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Approximate Inertial Manifold for a Class of the Kirchhoff Wave Equations with Nonlinear Strongly Damped Terms

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Abstract

This paper is devoted to the long time behavior of the solution to the initial boundary value problems for a class of the Kirchhoff wave equations with nonlinear strongly damped terms: $u_{tt}-\varepsilon_1\Delta u_t+\alpha|u_t|^{p-1}u_t+\beta|u|^{q-1}u-\phi\left(\|\nabla u\|^2\right)\Delta u=f(x)$. Firstly, in order to prove the smoothing effect of the solution, we make efficient use of the analytic property of the semigroup generated by the principal operator of the equation in the phase space. Then we obtain the regularity of the global attractor and construct the approximate inertial manifold of the equation. Finally, we prove that arbitrary trajectory of the Kirchhoff wave equations goes into a small neighbourhood of the approximate inertial manifold after large time.

Keywords

Kirchhoff Wave Equation, Global Attractor, The Smoothing Effect, The Regularity, Approximate Inertial Manifold

1. Introduction

It is well known that we are studying the long time behavior of the infinite dimensional dynamical systems of the nonlinear partial differential equations, and the concept of the inertial manifold plays an important role in this field. In 1985, G. Foias, G. R. Sell and R. Teman [1] first put forward the concept of the inertial manifold; it is an invariant finite dimensional Lipschitz manifold; it is exponentially attracting trajectory and contains the global attractor. But to ensure that existing conditions are very harsh for inertial manifolds (For instance, spectral interval condition), the existence of a large number of important partial differential equations is still not solved. Therefore, people naturally think of using an approximate, smooth and easy to solve the manifolds to approximate the global attractor and inertial manifolds, which is the approximate inertial manifold.

Approximate inertial manifolds are finite dimensional smooth manifolds, and each solution of the equation is in a finite time to its narrow field. In particular, the global attractor is also included in its neighbourhood. The existence of approximate inertial manifolds of a large number of dissipative partial differential equations has been studied [2]-[7].

In this paper, we are concerned a class of the Kirchhoff wave equations with nonlinear strongly damped terms referred to as follows:

$$u_{tt} - \varepsilon_1 \Delta u_t + \alpha |u_t|^{p-1} u_t + \beta |u|^{q-1} u - \phi\left(\|\nabla u\|^2\right)\Delta u = f(x) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u(x,0) = u_0(x); u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t)\big|_{\partial\Omega} = 0, \Delta u(x,t)\big|_{\partial\Omega} = 0, \quad x \in \Omega. \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $\varepsilon_1, \alpha, \beta$ are positive constants, and the assumptions on $\phi\left(\|\nabla u\|^2\right)$ will be specified later.

In [8], G. Kirchhoff firstly proposed the so called Kirchhoff string model in the study nonlinear vibration of an elastic string. Kirchhoff type wave equations have been studied by many scholars (see [9] [10] [11]). In reference [12], the long time behavior of solutions for the initial value problems (1.1) - (1.3), the existence of global attractor corresponding to the semigroup operator $S(t)$ and the dimension estimation of global attractor, have been researched.

In [13], Dai Zhengde, Guo Boling, Lin Guoguang studied the fractal structure of attractor for the generalized Kuramoto-Sivashinsky equations:

$$u_t + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} + f(u)_x + \varphi(u)_{xx} = g(u) + h(x), t > 0, x \in R, \tag{1.4}$$

$$u(x,0) = u_0(x), \tag{1.5}$$

$$u(x-D,t) = u(x+D,t), t > 0, x \in R. \tag{1.6}$$

where $\alpha \geq 0, \gamma > 0, D > 0$.

In [14], Li Yongsheng, Zhang Weiguo studied regularity and approximate of the attractor for the strongly damped wave equation:

$$u_{tt} - \alpha u_{xxt} - \beta u_{xx} + h(u)u_t + f(u) = g(x), t > 0, x \in (0,1), \tag{1.7}$$

$$u(0,t) = u(1,t) = 0, t \geq 0, \tag{1.8}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in (0,1). \tag{1.9}$$

where α, β are positive constants.

Luo Hong, Pu Zhilin and Chen Guanggan [15] studied regularity of the attractor and approximate inertial manifold for strongly damped nonlinear wave equation:

$$u_{tt} - \alpha u_{xxt} - \sigma(u_x)_x + f(u) = g(x), x \in (0,1), t \in [0,\infty), \tag{1.10}$$

$$u(0) = u_0, u_t(0) = u_1, \tag{1.11}$$

$$u(0,t)=u(1,t)=0. \tag{1.12}$$

where α is a positive constant.

Wang Lei, Dang Jinbao and Lin Guoguang [16] also studied the approximate inertial manifolds of the fractional nonlinear Schrödinger equation:

$$iu_t+(-\Delta)^{\alpha}u+\beta|u|^{\rho}u+i\delta u=f(x), x\in\Omega, t>0, \tag{1.13}$$

$$u(x,0)=u_0(x), x\in\Omega, \tag{1.14}$$

$$u(x+Le_i,t)=u(x,t), x\in\Omega, t>0. \tag{1.15}$$

where $\Omega=(0,L)^n, e_i=(0,\cdots,0,1,0,\cdots,0),(i=1,2,\cdots,n)$ is a standard orthogonal base, i is the imaginary unit. $\alpha>\frac{n}{2},\beta>0,\rho>0,\delta>0$.

Recently, Sufang Zhang, Jianwen Zhang [17] studied approximate inertial manifold of strongly damped wave equation:

$$u_{tt}-\Delta u-\Delta u_t-\alpha\Delta u_{tt}+f(u)=g(x,t)\in\Omega\times R^{+}, \tag{1.16}$$

$$u(x,0)=u_0, u_t(x,0)=u_1, x\in\Omega, \tag{1.17}$$

$$u(x,t)=0,(x,t)\in\partial\Omega\times R^{+}. \tag{1.18}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\alpha>0$ is a constant, the function $g\in L^2(\Omega)$.

There have many researches on approximate inertial manifolds for nonlinear wave equations (see [18]-[24]). In order to construct the approximate inertial manifolds for the initial boundary value problems, in the references [14] to [15], the regularity of the global attractor is studied, and then the approximate inertial manifold is constructed. In [18], Tian Lixin, Lin Yurui construct approximate inertial manifolds under spline wavelet basis in weakly damped forced KdV equation. In infinite-dimensional dynamical systems, Kirchhoff type wave equation is a class of very important equation. However, the approximate inertial manifold and inertial manifold of the Kirchhoff wave equation with nonlinear strong damping term are rarely studied. Based on the current research situation of Kirchhoff wave equations, in this paper, we first study the regularity of the global attractor for a class of the Kirchhoff wave equations with nonlinear strongly damped terms, and then construct its approximate inertial manifold.

The paper is arranged as follows. In Section 2, we state some assumptions, notations and the main results are stated. In Section 3, through the estimation of solution smoothness of higher order, then we obtain the regularity of the global attractor. In Section 4, by constructing a smooth manifold, namely the approximate inertial manifold, we approximate the global attractor for the problems (1.1) - (1.3).

2. Statement of Some Assumptions, Notations and Main Results

For convenience, we denote the norm and scalar product in $L^2(\Omega)$ by $\|.\|$ and $(.,.)$; $f=f(x)$, $L^p=L^p(\Omega)$, $H^k=H^k(\Omega)$, $H_0^k=H_0^k(\Omega)$, $\|\cdot\|=\|\cdot\|_{L^2}$, $\|\cdot\|_p=\|\cdot\|_{L^p}$.

Let $E=L^2(\Omega)$, where $\Omega\subset\mathbb{R}^N$ is a bounded domain, where the norm is defined as $\|.\|$. $A=-\Delta$ is an unbounded positive definite self adjoint operator. Let $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$, From reference [25], A^{-1} is compact, $D(A)$ is dense in *E*, so $E=span\{\omega_k\}_{k=1}^{\infty}$, where *E* is space by $\{\omega_1,\omega_2,\cdots,\omega_k\cdots\}$ as base generated. $A\omega_k=\lambda_k\omega_k$, where λ_k, ω_k are the eigenvalues and eigenvectors of *A*, $0<\lambda_1\le\lambda_2\le\lambda_3\le\cdots,\lambda_k\to\infty$, $k=1,2,3,\cdots$. Then ω_k consists of a set of standard orthogonal basis space *E*.

We present some assumptions and notations needed in the proof of our results as follows:

(G_1) From reference [12], we set some constants: $\varepsilon_1>0,\varepsilon>0,\gamma_1>0,\gamma_2>0,K\ge0$, such that $K-2\varepsilon\ge0$, $\varepsilon_1\varepsilon\le\phi\left(\|\nabla u\|^2\right)\le\frac{\gamma_1}{K-2\varepsilon}\left(1-\frac{K-2\varepsilon}{\gamma_1}e^{-(K-2\varepsilon)t}\right)$.

(G_2) Let $\phi(s)\in C^1([0,+\infty))$, and $\phi(0)=0,\sup|\phi'(s)|\le r_0$, $\forall s\in[0,+\infty)$.

Theorem 2.1 *From reference* [12], *due to* (G_1), (G_2) *hold*,

(i) Let $f(x)\in L^2(\Omega)$, then for each $u_0\in H^2(\Omega)\cap H_0^1(\Omega)$, $u_1\in L^2(\Omega)$, the problems (1.1)-(1.3) exist solution *u*, $u\in C_b([0,+\infty);D(A))$; $u_t\in C_b([0,+\infty);E)\cap L^2(0,T;H_0^1(\Omega))$, $\forall T>0$.

(ii) Let $f(x)\in H_0^1(\Omega)$, $S(t)$ is the semigroup operator for the problems (1.1) - (1.3), then the semigroup $S(t)$ exists a compact global attractor $\mathcal{A}_0$. So we can find a compact connected invariant set B to absorb all the bounded sets on $D(A)\times E$.

3. The Regularity of Global Attractor

In order to obtain the regularity of global attractor, we need to give a higher order uniform a priori estimates for the solution.

Let $v=u_t$, then the problem (1.1) can be reduced to the following form:

$$u_t=v, \tag{3.1}$$

$$v_t-\varepsilon_1\Delta v+\alpha|v|^{p-1}v+\beta|u|^{q-1}u-\phi\left(\|\nabla u\|^2\right)\Delta u=f(x). \tag{3.2}$$

Let

$$U=\begin{pmatrix}u\\v\end{pmatrix},\Lambda=\begin{pmatrix}0 & -I\\ -\phi\left(\|\nabla u\|^2\right)\Delta & -\varepsilon_1\Delta\end{pmatrix},FU=\begin{pmatrix}0\\F_1(u,v)\end{pmatrix},D(B)=[D(A)]^2. \tag{3.3}$$

where $F_1(u,v)=f(x)-\alpha|v|^{p-1}v-\beta|u|^{q-1}u$.

Further, we rewrite the problems (1.1) - (1.3):

$$\frac{dU}{dt}+\Lambda U=F(U),U(0)=U_0=\begin{pmatrix}u_0\\u_1\end{pmatrix}. \tag{3.4}$$

From references [26] [27], Λ is a linear dense closed operator on $D(A)\times E$, which is a sector operator and has a bounded inverse. Λ generates an analytic semigroup on $D(A)\times E$.

Lemma 3.1 From references [14] [15], due to (G_1), (G_2) hold, let $f\in L^2(\Omega),u(x,t)|_{\partial\Omega}=0$, then

Each $(u_0,u_1)\in D(A)\times E$, the solution to the problems (1.1) - (1.3) meet the following conditions:

$$u,u_t\in C^{\theta}\left((0,+\infty);D(A)\right),u_{tt}\in C^{\theta}\left((0,+\infty);E\right),\ \forall\,\theta\in(0,1). \tag{3.5}$$

And there exist $\tau_0>0,K_0>0$ such that the following inequalities are established:

$$\|u_t(t)\|_{D(A)}\le K_0,\|u_{tt}(t)\|\le R_1,\ \forall\,t\ge K_0. \tag{3.6}$$

where $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$, K_0 is independent of the initial value U_0.

Proof. By the first conclusion (i) of theorem 2.1, when $u_0\in D(A),u_1\in E$, the solution u meet: $u\in C_b\left([0,+\infty);D(A)\right)$, $u_t\in C_b\left([0,+\infty);L^2(\Omega)\right)$, $\forall\,T>0,u_t\in L^2\left(0,T;H_0^1(\Omega)\right)$. By the second conclusion (ii) of theorem 2.1, there exist $\tau>0,R_0>0$, when $t>\tau$,

$$\|u\|_{D(A)}\le R_0,\|u_t\|\le R_0. \tag{3.7}$$

Meanwhile, Δu is uniformly bounded in E, $t\in[0,+\infty)$.

$$FU=\left(0,F_1(u,v)\right)^{\mathrm{T}}=\left(0,f(x)-\alpha|v|^{p-1}v-\beta|u|^{q-1}u\right)^{\mathrm{T}}\in C_b\left([0,+\infty);D(A)\times E\right) \tag{3.8}$$

Then $F_1(u,v)\in C_b\left([0,T];D(A)\times E\right)\to L^p\left(0,T;D(A)\times E\right)$, $p=\dfrac{1}{1-\theta},\theta\in(0,1)$.

Based on the reference [27], the analytic properties of the semigroups generated by Λ and the Equation (3.4), immediately get $\forall\,0<t_0<T$, the solution $U(\cdot)\in C^{\theta}\left([t_0,T];D(A)\times E\right)$, furthermore, for the non-homogeneous term $F_1(u,v)$ in the Equation (3.4), $F_1(u,v)\in C^{\theta}\left([t_0,T];D(A)\times E\right)$, then $U(\cdot)\in C^{\theta}\left((t_0,T];D(\Lambda)\right)$, $U_t(\cdot),\Lambda U(\cdot)\in C^{\theta}\left((t_0,T);D(A)\times E\right)$, due to T,t_0 are arbitrary, $U(\cdot)\in C^{\theta}\left((0,+\infty);D(\Lambda)\right)$, $U_t(\cdot)\in C^{\theta}\left((0,+\infty);D(A)\times E\right)$.

Since $U(\tau)\in D(\Lambda),U_t(\tau)\in D(A)\times E$, we are now considering $\tau,U_t(\tau)$, respectively, as the initial time, initial value. Next, we consider the equation about $\mathcal{V}=U_t=(v,v_t)^{\mathrm{T}}$,

$$\mathcal{V}_t+\Lambda\mathcal{V}=F(U)_t=\left(0,-\alpha\left(|v|^{p-1}v\right)_t-\beta\left(|u|^{q-1}u\right)_t\right)^{\mathrm{T}}. \tag{3.9}$$

then

$$v_{tt}-\varepsilon_1\Delta v_t-\left(\phi\left(\|\nabla u\|^2\right)\Delta u\right)_t+\alpha\left(|v|^{p-1}v\right)_t+\beta\left(|u|^{q-1}u\right)_t=0, \tag{3.10}$$

$$v(x,\tau)=u_t(x,\tau)\in D(A), \tag{3.11}$$

$$v_t(x,\tau)=u_{tt}(x,\tau)\in E, \tag{3.12}$$

$$v(x,t)\big|_{\partial\Omega}=0,\Delta v(x,t)\big|_{\partial\Omega}=0,x\in\Omega,t\ge\tau. \tag{3.13}$$

Next, we multiply $v_t+\varepsilon v$ with both sides of the equation (3.10) and integrate over Ω to obtain

$$(v_{tt},v_t+\varepsilon v)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_t\|^2+\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega v\cdot v_t\mathrm{d}x\right)-\varepsilon\|v_t\|^2. \tag{3.14}$$

$$(-\varepsilon_1\Delta v_t,v_t+\varepsilon v)=\varepsilon_1\|\nabla v_t\|^2+\frac{\varepsilon_1\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2. \tag{3.15}$$

$$\begin{aligned}&\left(-\left(\phi\left(\|\nabla u\|^2\right)\Delta u\right)_t, v_t+\varepsilon v\right)\\&=\left(-\left(\phi\left(\|\nabla u\|^2\right)\Delta u\right)_t, v_t\right)+\left(-\left(\phi\left(\|\nabla u\|^2\right)\Delta u\right)_t, \varepsilon v\right)\end{aligned} \tag{3.16}$$

where from the hypothesis (G2),

$$\begin{aligned}&\left(-\left(\phi\left(\|\nabla u\|^2\right)\Delta u\right)_t, v_t\right)\\&=-\phi\left(\|\nabla u\|^2\right)_t\int_\Omega \Delta u v_t\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\phi\left(\|\nabla u\|^2\right)\|\nabla v\|^2\right]-\frac{1}{2}\phi\left(\|\nabla u\|^2\right)_t\cdot\|\nabla v\|^2\\&\geq-\frac{r_0}{2}\|\nabla u\|^2-\frac{r_0}{2}\|\nabla v_t\|^2+\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\phi\left(\|\nabla u\|^2\right)\|\nabla v\|^2\right]-\frac{r_0}{2}\|\nabla v\|^2.\end{aligned} \tag{3.17}$$

$$\begin{aligned}&\left(-\left(\phi\left(\|\nabla u\|^2\right)\Delta u\right)_t, \varepsilon v\right)\\&=-\varepsilon\phi\left(\|\nabla u\|^2\right)_t\cdot(\Delta u, v)+\varepsilon\phi\left(\|\nabla u\|^2\right)\|\nabla v\|^2\\&\geq-\frac{r_0\varepsilon}{2}\|\nabla u\|^2-\frac{r_0\varepsilon}{2}\|\nabla v\|^2+\varepsilon_1\varepsilon^2\|\nabla v\|^2.\end{aligned} \tag{3.18}$$

$$\alpha\left(\left(|v|^{p-1}v\right)_t, v_t+\varepsilon v\right)=\alpha\left(\left(|v|^{p-1}v\right)_t, v_t\right)+\alpha\varepsilon\left(\left(|v|^{p-1}v\right)_t, v\right). \tag{3.19}$$

$$\begin{aligned}\alpha\left(\left(|v|^{p-1}v\right)_t, v_t\right)&=\int_\Omega|v|^{p-1}v_tv_t\mathrm{d}x+\int_\Omega\left(\left(v^2\right)^{\frac{p-1}{2}}\right)_t vv_t\mathrm{d}x\\&=\alpha\int_\Omega|v|^{p-1}v_tv_t\mathrm{d}x+\alpha\int_\Omega\frac{p-1}{2}\left(\left(v^2\right)^{\frac{p-3}{2}}\right)2vv_tvv_t\mathrm{d}x\\&=\alpha\int_\Omega|v|^{p-1}v_tv_t\mathrm{d}x+\alpha(p-1)\int_\Omega|v|^{p-1}v_tv_t\mathrm{d}x\\&=\alpha p\int_\Omega|v|^{p-1}v_t^2\mathrm{d}x.\end{aligned} \tag{3.20}$$

$$\alpha\varepsilon\left(\left(|v|^{p-1}v\right)_t, v\right)=\alpha\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega|v|^{p-1}vv\mathrm{d}x\right)-\alpha\varepsilon\int_\Omega|v|^{p-1}vv_t\mathrm{d}x. \tag{3.21}$$

where $\alpha\varepsilon\int_\Omega|v|^{p-1}vv_t\mathrm{d}x\leq\alpha\varepsilon\left(\int_\Omega|v|^{2p}\mathrm{d}x\right)^{\frac{1}{2}}\cdot\left(\int_\Omega v_t^2\mathrm{d}x\right)^{\frac{1}{2}}=\alpha\varepsilon\|v\|_{2p}^p\|v_t\|$.

By using Gagliardo-Nirenberg's embedding inequality, Hölder's inequality:

$$\begin{aligned}\|v\|_{2p}^p\|v_t\|&\leq C_1\|\nabla v\|^{\frac{(p-1)n}{2}}\|v\|^{\frac{2p-(p-1)n}{2}}\|v_t\|\\&\leq\frac{C_1^2\alpha\varepsilon\|v_t\|^2}{2}+\frac{\|\nabla v\|^{(p-1)n}\|v\|^{2p-(p-1)n}}{2\alpha\varepsilon}\\&\leq\frac{C_1^2\alpha\varepsilon\|v_t\|^2}{2}+\frac{C_2\|v\|^2}{4\alpha\varepsilon}+\frac{C_2\|\nabla v\|^2}{4\alpha\varepsilon}+C_3\left(C_2,\alpha,\varepsilon\right)\end{aligned} \tag{3.22}$$

Similar to the relation (3.20):

$$\beta\left(\left(|u|^{q-1}u\right)_t, v_t+\varepsilon v\right)=\beta q\int_\Omega|u|^{q-1}vv_t\mathrm{d}x+\beta q\varepsilon\int_\Omega|u|^{q-1}v^2\mathrm{d}x. \tag{3.23}$$

By using Hölder's inequality, Young's inequality and Sobolev's embedding inequality:

$$\beta q\int_\Omega |u|^{q-1} vv_t \mathrm{d}x \le \beta q\int_\Omega |u|^{q-1}|v||v_t|\mathrm{d}x \le \beta q\int_\Omega |u|^{q-1}\left(\frac{|v|^2}{2}+\frac{|v_t|^2}{2}\right)\mathrm{d}x. \tag{3.24}$$

$$\frac{\beta q}{2}\int_\Omega |u|^{q-1}|v_t|^2 \mathrm{d}x \le \frac{\beta q}{2}\left(\int_\Omega |u|^{2(q-1)}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega |v_t|^4\mathrm{d}x\right)^{\frac{1}{2}} = \frac{\beta q}{2}\|u\|_{2(q-1)}^{q-1}\|v_t\|_4^2. \tag{3.25}$$

$$\|u\|_{2(q-1)}^{q-1} \le C_4\|\Delta u\|^{\frac{n(q-2)}{4}}\|u\|^{\frac{4(q-1)-n(q-2)}{4}}; \tag{3.26}$$

$$\|v_t\|_4^2 \le C_5\|\nabla v_t\|^{\frac{n}{2}}\|v_t\|^{\frac{4-n}{2}} \le \frac{\varepsilon_2\|\nabla v_t\|^2}{2}+\frac{\|v_t\|^2}{2\varepsilon_2}+C_6(\varepsilon_2,C_5). \tag{3.27}$$

In reference [12], $\|\Delta u\|,\|u\|$ are bounded by a priori estimates.

$$\|u\|_{2(q-1)}^{q-1}\|v_t\|_4^2 \le C_7\left(C_4,\|u\|_\infty,\|\Delta u\|_\infty\right)\left(\frac{\varepsilon_2\|\nabla v_t\|^2}{2}+\frac{\|v_t\|^2}{2\varepsilon_2}+C_6\right). \tag{3.28}$$

So we get:

$$\begin{aligned}\beta\left(\left(|u|^{q-1}u\right)_t, v_t+\varepsilon v\right) \ge{}& \left(\beta q\varepsilon-\frac{\beta q}{2}\right)\int_\Omega |u|^{q-1}v^2\mathrm{d}x-\frac{\beta qC_7\varepsilon_2}{2}\|\nabla v_t\|^2\\ &-\frac{\beta qC_7}{2\varepsilon_2}\|v_t\|^2-\beta qC_7C_6.\end{aligned} \tag{3.29}$$

From above, we have

$$\Phi_1=\frac{1}{2}\|v_t\|^2+\varepsilon\int_\Omega vv_t\mathrm{d}x+\frac{\varepsilon\varepsilon_1}{2}\|\nabla v\|^2+\alpha\varepsilon\int_\Omega |v|^{p-1}v^2\mathrm{d}x+\frac{1}{2}\phi\left(\|\nabla u\|^2\right)\|\nabla v\|^2. \tag{3.30}$$

$$\begin{aligned}\Psi_1={}&\varepsilon_1\|\nabla v_t\|^2-\varepsilon\|v_t\|^2-\phi\left(\|\nabla u\|^2\right)_t\int_\Omega \Delta uv_t\mathrm{d}x-\frac{1}{2}\phi\left(\|\nabla u\|^2\right)_t\cdot\|\nabla v\|^2\\ &-\varepsilon\phi\left(\|\nabla u\|^2\right)_t(\Delta u,v)+\varepsilon\phi\left(\|\nabla u\|^2\right)\|\nabla v\|^2+\alpha p\int_\Omega |v|^{p-1}v_t^2\mathrm{d}x\\ &+\beta q\varepsilon\int_\Omega |u|^{q-1}v^2\mathrm{d}x-\alpha\varepsilon\int_\Omega |v|^{p-1}vv_t\mathrm{d}x+\beta q\int_\Omega |u|^{q-1}vv_t\mathrm{d}x.\end{aligned} \tag{3.31}$$

Taking $\kappa_1>0$, then

$$\begin{aligned}\Psi_1-\kappa_1\Phi_1 \ge{}& \varepsilon_1\|\nabla v_t\|^2-\varepsilon\|v_t\|^2-\frac{r_0}{2}\|\Delta u\|^2-\frac{r_0}{2}\|v_t\|^2-\frac{r_0}{2}\|\nabla v\|^2-\frac{r_0\varepsilon}{2}\|\nabla u\|^2\\ &-\frac{r_0\varepsilon}{2}\|\nabla v\|^2+\varepsilon^2\varepsilon_1\|\nabla v\|^2-\frac{C_1^2\alpha^2\varepsilon^2}{2}\|v_t\|^2-\frac{C_2\|v\|^2}{4}-\frac{C_2\|\nabla v\|^2}{4}\\ &-C_3-\frac{\beta qC_7\varepsilon_2}{2}\|\nabla v_t\|^2-\frac{\beta qC_7}{2\varepsilon_2}\|v_t\|^2-\beta qC_7C_6-\frac{\kappa_1\|v_t\|^2}{2}\\ &-\frac{\kappa_1\varepsilon\|v\|^2}{2}-\kappa_1\alpha\varepsilon\int_\Omega |v|^{p-1}v^2\mathrm{d}x-\frac{\kappa_1\varepsilon_1\varepsilon\|\nabla v\|^2}{2}-\frac{\kappa_1\gamma_1\|\nabla v\|^2}{2(K-2\varepsilon)}.\end{aligned} \tag{3.32}$$

$$\begin{aligned}\int_\Omega |v|^{p-1}v^2\mathrm{d}x \le{}& \left(|\Omega|\right)^{\frac{1}{2}}\|v\|_{2(p+1)}^{p+1} \le C_8\left(\left(|\Omega|\right)^{\frac{1}{2}}\right)\|\nabla v\|^{\frac{np}{2}}\|v\|^{\frac{2p+2-np}{2}}\\ &\le C_9\frac{\|v\|^2}{2}+C_9\frac{\|\nabla v\|^2}{2}+C_{10}(C_8,C_9).\end{aligned} \tag{3.33}$$

At last, we get:

$$\Psi_1-\kappa_1\Phi_1\ge\left(\varepsilon_1-\frac{C_7\beta q\varepsilon_2}{2}\right)\|\nabla v_t\|^2-\left(\varepsilon+\frac{r_0}{2}+\frac{C_1^2\alpha^2\varepsilon^2}{2}+\frac{C_7\beta q}{2\varepsilon_2}+\frac{\kappa_1}{2}+\frac{\kappa_1\varepsilon}{2}\right)\|v_t\|^2$$
$$+\left(\varepsilon^2\varepsilon_1-\frac{r_0}{2}-\frac{r_0\varepsilon}{2}-\frac{\kappa_1\varepsilon_1\varepsilon}{2}-\frac{C_9\kappa_1\alpha\varepsilon}{2}-\frac{C_2}{4}-\frac{\kappa_1\gamma_1}{2(K-2\varepsilon)}\right)\|\nabla v\|^2 \tag{3.34}$$
$$-\left(\frac{C_2}{4}+\frac{\kappa_1\varepsilon}{2}+\frac{C_9\kappa_1\alpha\varepsilon}{2}\right)\|v\|^2-C.$$

Let $m_1=\varepsilon_1-\frac{C_7\beta q\varepsilon_2}{2}$; $m_2=\varepsilon+\frac{r_0}{2}+\frac{C_1^2\alpha^2\varepsilon^2}{2}+\frac{C_7\beta q}{2\varepsilon_2}+\frac{\kappa_1}{2}+\frac{\kappa_1\varepsilon}{2}$;

$$m_3=\varepsilon^2\varepsilon_1-\frac{r_0}{2}-\frac{r_0\varepsilon}{2}-\frac{\kappa_1\varepsilon_1\varepsilon}{2}-\frac{C_9\kappa_1\alpha\varepsilon}{2}-\frac{C_2}{4}-\frac{\kappa_1\gamma_1}{2(K-2\varepsilon)};$$

$$m_4=\frac{C_2}{4}+\frac{\kappa_1\varepsilon}{2}+\frac{C_9\kappa_1\alpha\varepsilon}{2}.$$

By using Poincaré's inequality, we get

$$\Psi_1-\kappa_1\Phi_1\ge(\lambda_1m_1-m_2)\|\nabla v_t\|^2+(\lambda_1m_3-m_4)\|v\|^2-C. \tag{3.35}$$

We take proper $\varepsilon,\varepsilon_1,\varepsilon_2,\gamma_1,\kappa_1,r_0,\alpha,\beta$, such that:

$$\begin{cases}\lambda_1m_1-m_2\ge 0\\ \lambda_1m_3-m_4\ge 0.\end{cases}$$

Then

$$\Psi_1-\kappa_1\Phi_1\ge -C. \tag{3.36}$$

From the relation (3.36), we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1(t)+\kappa_1\Phi_1(t)\le C,\ t\ge\tau. \tag{3.37}$$

By using Gronwall's inequality, we obtain:

$$\Phi_1(t)\le\Phi_1(\tau)\mathrm{e}^{-\kappa_1(t-\tau)}+\frac{C}{\kappa_1}\left(1-\mathrm{e}^{-\kappa_1(t-\tau)}\right),\ t\ge\tau. \tag{3.38}$$

Taking $\tau_0\gg\tau$, such that $\Phi_1(\tau)\mathrm{e}^{-\kappa_1(t-\tau)}\le 1$, then

$$\Phi_1(t)\le 1+\frac{C}{\kappa_1},\ \forall\, t\ge\tau_0. \tag{3.39}$$

where

$$\Phi_1=\frac{1}{2}\|v_t\|^2+\varepsilon\int_\Omega vv_t\mathrm{d}x+\frac{\varepsilon\varepsilon_1}{2}\|\nabla v\|^2+\alpha\varepsilon\int_\Omega|v|^{p-1}v^2\mathrm{d}x+\frac{1}{2}\phi\left(\|\nabla u\|^2\right)\|\nabla v\|^2$$
$$\ge\frac{1}{2}\|v_t\|^2+\varepsilon\int_\Omega vv_t\mathrm{d}x+\frac{\varepsilon\varepsilon_1}{2}\|\nabla v\|^2+\alpha\varepsilon\int_\Omega|v|^{p-1}v^2\mathrm{d}x+\frac{1}{2}\phi\left(\|\nabla u\|^2\right)\|\nabla v\|^2 \tag{3.40}$$
$$\ge\frac{(1-\varepsilon)}{2}\|v_t\|^2+\frac{(2\mu_1\varepsilon_1\varepsilon-C_9\mu_1\alpha\varepsilon-\varepsilon-C_9\alpha\varepsilon)}{2}\|v\|^2-\frac{C}{\kappa_1}.$$

Meanwhile, we once again take proper $\varepsilon,\varepsilon_1,\mu_1,\alpha$, such that:

$$\begin{cases} 1-\varepsilon>0 \\ 2\mu_1\varepsilon_1\varepsilon-C_9\mu_1\alpha\varepsilon-\varepsilon-C_9\alpha\varepsilon>0. \end{cases}$$

So there are $\tau_0>0, K_0>0$, which make the following inequalities:

$$\|u_t(t)\|_{D(A)}\le K_0, \|u_{tt}(t)\|\le K_0, \forall\, t\ge\tau_0. \tag{3.41}$$

where $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$, K_0 is independent of the initial value U_0.

Lemma 3.2 From references [14] [15], due to (G_1), (G_2) hold, let $f(0)=0$, $f\in D(A)=H^2(\Omega)\cap H_0^1(\Omega)$, then $\forall(u_0,u_1)\in D(A)\times E$, the solution to the problems (1.1) (1.3) meet the following conditions:

$$u,u_t\in C^\theta\left((0,+\infty);D(A^2)\right), u_{tt}\in C^\theta\left((0,+\infty);D(A)\right), \forall\,\theta\in(0,1). \tag{3.42}$$

And there exist $\tau_1>0$, $K_1>0$ such that the following inequalities are established:

$$\|u(t)\|_{D(A^2)}\le K_1, \|u_t(t)\|_{D(A^2)}\le K_1, \forall\, t\ge\tau_1. \tag{3.43}$$

Proof. Take proper T, such that $\forall\, 0<t_0<T$, $U(t_0)\in D(A)$, we are now considering the Equation (3.9), assume (G_1), (G_2) hold, $f\in D(A)$, $u,u_t\in C^\theta([t_0,T];D(A))$, $u_{tt}\in C^\theta([t_0,T];E)$, the nonlinear term $F(U(t))_t\in C^\theta([t_0,T];D(A)\times E)$. Based on the reference [27], the solution to the Equation (3.9):
$\mathcal{V}(\cdot),\mathcal{V}_t(\cdot),\Lambda\mathcal{V}(\cdot)\in C^\theta([t_0,T];D(A)\times E)$. From (3.4), we get
$U(\cdot)\in C^\theta\left([t_0,T];D(\Lambda^2)\right)$, due to T,t_0 are arbitrary, $U(\cdot)\in C^\theta\left((0,+\infty);D(\Lambda^2)\right)$,
$U(\cdot)_t\in C^\theta\left((0,+\infty);D(\Lambda)\right)$, and then we can get $u,u_t\in C^\theta\left((0,+\infty);D(A^2)\right)$,
$u_{tt}\in C^\theta\left((0,+\infty);D(A)\right)$, $\forall\,\theta\in(0,1)$.

Similar to lemma (3.1), we are now considering $\tau_0,U_t(\tau_0)$, respectively, as the initial time, initial value. Next, and once again, we consider the Equations (3.9) - (3.13), multiplying $-\Delta v_t-\varepsilon\Delta v$ with both sides of the equation (3.10) and integrating over Ω to obtain

$$(v_{tt},-\Delta v_t-\varepsilon\Delta v)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v_t\|^2+\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega\nabla v\cdot\nabla v_t\mathrm{d}x\right)-\varepsilon\|\nabla v_t\|^2. \tag{3.44}$$

$$(-\varepsilon_1\Delta v_t,-\Delta v_t-\varepsilon\Delta v)=\varepsilon_1\|\Delta v_t\|^2+\frac{\varepsilon_1\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta v\|^2. \tag{3.45}$$

$$\begin{aligned}&\left(-\left(\phi\left(\|\nabla u\|^2\right)\Delta u\right)_t,-\Delta v_t-\varepsilon\Delta v\right)\\ &=\left(-\left(\phi\left(\|\nabla u\|^2\right)\Delta u\right)_t,-\Delta v_t\right)+\left(-\left(\phi\left(\|\nabla u\|^2\right)\Delta u\right)_t,-\varepsilon\Delta v\right)\end{aligned} \tag{3.46}$$

where from the hypothesis (G2),

$$\begin{aligned}&\left(-\left(\phi\left(\|\nabla u\|^2\right)\Delta u\right)_t,-\Delta v_t\right)\\ &=\left(\phi\left(\|\nabla u\|^2\right)\right)_t(\Delta u,\Delta v_t)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\phi\left(\|\nabla u\|^2\right)\|\Delta v\|^2\right]-\frac{1}{2}\left(\phi\left(\|\nabla u\|^2\right)\right)_t\|\Delta v\|^2\\ &\ge-\frac{r_0}{2}\|\Delta u\|^2-\frac{r_0}{2}\|\Delta v_t\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\phi\left(\|\nabla u\|^2\right)\|\Delta v\|^2\right]-\frac{r_0}{2}\|\Delta v\|^2.\end{aligned}$$

$$\begin{aligned}&\left(-\left(\phi\left(\|\nabla u\|^2\right)\Delta u\right)_t,-\varepsilon\Delta v\right)\\&=\left(\phi\left(\|\nabla u\|^2\right)\right)_t\left(\Delta u,\varepsilon\Delta v\right)+\varepsilon\phi\left(\|\nabla u\|^2\right)\|\Delta v\|^2\\&\geq-\frac{r_0\varepsilon}{2}\|\Delta u\|^2-\frac{r_0\varepsilon}{2}\|\Delta v\|^2+\varepsilon^2\varepsilon_1\|\Delta v\|^2.\end{aligned}$$

Similar to lemma 3.1

$$\begin{aligned}&\alpha\left(\left(|v|^{p-1}v\right)_t,-\Delta v_t-\varepsilon\Delta v\right)\\&=\alpha\left(\left(|v|^{p-1}v\right)_t,-\Delta v_t\right)+\alpha\left(\left(|v|^{p-1}v\right)_t,-\varepsilon\Delta v\right)\\&=-\alpha p\int_\Omega|v|^{p-1}v_t\Delta v_t\mathrm{d}x-\alpha p\varepsilon\int_\Omega|v|^{p-1}v_t\Delta v\mathrm{d}x\end{aligned}\tag{3.47}$$

$$\begin{aligned}&\beta\left(\left(|u|^{q-1}u\right)_t,-\Delta v_t-\varepsilon\Delta v\right)\\&=\beta\left(\left(|u|^{q-1}u\right)_t,-\Delta v_t\right)+\beta\left(\left(|u|^{q-1}u\right)_t,-\varepsilon\Delta v\right)\\&=-\beta q\int_\Omega|u|^{q-1}v\Delta v_t\mathrm{d}x-\beta q\varepsilon\int_\Omega|u|^{q-1}v\Delta v\mathrm{d}x.\end{aligned}\tag{3.48}$$

By using Hölder's inequality, Young's inequality and Sobolev's embedding inequality:

$$\begin{aligned}&\alpha p\int_\Omega|v|^{p-1}v_t\Delta v_t\mathrm{d}x\\&\leq\alpha p\left(\int_\Omega|v|^{2(p-1)}|\Delta v_t|\mathrm{d}x\right)^{\frac12}\left(\int_\Omega|\Delta v_t||v_t|^2\mathrm{d}x\right)^{\frac12}\\&\leq\frac{\alpha p}{2}\int_\Omega|v|^{2(p-1)}|\Delta v_t|\mathrm{d}x+\frac{\alpha p}{2}\int_\Omega|\Delta v_t||v_t|^2\mathrm{d}x\\&\leq\frac{\alpha p}{2}\left(\int_\Omega|v|^{4(p-1)}\mathrm{d}x\right)^{\frac12}\left(\int_\Omega|\Delta v_t|^2\mathrm{d}x\right)^{\frac12}+\frac{\alpha p}{2}\left(\int_\Omega|v_t|^4\mathrm{d}x\right)^{\frac12}\left(\int_\Omega|\Delta v_t|^2\mathrm{d}x\right)^{\frac12}\\&=\frac{\alpha p}{2}\|v\|_{4(p-1)}^{2(p-1)}\|\Delta v_t\|+\frac{\alpha p}{2}\|v_t\|_4^2\|\Delta v_t\|.\end{aligned}$$

$$\begin{aligned}&\frac{\alpha p}{2}\|v\|_{4(p-1))}^{2(p-1)}\|\Delta v_t\|\\&\leq\frac{\alpha p}{2}C_1\left(\|v\|_\infty\right)\|\Delta v\|^{\frac{(2p-3)n}{4}}\|\Delta v_t\|\\&\leq\frac{r_0\|\Delta v\|^2}{8}+C_2\left(C_1,\alpha,p,r_0\right)+\frac{\alpha p\|\Delta v_t\|^2}{4}\end{aligned}$$

$$\begin{aligned}&\frac{\alpha p}{2}\|v_t\|_4^2\|\Delta v_t\|\\&\leq\frac{\alpha p}{4}C_3\left(\|v_t\|_\infty\right)\|\nabla v_t\|^n+\frac{\alpha p}{4}\|\Delta v_t\|^2\\&\leq\frac{\varepsilon_1\|\nabla v_t\|^2}{8}+C_4\left(C_3,\alpha,\varepsilon_1,p\right)+\frac{\alpha p}{4}\|\Delta v_t\|^2.\end{aligned}$$

$$\begin{aligned}&\alpha p\int_\Omega|v|^{p-1}v_t\Delta v_t\mathrm{d}x\\&\leq\frac{r_0\|\Delta v\|^2}{8}+C_2\left(C_1,\alpha,p,r_0\right)+\frac{\alpha p\|\Delta v_t\|^2}{2}+\frac{\varepsilon_1\|\nabla v_t\|^2}{8}+C_4\left(C_3,\alpha,\varepsilon_1,p\right)\end{aligned}$$

Through similar methods above

$$\alpha p\varepsilon\int_\Omega |v|^{p-1} v_t \Delta v \mathrm{d}x$$

$$\le \frac{\alpha p\varepsilon}{2} C_5\left(\|v\|_\infty\right)\|\Delta v\|^{\frac{(2p-3)n}{4}}\|\Delta v\| + \frac{\alpha p\varepsilon}{4} C_6\left(\|v_t\|_\infty\right)\|\nabla v_t\|^n + \frac{\alpha p\varepsilon}{4}\|\Delta v\|^2$$

$$\le \frac{r_0\|\Delta v\|^2}{4} + C_7\left(C_5,\alpha,p,\varepsilon,r_0\right) + \frac{\alpha p\varepsilon}{4}\|\Delta v\|^2 + \frac{\varepsilon_1\|\nabla v_t\|^2}{8} + C_8\left(C_6,\alpha,\varepsilon,\varepsilon_1,p\right)$$

$$\beta q\int_\Omega |u|^{q-1} v \Delta v_t \mathrm{d}x$$

$$\le \frac{\beta q}{4} C_9\left(\|u\|_\infty\right)\|\Delta u\|^{\frac{(2q-3)n}{2}} + \frac{\beta q}{4}\|\Delta v_t\|^2 + \frac{\beta q}{4} C_{10}\left(\|v\|_\infty\right)\|\nabla v\|^n + \frac{\beta q}{4}\|\Delta v_t\|^2$$

$$\le \frac{\beta q}{8}\|\Delta u\|^2 + C_{11}\left(C_9,\beta,q\right) + \frac{\beta q}{2}\|\Delta v_t\|^2 + \frac{r_0\|\nabla v\|^2}{8} + C_{12}\left(C_{10},\beta,q,r_0\right).$$

$$\beta q\varepsilon\int_\Omega |u|^{q-1} v \Delta v \mathrm{d}x$$

$$\le \frac{\beta q\varepsilon}{4} C_{13}\left(\|u\|_\infty\right)\|\Delta u\|^{\frac{(2q-3)n}{2}} + \frac{\beta q\varepsilon}{4}\|\Delta v\|^2 + \frac{\beta q\varepsilon}{4} C_{14}\left(\|v\|_\infty\right)\|\nabla v\|^n + \frac{\beta q\varepsilon}{4}\|\Delta v\|^2$$

$$\le \frac{\beta q\varepsilon}{8}\|\Delta u\|^2 + C_{15}\left(C_{13},\beta,q,\varepsilon\right) + \frac{\beta q\varepsilon}{2}\|\Delta v\|^2 + \frac{r_0\|\nabla v\|^2}{8} + C_{16}\left(C_{14},\beta,q,\varepsilon,r_0\right).$$

From above, we have

$$\Phi_2 = \frac{1}{2}\|\nabla v_t\|^2 + \varepsilon\int_\Omega \nabla v \nabla v_t \mathrm{d}x + \frac{\varepsilon\varepsilon_1}{2}\|\Delta v\|^2 + \frac{1}{2}\phi\left(\|\nabla u\|^2\right)\|\Delta v\|^2. \tag{3.49}$$

$$\begin{aligned}\Psi_2 = {} & \varepsilon_1\|\Delta v_t\|^2 - \varepsilon\|\nabla v_t\|^2 + \phi\left(\|\nabla u\|^2\right)_t\int_\Omega \Delta u \Delta v_t \mathrm{d}x - \frac{1}{2}\phi\left(\|\nabla u\|^2\right)_t \cdot \|\Delta v\|^2 \\ & + \varepsilon\phi\left(\|\nabla u\|^2\right)_t\left(\Delta u,\Delta v\right) + \varepsilon\phi\left(\|\nabla u\|^2\right)\|\Delta v\|^2 - \beta q\varepsilon\int_\Omega |u|^{q-1} v \Delta v \mathrm{d}x \\ & - \beta q\int_\Omega |u|^{q-1} v \Delta v_t \mathrm{d}x - \alpha p\int_\Omega |v|^{p-1} v_t \Delta v_t \mathrm{d}x - \alpha p\varepsilon\int_\Omega |v|^{p-1} v_t \Delta v \mathrm{d}x.\end{aligned} \tag{3.50}$$

Taking $\kappa_2 > 0$, then

$$\kappa_2\Phi_2 \le \frac{\kappa_2}{2}\|\nabla v_t\|^2 + \frac{\kappa_2\varepsilon}{2}\|\nabla v\|^2 + \frac{\kappa_2\varepsilon}{2}\|\nabla v_t\|^2 + \frac{\kappa_2\gamma_1}{2(K-2\varepsilon)}\|\Delta v\|^2.$$

At last, we get:

$$\begin{aligned}\Psi_2 - \kappa_2\Phi_2 \ge {} & \left(\varepsilon_1 - \frac{r_0}{2} - \frac{\beta q}{2} - \frac{\alpha p}{2}\right)\|\Delta v_t\|^2 - \left(\varepsilon + \frac{\varepsilon_1}{4} + \frac{\kappa_2}{2} + \frac{\kappa_2\varepsilon}{2}\right)\|\nabla v_t\|^2 \\ & + \left(\varepsilon^2\varepsilon_1 - \frac{r_0}{2} - \frac{r_0\varepsilon}{2} - \frac{\beta q\varepsilon}{2} - \frac{3r_0}{8} - \frac{\alpha p\varepsilon}{4} - \frac{\kappa_2\varepsilon_1\varepsilon}{2} - \frac{\kappa_2\gamma_1}{2(K-2\varepsilon)}\right)\|\Delta v\|^2 \\ & - \left(\frac{r_0}{4} + \frac{\kappa_2\varepsilon}{2}\right)\|\nabla v\|^2 - C.\end{aligned} \tag{3.51}$$

Let $n_1 = \varepsilon_1 - \frac{r_0}{2} - \frac{\beta q}{2} - \frac{\alpha p}{2}$; $n_2 = \varepsilon + \frac{\varepsilon_1}{4} + \frac{\kappa_2}{2} + \frac{\kappa_2\varepsilon}{2}$;

$$n_3 = \varepsilon^2\varepsilon_1 - \frac{r_0}{2} - \frac{r_0\varepsilon}{2} - \frac{\beta q\varepsilon}{2} - \frac{3r_0}{8} - \frac{\alpha p\varepsilon}{4} - \frac{\kappa_2\varepsilon_1\varepsilon}{2} - \frac{\kappa_2\gamma_1}{2(K-2\varepsilon)};$$

$$n_4 = \frac{r_0}{4} + \frac{\kappa_2 \varepsilon}{2}.$$

By using Poincaré's inequality, we get

$$\Psi_2 - \kappa_2 \Phi_2 \ge (\lambda_1 n_1 - n_2)\|\nabla v_t\|^2 + (\lambda_1 n_3 - n_4)\|v\|^2 - C. \tag{3.52}$$

We take proper ε, ε_1, γ_1, κ_2, α, r_0, α,, such that:

$$\begin{cases} \lambda_1 n_1 - n_2 \ge 0 \\ \lambda_1 n_3 - n_4 \ge 0. \end{cases}$$

Then

$$\Psi_2 - \kappa_2 \Phi_2 \ge -C. \tag{3.53}$$

From the relation (3.53), we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_2(t) + \kappa_2 \Phi_2(t) \le C, t \ge \tau_0. \tag{3.54}$$

By using Gronwall's inequality, we obtain:

$$\Phi_2(t) \le \Phi_2(\tau_0)\mathrm{e}^{-\kappa_2(t-\tau_0)} + \frac{C}{\kappa_2}\left(1 - \mathrm{e}^{-\kappa_2(t-\tau_0)}\right), t \ge \tau_0. \tag{3.55}$$

Taking $T_1 \gg \tau_0$, such that $\Phi_2(T_0)\mathrm{e}^{-\kappa_2(t-\tau_0)} \le 1$, then

$$\Phi_2(t) \le 1 + \frac{C}{\kappa_2}, \forall\, t \ge T_1. \tag{3.56}$$

where

$$\begin{aligned} \Phi_2 &= \frac{1}{2}\|\nabla v_t\|^2 + \varepsilon \int_\Omega \nabla v \nabla v_t \mathrm{d}x + \frac{\varepsilon \varepsilon_1}{2}\|\Delta v\|^2 + \frac{1}{2}\phi\left(\|\nabla u\|^2\right)\|\Delta v\|^2 \\ &\ge \frac{(1-\varepsilon)}{2}\|\nabla v_t\|^2 + \frac{(\lambda_1 \varepsilon_1 \varepsilon - \varepsilon)}{\lambda_1}\|\Delta v\|^2 \\ &\ge \frac{(1-\varepsilon)}{2}\|\nabla v_t\|^2 + \frac{(\lambda_1 \varepsilon_1 \varepsilon - \varepsilon)}{\lambda_1}\|\Delta v\|^2 - \frac{C}{\kappa_2}. \end{aligned} \tag{3.57}$$

Meanwhile, we once again take proper ε, ε_1, such that:

$$\begin{cases} 1 - \varepsilon > 0 \\ \lambda_1 \varepsilon_1 \varepsilon - \varepsilon > 0. \end{cases}$$

So there are $T_1 > 0, R_1 > 0$, which make the following inequalities:

$$\|Au(t)\| \le R_1, \left\|A^{\frac{1}{2}} u_t(t)\right\| \le R_1, \forall\, t \ge T_1. \tag{3.58}$$

where R_1 is independent of the initial value U_0.

Similar to above discussions, there are $T_2 \gg T_1$, $R_2 > 0$, which make the following inequalities:

$$\left\|A^{\frac{3}{2}} u_t(t)\right\| \le R_2, \|Au_{tt}(t)\| \le R_2, \forall\, t \ge T_2. \tag{3.59}$$

where R_2 is independent of the initial value U_0.

Using the original Equation (1.1), we obtain

$$A\left(\varepsilon_1 u_t+\phi\left(\left\|A^{\frac{1}{2}}u\right\|^2\right)u\right)$$
$$=f(x)-u_{tt}-\alpha|u_t|^{p-1}u_t-\beta|u|^{q-1}u\in C_b\left([T_2,+\infty);D(A)\right) \tag{3.60}$$

Next, using the elliptic property of the operator A, we get:

$$\left\|\varepsilon_1 u_t+\phi\left(\left\|A^{\frac{1}{2}}u\right\|^2\right)u\right\|_{D(A^2)}$$
$$\le\|Af(x)\|+\|Au_{tt}\|+\left\|A\left(\alpha|u_t|^{p-1}u_t\right)\right\|+\left\|A\left(\beta|u|^{q-1}u\right)\right\|\le R_3,\ \forall t\ge T_2 \tag{3.61}$$

where R_3 is independent of the initial value U_0.

So there are $\tau_1\gg T_2$, $K_1>0$, which make the following inequalities:

$$\|u(t)\|_{D(A^2)}\le K_1,\|u_t(t)\|_{D(A^2)}\le K_1,\ \forall t\ge\tau_1. \tag{3.62}$$

where K_1 is independent of the initial value U_0.

According to Lemmas 3.1, 3.2, we can get the following theorem :

Theorem 3.1 *From reference* [14], *let* $S(t)$ *is the semigroup operator for the problems* (1.1) - (1.3), *then the semigroup* $S(t)$ *exists a compact global attractor* $\mathcal{A}_1$ *in* $D(A^2)$, *and* $\mathcal{A}_1=\mathcal{A}_0$.

The proof of theorem 3.1 see ref. [14], is omitted here.

4. The Approximate Inertial Manifold for the Global Attractor

In this section, we first construct a smooth manifold $\mathcal{M}_1=\mathrm{graph}(\psi_0)$, and then prove that $\mathcal{M}_1$ is an approximate inertial manifold of the semigroup $S(t)$, namely, the arbitrary trajectory of the Kirchhoff wave equations goes into a small neighbourhood of the approximate inertial manifold after large time.

Let $E_N=span\{\omega_k\}_{k=1}^N$, P_N is an orthogonal projection from the space E to the subspace spanned by $E_N=span\{\omega_k\}_{k=1}^N$, $Q_N=I-P_N$, so that u is decomposed as the sum $u=p+q$.

For the solution u of the problems (1.1) - (1.3), let $p=P_Nu$, $p_t=P_Nu_t$, $q=Q_Nu$, $q_t=Q_Nu_t$. Then $\xi=(p,p_t)^{\mathrm{T}}$, $\zeta=(q,q_t)^{\mathrm{T}}$, $g(u)=|u|^{q-1}u$, $h(u_t)=|u_t|^{p-1}u_t$. We use P_N and Q_N to act the problem (1.1) respectively.

$$p_{tt}+\varepsilon_1 Ap_t+\phi\left(\left\|A^{\frac{1}{2}}u\right\|^2\right)Ap+P_N\left(\beta g(p+q)+\alpha h(p_t+q_t)\right)=P_Nf(x),\quad x\in\Omega, \tag{4.63}$$

$$q_{tt}+\varepsilon_1 Aq_t+\phi\left(\left\|A^{\frac{1}{2}}u\right\|^2\right)Aq+Q_N\left(\beta g(p+q)+\alpha h(p_t+q_t)\right)=Q_Nf(x),\quad x\in\Omega. \tag{4.64}$$

Let $\widetilde{P_N}=\begin{pmatrix}P_N & 0\\ 0 & P_N\end{pmatrix}$, $\widetilde{Q_N}=\begin{pmatrix}Q_N & 0\\ 0 & Q_N\end{pmatrix}$. Then the problems (4.63) - (4.64) can be written as:

$$\xi_t+\Lambda\xi=\widetilde{P_N}F(\xi+\zeta), \tag{4.65}$$

$$\zeta_t + \Lambda\zeta = \widetilde{Q_N} F(\xi+\zeta). \tag{4.66}$$

From above, we have $\forall U_0 \in D(A)\times E$, there exist τ_1, $K_1 > 0$, is independent of the initial value U_0, and then $U(\cdot)\in C_b([\tau_1,+\infty), D(A)\times E)$, $\|u(t)\|_{D(A^2)} \le K_1, \|u_t(t)\|_{D(A^2)} \le K_1, \forall t \ge \tau_1$. So for $q = Q_N u$, $q_t = Q_N u_t$, we obtain

$$\|q\| \le K_1\lambda_{N+1}^{-2}, \|q_t\| \le K_1\lambda_{N+1}^{-2}, \forall t \ge \tau_1. \tag{4.67}$$

Theorem 4.1 *From references* [14] [15] [16], *according to lemmas* 3.1, 3.2 *and the theorem* 3.1, *let* $\mathcal{M}_0 = \widetilde{P_N}(D(A)\times E)$ *is the N dimensional linear subspace of* $D(A)\times E$, *there exists* $\tau_1 > 0$, τ_1 *is sufficiently large. When* $t > \tau_1$, *arbitrary trajectory arising from the* U_0 *for the Kirchhoff wave equations, which track into a* $K_1\lambda_{N+1}^{-2}$ *sphere in* $\mathcal{M}_0$. *Namely,* $dist_{D(A)\times E}(S(t)U_0, \mathcal{M}_0) \le K_1\lambda_{N+1}^{-2}$. *Meanwhile, the* $\mathcal{M}_0$ *is called a N dimensional flat approximate inertial manifold of the semigroup* $S(t)$.

Remark 4.1. For the problem (4.66), if we do not consider ζ_t and ζ contained in the nonlinear terms, for $\xi \in (E_N)^2$, we define mapping $\psi_0:\ \xi \mapsto \psi_0(\xi)$. $\zeta_0 := \psi_0(\xi)$ is the solution of the Equation (4.68):

$$\Lambda\zeta_0 = \widetilde{Q_N} F(\xi). \tag{4.68}$$

Then $\psi_0:\ (E_N)^2 \to (Q_N E)^2$ is a smooth map, its image is $\mathcal{M}_1 = \text{graph}(\psi_0) = \left\{\xi + \psi_0(\xi) \mid \xi \in (E_N)^2\right\}$, which is a approximate inertial manifold of the semigroup $S(t)$.

Theorem 4.2 *From references* [14] [15] [16], *according to lemmas* 3.1, 3.2 *and the theorems* 3.1, 4.1, *then* $\forall U_0 \in D(A)\times E$, *there exists* $\tau_1 > 0$, *when* $t > \tau_1$, *arbitrary trajectory arising from the* U_0 *for the Kirchhoff wave equations, which track into a* $K_1\lambda_{N+1}^{-1}$ *neighborhood in* $\mathcal{M}_1$. *Namely,* $dist_{D(A)\times E}(S(t)U_0, \mathcal{M}_1) \le K_1\lambda_{N+1}^{-1}$. *Meanwhile, the* $\mathcal{M}_1$ *is a approximate inertial manifold of the semigroup* $S(t)$. *Furthermore,* $\forall U_0 \in D(A)\times E$, *there exists* $\tau_n > 0$, τ_n *is sufficiently large,* $n \ge 1$. *When* $t > \tau_n$, *arbitrary trajectory arising from the* U_0 *for the Kirchhoff wave equations, which track into a* $K_n\lambda_{N+1}^{-n}$ *neighborhood in* $\mathcal{M}_n$. *Namely,* $dist_{D(A)\times E}(S(t)U_0, \mathcal{M}_n) \le \widetilde{C_n} K_n\lambda_{N+1}^{-n}$. *The* $\mathcal{M}_n$ *is a very precise approximate inertial manifold of the semigroup* $S(t)$.

Proof. Firstly, let $U(t) = S(t)U_0$, then $\xi(t) = (p(t), p_t(t))^{\mathrm{T}} := \widetilde{P_N} U(t)$, $\zeta(t) = (q(t), q_t(t))^{\mathrm{T}} := \widetilde{Q_N} U(t)$ are the solutions of the problems (4.65) - (4.66), and then let $\zeta_0(t) = (q_0(t), q_{0t}(t))^{\mathrm{T}} := \psi_0(\xi(t))$. $W(t) = (\omega(t), \tilde{\omega}(t))^{\mathrm{T}}$.

From the relation (4.68), we can obtain:

$$\phi\left(\left\|A^{\frac{1}{2}}u\right\|^2\right) A q_0 = Q_N\left(f - \beta g(p) - \alpha h(p_t)\right) \tag{4.69}$$

$$q_{0t} = 0. \tag{4.70}$$

Then from the hypothesis (G_1), $\varepsilon_1\varepsilon \le \phi\left(\left\|A^{\frac{1}{2}}u\right\|^2\right)$.

$$\zeta_0 = \begin{pmatrix} q_0 \\ q_{0t} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\phi\left(\left\|A^{\frac{1}{2}}u\right\|^2\right)} A^{-1} Q_N\left(f - \beta g(p) - \alpha h(p_t)\right) \\ 0 \end{pmatrix}. \tag{4.71}$$

$$W(t)=(\omega(t),\tilde{\omega}(t))^{\mathrm{T}}=U(t)-(\xi(t)+\zeta_0(t))=\zeta(t)-\zeta_0(t). \tag{4.72}$$

$$dist_{D(A)\times E}(S(t)U_0,\mathcal{M}_1)\le\|W(t)\|_{D(A)\times E}. \tag{4.73}$$

We put $W(t)$ into the relation (4.68), the following relations can be obtained immediately,

$$\begin{aligned}\phi\left(\left\|A^{\frac{1}{2}}u\right\|^2\right)A\omega&=\phi\left(\left\|A^{\frac{1}{2}}u\right\|^2\right)Aq-\phi\left(\left\|A^{\frac{1}{2}}u\right\|^2\right)Aq_0\\&=Q_N\left[(\beta g(p)+\alpha h(p_t))-(\beta g(p+q)+\alpha h(p_t+q_t))\right]-q_{tt}-\varepsilon_1 Aq_t\end{aligned} \tag{4.74}$$

$$\tilde{\omega}=q_t. \tag{4.75}$$

Therefore

$$\begin{aligned}\phi\left(\left\|A^{\frac{1}{2}}u\right\|^2\right)\left\|A^2\omega\right\|&\le\left\|AQ_N\left[\beta g(p)-\beta g(p+q)\right]\right\|\\&\quad+\left\|AQ_N\left[\alpha h(p_t)-\alpha h(p_t+q_t)\right]\right\|+\left\|Aq_{tt}\right\|+\varepsilon_1\left\|A^2q_t\right\|\\&\le C_1(\varepsilon,\alpha,\beta,\varepsilon_1)K_1,\ t\ge\tau_1.\end{aligned} \tag{4.76}$$

$$\|A\tilde{\omega}\|\le K_1,\ t\ge\tau_1. \tag{4.77}$$

Then

$$\|A\omega\|\le C_1K_1\lambda_{N+1}^{-1},\ \|\tilde{\omega}\|\le K_1\lambda_{N+1}^{-1},\ t\ge\tau_1. \tag{4.78}$$

So, we obtain

$$dist_{D(A)\times E}(S(t)U_0,\mathcal{M}_1)\le\|A\omega\|+\|\tilde{\omega}\|\le(C_1+1)K_1\lambda_{N+1}^{-1}:=\widetilde{C_1}K_1\lambda_{N+1}^{-1},\ t\ge\tau_1. \tag{4.79}$$

A similar method in reference [14], we immediately get the semigroup $S(t)$ exists a compact global attractor $\mathcal{A}_n$ in $D(A^n)$, and $\mathcal{A}_0=\mathcal{A}_1=\cdots=\mathcal{A}_n$, and then $\forall U_0\in D(A)\times E$, there exists $\tau_n>0$, τ_n is sufficiently large, $n\ge1$. When $t>\tau_n$, arbitrary trajectory arising from the U_0 for the Kirchhoff wave equations, which track into a $K_n\lambda_{N+1}^{-n}$ neighborhood in $\mathcal{M}_n$.

$$dist_{D(A)\times E}(S(t)U_0,\mathcal{M}_n)\le\widetilde{C_n}K_n\lambda_{N+1}^{-n}. \tag{4.80}$$

where the $\mathcal{M}_n$ is a smooth manifold that we construct, which is very precise, to approximate inertial manifold of the semigroup $S(t)$.

Remark 4.2. This article is based on the references [14] [15] [16], by estimating the higher regularity of the global attractor, then we construct its approximate inertial manifold. Approximate inertial manifold, which is a kind of nonlinear, finite dimensional and has certain smoothness. It is of great significance to study the long time behavior of the dissipative equations and the structure of the attractors. On the basis of this article, then we are likely to consider the inertial manifold of the global attractor for the problems (1.1) - (1.3).

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