omega)$  is given by

$$\begin{aligned} R(\tau, \omega) &= 2e^{-\varepsilon_1(t-\tau)} \left( \|u_0\|_2^2 + \|u_1 + \varepsilon u_0\|^2 + \|q\|^2 |W|^2 + \int_{\Omega} F(u_0) dx + |\nabla u_0|^2 - p \right)^2 \\ &\quad + r_1^2(\omega). \end{aligned}$$

Besides it is easy to deduce a similar absorption result for

$$\phi(-1) = (u(-1), u_t(-1) + \varepsilon u(-1))^T$$

instead of  $\psi(-1)$ .

**Proof.** We take the inner product in  $E$  of (4.2) with  $\psi = (u, \tilde{z})^T$ , where  $\tilde{z} = u_t + \varepsilon u - q(x)W$ , we get

$$\frac{1}{2} \frac{d}{dt} \|\psi\|_E^2 + (Q\psi, \psi)_E = (\tilde{F}(\psi, \omega), \psi)_E, \quad \forall t \geq \tau, \tag{4.6}$$

where

$$\begin{aligned} (\tilde{F}(\psi, \omega), \psi)_E &= ((u, q(x)W)) - b(u^+, \tilde{z}) - (f(u), \tilde{z}) \\ &\quad - (A^2q(x)W, \tilde{z}) + \varepsilon(q(x)W, \tilde{z}) + \left( (p - |\nabla u|^2)Au, \tilde{z} \right). \end{aligned} \tag{4.7}$$

We deal with the terms in (4.7) one by one as follows:

$$((u, q(x)W)) \leq \frac{\varepsilon}{4} \|u\|_2^2 + \frac{1}{\varepsilon} \|q\|_2^2 |W(t)|^2; \tag{4.8}$$

$$\begin{aligned}
 -b(u^+, \tilde{z}) &= -b(u^+, u_t + \varepsilon u - q(x)W) \\
 &= -\frac{1}{2} \frac{d}{dt} b \|u^+\|^2 - \varepsilon b \|u^+\|^2 + b(u^+, q(x)W)
 \end{aligned} \tag{4.9}$$

$$\begin{aligned}
 &\leq -\frac{1}{2} \frac{d}{dt} b \|u^+\|^2 - \frac{\varepsilon b}{2} \|u^+\|^2 + \frac{b \|q\|^2}{2\varepsilon} |W|^2; \\
 \varepsilon(q(x)W, \tilde{z}) &\leq \frac{\lambda}{4} \|\tilde{z}\|^2 + \frac{\varepsilon^2}{\lambda} \|q\|^2 |W|^2;
 \end{aligned} \tag{4.10}$$

$$\left| -(A^2 q(x)W, \tilde{z}) \right| \leq \frac{\|q\|_4^2}{\lambda} |W|^2 + \frac{\lambda}{4} \|\tilde{z}\|^2. \tag{4.11}$$

By using (1.2)-(1.3) and the Hölder inequality, we get

$$\begin{aligned}
 (f(u), q(x)W) &\leq C_0 \int_{\Omega} (1 + |u|^k) q(x)W(t) dx \\
 &\leq C_0 \|q\| \|W\| + C_0 \left( \int_{\Omega} |u|^{k+1} dx \right)^{\frac{k}{k+1}} \|q\|_{k+1} |W| \\
 &\leq C_0 \|q\| \|W\| + C_0 C_1^{-\frac{k}{k+1}} \left( \int_{\Omega} (F(u) + C_1) dx \right)^{\frac{k}{k+1}} \|q\|_{k+1} |W| \\
 &\leq C_0 \|q\| \|W\| + \frac{\varepsilon C_0 C_1^{-1}}{2} \int_{\Omega} F(u) dx + \frac{C_0}{2\varepsilon} \|q\|_{k+1}^{k+1} |W|^{k+1} + \frac{\varepsilon C_0 |\Omega|}{2}.
 \end{aligned} \tag{4.12}$$

Inequality (4.12) together with (1.4) yields

$$\begin{aligned}
 -(f(u), \tilde{z}) &= -(f(u), u_t + \varepsilon u - q(x)W) \\
 &\leq -\frac{d}{dt} \int_{\Omega} F(u) dx - \varepsilon C_2 \int_{\Omega} F(u) dx + \varepsilon C_2 |\Omega| + (f(u), q(x)W) \\
 &\leq -\frac{d}{dt} \int_{\Omega} F(u) dx - \frac{\varepsilon(2C_2 - C_0 C_1^{-1})}{2} \int_{\Omega} F(u) dx + C_0 \|q\| \|W\| \\
 &\quad + \frac{C_0}{2\varepsilon} \|q\|_{k+1}^{k+1} |W|^{k+1} + \frac{\varepsilon(C_0 + 2C_2)}{2} |\Omega|.
 \end{aligned} \tag{4.13}$$

$$\begin{aligned}
 ((p - |\nabla u|^2)Au, \tilde{z}) &= (p - |\nabla u|^2)(Au, u_t + \varepsilon u - q(x)W) \\
 &= (p - |\nabla u|^2) \left[ \frac{1}{2} \frac{d|\nabla u|^2}{dt} + \varepsilon |\nabla u|^2 - (Au, q(x)W) \right] \\
 &= -\frac{1}{4} \frac{d}{dt} (|\nabla u|^2 - p)^2 - \varepsilon (|\nabla u|^2 - p)^2 - \varepsilon p (|\nabla u|^2 - p) \\
 &\quad + (|\nabla u|^2 - p)(Au, q(x)W).
 \end{aligned} \tag{4.14}$$

$$(|\nabla u|^2 - p)(Au, q(x)W) \leq \frac{\varepsilon}{4} (|\nabla u|^2 - p)^2 + \frac{\varepsilon}{2} |\nabla u|^4 + \frac{1}{2\varepsilon^3} |\nabla q|^4 |W|^4. \tag{4.15}$$

Collecting with (4.6)-(4.15) and Lemma 4.1, we get that

$$\begin{aligned}
 &\frac{d}{dt} \left[ \|\psi\|_E^2 + b \|u^+\|^2 + 2 \int_{\Omega} F(u) dx + \frac{1}{2} (|\nabla u|^2 - p)^2 + 2C_1 |\Omega| \right] \\
 &+ \varepsilon_1 \left[ \|\psi\|_E^2 + b \|u^+\|^2 + 2 \int_{\Omega} F(u) dx + \frac{1}{2} (|\nabla u|^2 - p)^2 + 2C_1 |\Omega| \right] \\
 &\leq M \left( 1 + |W(t)| + |W(t)|^2 + |W(t)|^4 + |W(t)|^{k+1} \right),
 \end{aligned}$$



where  $\varepsilon_1 = \min \left\{ \varepsilon, \frac{2C_2 - C_0 C_1^{-1}}{2} \varepsilon \right\}$ ,  $C_2 > \frac{C_0 C_1^{-1}}{2}$ , and

$$M = \max \left\{ \varepsilon p^2 + \varepsilon (C_0 + 2C_2 + 2C_1) |\Omega|, C_0 \|q\|, \frac{\|q\|_2^2}{\varepsilon} + \left( \frac{\varepsilon^2}{\lambda} + \frac{b}{2\varepsilon} \right) \|q\|^2 + \frac{\|q\|_4^2}{\lambda}, \frac{C_0 \|q\|_{k+1}^{k+1}}{2\varepsilon}, \frac{\|\nabla q\|^4}{2\varepsilon^3} \right\}.$$

By the Gronwall lemma, we conclude that

$$\begin{aligned} & \|\psi(t, \omega, \psi(\tau, \omega))\|_E^2 \\ & \leq e^{-\varepsilon_1(t-\tau)} \left( \|\psi(\tau, \omega)\|_E^2 + b \|u_0^+\|^2 + 2 \int_{\Omega} F(u_0) dx + \frac{1}{2} (|\nabla u_0|^2 - p)^2 + 2C_1 |\Omega| \right) \\ & \quad + M \int_{\tau}^t e^{-\varepsilon_1(t-s)} \left( 1 + |W(s)| + |W(s)|^2 + |W(s)|^4 + |W(s)|^{k+1} \right) ds \\ & \leq 2e^{-\varepsilon_1(t-\tau)} \left[ \|u_0\|_2^2 + \|u_1 + \varepsilon u_0\|^2 + \|q\|^2 |W(\tau)|^2 + b \|u_0^+\|^2 \right. \\ & \quad \left. + \int_{\Omega} F(u_0) dx + \frac{1}{4} (|\nabla u_0|^2 - p)^2 \right] \\ & \quad + M \int_{\tau}^t e^{-\varepsilon_1(t-s)} \left( 1 + |W(s)| + |W(s)|^2 + |W(s)|^4 + |W(s)|^{k+1} \right) ds. \end{aligned} \tag{4.16}$$

Let

$$\begin{aligned} r_0^2(\omega) &= 2 \left( 1 + \sup_{\tau \leq -1} e^{\varepsilon_1 \tau} \|q\|^2 |W(\tau)|^2 \right) + \frac{M}{\varepsilon_1} \\ & \quad + M \int_{-\infty}^{-1} e^{-\varepsilon_1(-1-s)} \left( |W(s)| + |W(s)|^2 + |W(s)|^4 + |W(s)|^{k+1} \right) ds, \\ r_1^2(\omega) &= \frac{M}{\varepsilon_1} + M \int_{-\infty}^0 e^{\varepsilon_1 s} \left( |W(s)| + |W(s)|^2 + |W(s)|^4 + |W(s)|^{k+1} \right) ds, \end{aligned}$$

where  $r_0^2(\omega)$  and  $r_1^2(\omega)$  are finite  $P$ -a.s., we get a bounded set  $B$  of  $E$ , we choose  $T(B) \leq -1$  such that

$$e^{-\varepsilon_1(-1-\tau)} \left( \|u_0\|_2^2 + \|u_1 + \varepsilon u_0\|^2 + b \|u_0^+\|^2 + \int_{\Omega} F(u_0) dx + C_1 |\Omega| + \frac{1}{4} (|\nabla u_0|^2 - p)^2 \right) \leq 1 \tag{4.17}$$

for all  $\forall (u_0, u_1 + \varepsilon u_0)^T \in B$ , and

$$e^{\varepsilon_1 \tau} \left( \|u_0\|_2^2 + \|u_1 + \varepsilon u_0\|^2 + b \|u_0^+\|^2 + \int_{\Omega} F(u_0) dx + C_1 |\Omega| + \frac{1}{4} (|\nabla u_0|^2 - p)^2 \right) \leq 1 \tag{4.18}$$

for all  $\forall (u_0, u_1 + \varepsilon u_0)^T \in B$ , and for all  $\tau \leq T(B)$ .  $\square$

Let  $u(t)$  be a solution of problem (1.1) with initial value  $(u_0, u_1 + \varepsilon u_0)^T \in B$ . we make the decomposition  $u(t) = y_1(t) + y_2(t)$ , where  $y_1(t)$  and  $y_2(t)$  satisfy

$$\begin{cases} y_{1tt} + \Delta^2 y_1 + \Delta^2 y_{1t} + (p - |\nabla u|^2) \Delta y_1 = 0, & \Omega \times [\tau, +\infty), \tau \in \mathbb{R}, \\ y_1(x, t) = \nabla y_1(x, t) = 0, & x \in \partial\Omega, t \geq \tau, \\ y_1(x, \tau) = u_0(x), y_{1t}(x, \tau) = u_1(x), & x \in \Omega. \end{cases} \tag{4.19}$$

and

$$\begin{cases} y_{2t} + \Delta^2 y_2 + \Delta^2 y_t + (p - |\nabla u|^2) \Delta y_2 + bu^+ + f(u) = q(x) \dot{W}, & \Omega \times [\tau, +\infty), \tau \in \mathbb{R}, \\ y_2(x, t) = \nabla y_2(x, t) = 0, & x \in \partial\Omega, t \geq \tau, \\ y_2(x, \tau) = 0, y_{2\tau}(x, \tau) = 0, & x \in \Omega. \end{cases} \tag{4.20}$$

**Lemma 4.3.** Let  $p < \frac{\sqrt{\lambda}}{4}$ ,  $B$  be a bounded non-random subset of  $E$ ,  $\forall (u_0, u_1 + \varepsilon u_0)^T \in B$

$$\begin{aligned} \|Y_1(0)\|_E^2 &= \|y_1(0)\|_2^2 + \|y_{1\tau}(0) + \varepsilon y_1(0)\|^2 \\ &\leq \frac{e^{\varepsilon\tau}}{(1-\varepsilon)C(p)} \left( \|u_0\|_2^2 - (p - |\nabla u_0|^2) |\nabla u_0|^2 + \|u_1 + \varepsilon u_0\|^2 \right), \end{aligned} \tag{4.21}$$

where  $Y_1 = (y_1, y_{1\tau} + \varepsilon y_1)^T$  satisfies (4.19),  $C(p) = \begin{cases} 0, & p \leq 0, \\ 1 - \frac{p}{\sqrt{\lambda}}, & p < \frac{\sqrt{\lambda}}{4}. \end{cases}$

**Proof.** Let  $v = y_{1\tau} + \varepsilon y_1$ , taking the scalar product in  $H$  of (4.19), we get that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left[ \|v\|^2 + (1-\varepsilon) \|y_1\|_2^2 + (|\nabla u_0|^2 - p) |\nabla y_1|^2 \right] \\ &+ \|v\|_2^2 - \varepsilon \|v\|^2 + \varepsilon (1-\varepsilon) \|y_1\|_2^2 + \varepsilon (|\nabla u_0|^2 - p) |\nabla y_1|^2 + \varepsilon^2 (y_1, v) = 0, \end{aligned} \tag{4.22}$$

using the Hölder inequality and the Young inequality, we get that

$$\begin{aligned} &\|v\|_2^2 - \varepsilon \|v\|^2 + \varepsilon (1-\varepsilon) \|y_1\|_2^2 + \varepsilon^2 (y_1, v) \\ &\geq \varepsilon (1-\varepsilon) \|y_1\|_2^2 + \|v\|_2^2 - \varepsilon \|v\|^2 - \frac{\varepsilon(1-\varepsilon)}{2} \|y_1\|_2^2 - \frac{\varepsilon^3}{2(1-\varepsilon)\lambda} \\ &\geq \frac{\varepsilon(1-\varepsilon)}{2} \|y_1\|_2^2 + \left( \lambda - \varepsilon - \frac{\varepsilon}{2\lambda} \right) \|v\|^2 \geq \frac{\varepsilon(1-\varepsilon)}{2} \|y_1\|_2^2 + \frac{\varepsilon}{2} \|v\|^2, \end{aligned} \tag{4.23}$$

we have that

$$\begin{aligned} &\frac{d}{dt} \left[ \|v\|^2 + (1-\varepsilon) \|y_1\|_2^2 + (|\nabla u_0|^2 - p) |\nabla y_1|^2 \right] \\ &+ \varepsilon \left[ \|v\|^2 + (1-\varepsilon) \|y_1\|_2^2 + 2(|\nabla u_0|^2 - p) |\nabla y_1|^2 \right] \leq 0. \end{aligned}$$

Let  $\mathcal{G}(t) = \|v\|^2 + (1-\varepsilon) \|y_1\|_2^2 + (|\nabla u_0|^2 - p) |\nabla y_1|^2$ , we can get that

$$\mathcal{G}(t) \geq (1-\varepsilon)C(p) \|y_1\|_2^2 + \|v\|^2 + |\nabla u|^2 |\nabla y_1|^2 > (1-\varepsilon)C(p) \|Y_1\|_E^2 > 0$$

and for  $p < \frac{\sqrt{\lambda}}{4}$  we get

$$\varepsilon \left[ \|v\|^2 + (1-\varepsilon) \|y_1\|_2^2 + 2(|\nabla u_0|^2 - p) |\nabla y_1|^2 \right] \geq \frac{\varepsilon}{2} \mathcal{G}(t),$$

so

$$\frac{d}{dt} \mathcal{G}(t) + \frac{\varepsilon}{2} \mathcal{G}(t) \leq 0,$$

By the Gronwall lemma, getting that (4.21).  $\square$

**Lemma 4.4.** Assume that (1.2) holds, there exists a random radius  $r_2(\omega)$ , such that for  $P$ -a.s.  $\omega \in \Omega$ ,

$$\left\| A^{\frac{1}{2}} Y_2(0, \omega; Y_2(\tau, \omega)) \right\|_E^2 \leq r_2^2(\omega), \tag{4.24}$$

where  $Y_2 = (y_2, y_{2t} + \varepsilon y_2 - q(x)W)^T$  satisfies (4.20).

**Proof.** Let  $Y_2 = (y_2, y_{2t} + \varepsilon y_2 - q(x)W)^T$ , Equation (4.20) can be written to

$$Y_{2t} + QY_2 = H(Y_2, \omega), \quad Y_2(\tau) = (0, -q(x)W(\tau))^T, \tag{4.25}$$

where

$$H(Y_2, \omega) = \begin{pmatrix} q(x)W \\ (p - |\nabla u|^2)Ay_2 - bu^+ - f(u) + (\varepsilon - A^2)q(x)W \end{pmatrix}.$$

Taking the scalar product in  $E$  of (4.25) with  $AY_2$ , we get that

$$\frac{1}{2} \frac{d}{dt} \left\| A^{\frac{1}{2}} Y_2 \right\|_E^2 + (QY_2, AY_2)_E = (H(Y_2, \omega), AY_2)_E, \tag{4.26}$$

where

$$\begin{aligned} & (H(Y_2, \omega), AY_2)_E \\ &= ((Ay_2, q(x)W)) + ((p - |\nabla u|^2)Ay_2, A(y_{2t} + \varepsilon y_2 - q(x)W)) \\ &+ (-bu^+ - f(u) + (\varepsilon - A^2)q(x)W, A(y_{2t} + \varepsilon y_2 - q(x)W)). \end{aligned} \tag{4.27}$$

Due to Lemma 4.1, we get that

$$(QY_2, AY_2)_E \geq \frac{\varepsilon}{2} \left\| A^{\frac{1}{2}} Y_2 \right\|_E^2 + \frac{\varepsilon}{4} \left\| A^{\frac{1}{2}} y_2 \right\|_2^2 + \frac{\lambda}{2} \left\| A^{\frac{1}{2}} (y_{2t} + \varepsilon y_2 - q(x)W) \right\|_2^2. \tag{4.28}$$

Using the Young inequality, we have that

$$((Ay_2, q(x)W)) \leq \frac{\varepsilon}{4} \left\| A^{\frac{1}{2}} y_2 \right\|_2^2 + \frac{1}{\varepsilon} \left\| A^{\frac{1}{2}} q \right\|_2^2 |W|^2; \tag{4.29}$$

$$\begin{aligned} & \left| -(bu^+, A(y_{2t} + \varepsilon y_2 - q(x)W)) \right| \\ & \leq \frac{2b^2}{\lambda} \left\| A^{\frac{1}{2}} u^+ \right\|_2^2 + \frac{\lambda}{8} \left\| A^{\frac{1}{2}} (y_{2t} + \varepsilon y_2 - q(x)W) \right\|_2^2; \end{aligned} \tag{4.30}$$

$$\begin{aligned} & \left| -(A^2 q(x)W, A(y_{2t} + \varepsilon y_2 - q(x)W)) \right| \\ & \leq \frac{2}{\lambda} \left\| A^{\frac{5}{2}} q \right\|_2^2 |W|^2 + \frac{\lambda}{8} \left\| A^{\frac{1}{2}} (y_{2t} + \varepsilon y_2 - q(x)W) \right\|_2^2; \end{aligned} \tag{4.31}$$

$$\begin{aligned} & (\varepsilon q(x)W, A(y_{2t} + \varepsilon y_2 - q(x)W)) \\ & \leq \frac{2\varepsilon^2}{\lambda} \left\| A^{\frac{1}{2}} q \right\|_2^2 |W|^2 + \frac{\lambda}{8} \left\| A^{\frac{1}{2}} (y_{2t} + \varepsilon y_2 - q(x)W) \right\|_2^2. \end{aligned} \tag{4.32}$$

By (1.2), (4.5) and Sobolev embedding theorem, we obtain that  $f(u)$  is

uniformly bounded in  $L^\infty$ , that is, there exists a constant  $M > 0$  such that

$$|f'(s)|_{L^\infty} \leq M. \tag{4.33}$$

Combining with (4.33), the Sobolev embedding theorem and the Young inequality, we have that

$$\begin{aligned} & \left| -(f(u), A(y_{2t} + \varepsilon y_2 - q(x)W)) \right| \\ & \leq \left\| A^{\frac{1}{2}} f(u) \right\| \left\| A^{\frac{1}{2}} (y_{2t} + \varepsilon y_2 - q(x)W) \right\| \\ & \leq \frac{2\mu M^2}{\lambda} \|u\|_2^2 + \frac{\lambda}{8} \left\| A^{\frac{1}{2}} (y_{2t} + \varepsilon y_2 - q(x)W) \right\|^2, \end{aligned} \tag{4.34}$$

where  $\mu$  is a positive constant.

$$\begin{aligned} & -\left( (p - |\nabla u|^2) A y_2, A(y_{2t} + \varepsilon y_2 - q(x)W) \right) \\ & = \frac{1}{2} \frac{d}{dt} \left( |\nabla u|^2 - p \right) \|y_2\|_2^2 + \varepsilon \left( |\nabla u|^2 - p \right) \|y_2\|_2^2 + (p - |\nabla u|^2) (A y_2, q(x)W) \\ & \geq \frac{1}{2} \frac{d}{dt} \left( |\nabla u|^2 - p \right) \|y_2\|_2^2 + \varepsilon \left( |\nabla u|^2 - p \right) \|y_2\|_2^2 - (|p| + |\nabla u|^2) \|y_2\|_2 \|Aq\| \|W\| \\ & \geq \frac{1}{2} \frac{d}{dt} \left( |\nabla u|^2 - p \right) \|y_2\|_2^2 + \frac{\varepsilon}{2} \left( |\nabla u|^2 - p \right) \|y_2\|_2^2 - \frac{\varepsilon}{2} |p| \|y_2\|_2^2 \\ & \quad - \frac{1}{2} \|Aq\|^2 \|W\|^2 - \frac{1}{2} (|p| + |\nabla u|^2)^2 \|y_2\|_2^2 \\ & = \frac{1}{2} \frac{d}{dt} \left( |\nabla u|^2 - p \right) \|y_2\|_2^2 + \frac{\varepsilon}{2} \left( |\nabla u|^2 - p \right) \|y_2\|_2^2 \\ & \quad - \frac{1}{2} \left( \varepsilon |p| + (|p| + |\nabla u|^2)^2 \right) \|y_2\|_2^2 - \frac{1}{2} \|Aq\|^2 \|W\|^2 \end{aligned} \tag{4.35}$$

Let  $E(t) = \left\| A^{\frac{1}{2}} Y_2 \right\|_{E_1}^2 + \left( |\nabla u|^2 - p \right) \|y_2\|_2^2$ , by the Poincaré and  $C(p)$ , we get that,  $E(t) \geq C(p) \left\| A^{\frac{1}{2}} Y_2 \right\|_{E_1}^2 > 0$ . Using (4.27)-(4.35) and (4.5), for  $\tau \leq T(\omega)$ , from (4.27) we get that

$$\begin{aligned} & \frac{d}{dt} E(t) + \varepsilon E(t) \\ & \leq C \left( \varepsilon |p| + (|p| + R(s, \omega))^2 + \frac{4b^2 + 4\mu M^2}{\lambda} \right) R(s, \omega) \\ & \quad + \left( \frac{2}{\varepsilon} \left\| A^{\frac{1}{2}} q \right\|_2^2 + \frac{4}{\lambda} \left\| A^{\frac{5}{2}} q \right\|_2^2 + \frac{4\varepsilon^2}{\lambda} \left\| A^{\frac{1}{2}} q \right\|_2^2 + \frac{1}{2} \|Aq\|^2 \right) |W(t)|^2, \quad \tau \leq t \leq 0. \end{aligned}$$

Using the Gronwall lemma, we get that

$$\begin{aligned} E(0) & \leq e^{\varepsilon \tau} \left\| A^{\frac{1}{2}} q \right\|_2^2 + C \int_\tau^0 e^{\varepsilon s} \left( \varepsilon |p| + (|p| + R(s, \omega))^2 + \frac{4b^2 + 4\mu M^2}{\lambda} \right) R(s, \omega) ds \\ & \quad + \left( \frac{2}{\varepsilon} \left\| A^{\frac{1}{2}} q \right\|_2^2 + \frac{4}{\lambda} \left\| A^{\frac{5}{2}} q \right\|_2^2 + \frac{4\varepsilon^2}{\lambda} \left\| A^{\frac{1}{2}} q \right\|_2^2 + \frac{1}{2} \|Aq\|^2 \right) \int_\tau^0 e^{\varepsilon s} |W(s)|^2 ds. \end{aligned} \tag{4.36}$$

Set

$$r_2^2(\omega) = \left( \frac{2}{\varepsilon} \left\| A^{\frac{1}{2}} q \right\|_2^2 + \frac{4}{\lambda} \left\| A^{\frac{5}{2}} q \right\|^2 + \frac{4\varepsilon^2}{\lambda} \left\| A^{\frac{1}{2}} q \right\|^2 + \frac{1}{2} \|Aq\|^2 \right) \int_{-\infty}^0 e^{\varepsilon s} |W(s)|^2 ds$$

$$+ \left\| A^{\frac{1}{2}} q \right\|^2 \sup_{\tau \leq 0} e^{\varepsilon \tau} |W(\tau)|^2$$

$$+ C \int_{-\infty}^0 e^{\varepsilon s} \left( \varepsilon |p| + (|p| + R(s, \omega))^2 + \frac{4b^2 + 4\mu M^2}{\lambda} \right) R(s, \omega) ds.$$

Since  $\lim_{t \rightarrow \infty} \frac{W(t)}{t} = 0$ ,  $r_2^2(\omega)$  is finite P-a.s., together with 4.18 and 4.36, we get that

$$\left\| A^{\frac{1}{2}} Y_2(0, \omega; Y_2(\tau, \omega)) \right\|_E^2 \leq \frac{r_2^2(\omega)}{C(p)}.$$

This completes the proof of Lemma 4.4.  $\square$

**Theorem 4.5.** Let  $p < \frac{\sqrt{\lambda}}{4}$ , (1.2)-(1.4) hold,  $q(x) \in H^3(\Omega) \cap H_0^2(\Omega)$ , then the random dynamical system  $S_\varepsilon(t, \omega)$  possesses a nonempty compact random attractor  $A$ .

**Proof.** Let  $B_1(\omega)$  be the ball of  $H^3(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))$  of radius  $\frac{r_2(\omega)}{\sqrt{C(p)}}$ , by the compact embedding  $H^3(\Omega)$ , it follows that  $B_1(\omega)$  is

compact in  $E$ . for every bounded non-random set  $B$  of  $E$  and any  $\psi(0) \in \tilde{S}_\varepsilon(t, \theta_{-t}\omega)B$ , by Lemma 4.4, we know that  $Y_2(0) = \psi(0) - Y_1(0) \in B_1(\omega)$ . Therefore, for  $\tau \leq 0$ ,

$$\inf_{l(0) \in B_1(\omega)} \left\| \psi(0) - l(0) \right\|_E^2 \leq \|Y_1(0)\|_E^2$$

$$\leq \frac{e^{\varepsilon \tau}}{(1-\varepsilon)C(p)} \left( \|u_0\|_2^2 - (p - |\nabla u_0|^2) |\nabla u_0|^2 + \|u_1 + \varepsilon u_0\|^2 \right).$$

So, for all  $t \geq 0$ ,

$$d\left(\tilde{S}_\varepsilon(t, \theta_{-t}\omega)B, B_1(\omega)\right)$$

$$\leq \frac{e^{-\varepsilon t}}{(1-\varepsilon)C(p)} \left( \|u_0\|_2^2 - (p - |\nabla u_0|^2) |\nabla u_0|^2 + \|u_1 + \varepsilon u_0\|^2 \right).$$

From relation (4.3) between  $S_\varepsilon(t, \omega)$  and  $\tilde{S}_\varepsilon(t, \omega)$ , we can obtain that for any non-random bounded  $B \subset EP$ -a.s.,

$$d\left(\tilde{S}_\varepsilon(t, \theta_{-t}\omega)B, B_1(\omega)\right) \rightarrow 0, \text{ as } t \rightarrow +\infty.$$

Hence, the RDS  $S_\varepsilon(t, \omega)$  associated with (3.7) possesses a uniformly attracting compact set  $B_1(\omega) \subset E$ . Using Theorem 2.1, we complete the proof.

### Acknowledgements

This work is supported by the National Natural Sciences Foundation of People's

Republic of China under Grant 11561076.

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