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Result on the Convergence Behavior of Solutions of Certain System of Third-Order Nonlinear Differential Equations

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Abstract

Convergence behaviors of solutions arising from certain system of third-order nonlinear differential equations are studied. Such convergence of solutions corresponding to extreme stability of solutions when $P \neq 0$ relates a pair of solutions of the system considered. Using suitable Lyapunov functionals, we prove that the solutions of the nonlinear differential equation are convergent. Result obtained generalizes and improves some known results in the literature. Example is included to illustrate the result.

Keywords

Nonlinear Differential Equations, Third Order, Convergence of Solutions, Lyapunov Method

1. Introduction

We shall consider here systems of real differential equations of the form

$$\dddot{X} + \Psi\left(X, \dot{X}\right)\ddot{X} + \Phi\left(\dot{X}\right) + H\left(X\right) = P\left(t, X, \dot{X}, \ddot{X}\right) \tag{1}$$

which is equivalent to the system

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= Z \\ \dot{Z} &= -\Psi\left(X, Y\right)Z - \Phi\left(Y\right) - H\left(X\right) + P\left(t, X, Y, Z\right) \end{aligned} \tag{2}$$

where Φ and H are continuous vector functions and Ψ is an $n \times n$-positive definite continuous symmetric

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matrix function, for the argument displayed explicitly and the dots here as elsewhere stand for differentiation with respect to the independent variable *t*, $t \in \mathbb{R}^+$; $\mathbb{R}^+$ denote the real interval $0 \le t < \infty$. $X \in \mathbb{R}^n$ and $P : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ in Equation (1). $J\Phi(Y)$, $JH(X)$ are the Jacobian matrices corresponding to the vector functions $\Phi(Y)$ and $H(X)$ respectively exist and are symmetric, positive definite and continuous.

So far in the literature, much attention has been drawn to the boundedness of solutions of ordinary scalar and vector nonlinear differential equations of third order. The book of Reissig *et al.* [1], the papers by Abou-El-Ela [2], Afuwape [3] [4], Chukwu [5], Ezeilo [6], Ezeilo and Tejumola [7], Meng [8], Omeike [9], Omeike and Afuwape [10], Tiryaki [11], Tunc [12] [13], Tunc and Ates [14], Tunc and Mohammed [15] and the references cited therein have comprehensive treatment of the subject. Throughout the results present in the book of Reissig *et al.* [1] and the papers mentioned above, Lyapunov's second (direct) method has been used as a basic tool to verify the results established in these works. Equations of the form (1) in which $\Psi(X, \dot{X}) = A$ and $\Phi(\dot{X}) = G(\dot{X})$ have been studied by [16] [17]. They have obtained some results related to the convergence properties of solutions as well as Afuwape in [18]. Very recently, Tunc and Gozen [19] studied the convergence of solution of the equation

$$\dddot{X} + F(\ddot{X}) + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X})$$

by extending the result of [17] to the special case $F(\ddot{X}) = A\ddot{X}$ of [17]. Also recently, Olutimo [20] studied the equation

$$\dddot{X} + \Psi(X, \dot{X})\ddot{X} + \Phi(\dot{X}) + cX = P(t, X, \dot{X}, \ddot{X})$$

a variant of (1), where *c* is a positive constant and obtained some results which guarantee the convergence of the solutions. With respect to our observation in the literature, no work based on (1) was found. The result to be obtained here is different from that in Olutimo [20] and the papers mentioned above. The intuitive idea of convergence of solutions also known as the extreme stability of solutions occurs when the difference between two equilibrium positions tends to zero as time increases infinitely is of practical importance. This intuitive idea is also applicable to nonlinear differential system. The Lyapunov's second method allows us to predict the convergence property of solutions of nonlinear physical system. Result obtained generalizes and improves some known results in the literature. Example is included to illustrate the result.

Definition

Definition 1.1. *Any two solutions* $X_1(t)$, $X_2(t)$ *of* (1) *are said to converge if*

$$\|X_2(t) - X_1(t)\| \to 0, \ \|\dot{X}_2(t) - \dot{X}_1(t)\| \to 0 \ \text{and} \ \|\ddot{X}_2(t) - \ddot{X}_1(t)\| \to 0, \ \text{as } t \to \infty.$$

If the relations above are true of each other (arbitrary) pair of solutions of (1), we shall describe this saying that all solutions of (1) converge.

2. Some Preliminary Results

We shall state for completeness, some standard results needed in the proofs of our results.

Lemma 1. *Let D be a real symmetric* $n \times n$ *matrices. Then for any* $X \in \mathbb{R}^n$.

$$\delta_d \|X\|^2 \le \langle DX, X \rangle \le \Delta_d \|X\|$$

where δ_d and Δ_d are the least and greatest eigenvalues of *D*, respectively.

Proof of Lemma 1. See [3] [7].

Lemma 2. *Let* Q, D *be real symmetric commuting* $n \times n$ *matrices. Then,*

1) The eigenvalues $\lambda_i(QD), (i = 1, 2, \cdots, n)$ of the product matrix QD are all real and satisfy

$$\min_{1 \le j,k \le n} \lambda_j(Q)\lambda_k(D) \le \lambda_i(QD) \le \max_{1 \le j,k \le n} \lambda_j(Q)\lambda_k(D).$$

2) The eigenvalues $\lambda_i(Q + D), (i = 1, 2, \cdots, n)$ of the sum of *Q* and *D* are all real and satisfy

$$\left\{\min_{1\le j,k\le n}\lambda_j(Q)+\min_{1\le j,k\le n}\lambda_k(D)\right\}\le\lambda_i(Q+D)\le\left\{\max_{1\le j,k\le n}\lambda_j(Q)+\max_{1\le j,k\le n}\lambda_k(D)\right\}$$

where $\lambda_j(Q)$ and $\lambda_k(D)$ are respectively the eigenvalues of *Q* and *D*.

Proof of Lemma 2. See [3] [7].

Lemma 3. *Subject to earlier conditions on* $\Psi(X,Y)$ *the following is true*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\langle\tau\Psi(\tau X,Y)Y,Y\rangle\mathrm{d}\tau=\langle\Psi(X,Y)Y,Z\rangle$$

where δ_d and Δ_d are the least and greatest eigenvalues of *D*, respectively.

Proof of Lemma 3. See [20].

Lemma 4. *Subject to earlier conditions on* Φ *and that* $\Phi(0)=0$, *then*

1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\langle\Phi(\tau Y),Y\rangle\mathrm{d}\tau=\langle\Phi(Y),Z\rangle.$$

2)

$$\langle\Phi(\tau Y),Y\rangle=\int_0^1\langle\tau J(\Phi(\tau Y),Y)\rangle\mathrm{d}\tau.$$

Proof of Lemma 4. See [20].

Lemma 5. *Subject to earlier conditions on* $H(X)$ *and that* $H(0)=0$, *then*

1)

$$\delta_h\|X\|^2\le 2\int_0^1\langle H(\tau X),X\rangle\mathrm{d}\tau\le\Delta_h\|X\|^2.$$

2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\langle H(\tau X),X\rangle\mathrm{d}\tau=\langle H(X),Y\rangle.$$

Proof of Lemma 5. See [3] [7] [11].

3. Statement of Results

Throughout the sequel $JH(X),J\Phi(Y)$ are the Jacobian matrices $\left(\frac{\partial h_i}{\partial x_j}\right),\left(\frac{\partial y_i}{\partial y_j}\right)$ corresponding to the vector functions $H(X),\Phi(Y)$, respectively.

Our main result which gives an estimate for the solutions of (1) is the following:

Theorem 1. *Assume that* $\Psi(X,Y),JH(X)$ *and* $J\Phi(Y)$, *for all* X,Y *in* $\mathbb{R}^n$ *are all symmetric. Jacobian matrices* $JH(X),J\Phi(Y)$ *exist, positive definite and continuous. Furthermore, there are positive constants* $\alpha_o,\beta_o,\gamma_o,\alpha_1,\beta_1,\gamma_1$ *such that the following conditions are satisfied.*

Suppose that $\Phi(0)=0=H(0)$ and that

1) The $n\times n$ continuous matrices $\Psi(X,Y)$, $JH(X)$ and $J\Phi(Y)$ are symmetric, associative and commute pairwise. Then eigenvalues $\lambda_i(\Psi(X,Y))$ of Ψ, $\lambda_i(J\Phi(Y))$ of $J\Phi(Y)$ and $\lambda_i(JH(X))$ of $JH(X)$ $(i=1,2)$, satisfy

$$\alpha_o\le\lambda_i(\Psi(X,Y))\le\alpha_1$$

$$\beta_o\le\lambda_i(J\Phi(Y))\le\beta_1$$

$$\gamma_o\le\lambda_i(JH(X))\le\gamma_1.$$

2) *P* satisfies

$$\|P(t,X_2,Y_2,Z_2)-P(t,X_1,Y_1,Z_1)\|\le\Delta_o\left\{\|X_2-X_1\|^2+\|Y_2-Y_1\|^2+\|Z_2-Z_1\|^2\right\}^{\frac{1}{2}}\tag{3}$$

for any X_i, Y_i, Z_i $(i = 1, 2)$ in $\mathbb{R}^n$, and Δ_o is a finite constant. Then, there exists a finite constant $\epsilon > 0$ such that any two solutions $X_1(t), X_2(t)$ of (2) necessarily converge if $\Delta_o < \epsilon$.

Our main tool in the proof of the result is the function $V = V(X,Y,Z)$ defined for any X,Y,Z in $\mathbb{R}^n$ by

$$2V = 2V_1 + 2V_2 \tag{4}$$

where

$$2V_1 = 2\int_0^1 \langle H(\tau X), X\rangle \mathrm{d}\tau + 2\int_0^1 \langle \tau\Psi(X,\tau Y)Y,Y\rangle \mathrm{d}\tau + 2\delta\int_0^1 \langle \Phi(\tau Y),Y\rangle \mathrm{d}\sigma + \delta\langle Z,Z\rangle + 2\langle Y,Z\rangle + 2\delta\langle Y,H(X)\rangle,$$

$$2V_2 = 2\alpha_o\int_0^1 \langle H(\tau X), X\rangle \mathrm{d}\tau + 2\alpha_o\int_0^1 \langle \tau\Psi(X,\tau Y)Y,Y\rangle \mathrm{d}\tau + \mu\alpha_o\beta_o^2\langle X,X\rangle + 2\int_0^1 \langle \Phi(\tau Y,Y)\rangle \mathrm{d}\tau + \langle Z,Z\rangle + 2\mu\alpha_o^2\beta_o\langle X,Y\rangle + +2\mu\alpha_o\beta_o\langle X,Z\rangle + 2\alpha_o\langle Y,Z\rangle + 2\langle Y,H(X)\rangle - \mu\alpha_o\beta_o\langle Y,Y\rangle$$

and $\delta > 0$ is a fixed constant chosen such that

$$\frac{1}{\alpha_o} < \delta < \frac{\beta_o}{\gamma_1}, \tag{5}$$

$$\mu < \min\left\{\frac{1}{\alpha_o}, \frac{\alpha_o}{\beta_o}, \frac{(\alpha_o\beta_o - \gamma_1) + (\beta_o - \delta\gamma_1)}{\alpha_o\beta_o\left\{\alpha_o + \gamma_o^{-1}(\beta_1 - \beta_o)^2\right\}}, \frac{\alpha_o\delta - 1}{\alpha_o\beta_o\gamma_o^{-1}(\alpha_1 - \alpha_o)^2}\right\}, \tag{6}$$

μ chosen such that $0 < \mu < 1$.

The following result is immediate from (4).

Lemma 6. *Assume that, all the hypotheses on matrix* $\Psi(X,Y)$ *and vectors* $\Phi(Y)$ *and* $H(X)$ *in Theorem* 1 *are satisfied. Then there exist positive constants* D_1 *and* D_2 *such that*

$$D_1\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right) \le 2V \le D_2\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right). \tag{7}$$

Proof of Lemma 6. In the proof of the lemma, the main tool is the function $V = V(X,Y,Z)$ in (4). This function, after re-arrangement, can be re-written as

$$2V_1 = 2\int_0^1 \langle H(\tau X), X\rangle \mathrm{d}\tau + \delta\left\|Z + \delta^{-1}Y\right\|^2 + 2\left\{\int_0^1 \langle \tau\Psi(X,\tau Y)Y,Y\rangle \mathrm{d}\tau - \delta^{-1}\langle Y,Y\rangle\right\} + \delta\int_0^1 \langle \Phi(\tau Y),Y\rangle \mathrm{d}\tau + 2\delta\langle Y,H(X)\rangle.$$

Since

$$\int_0^1 \langle \Phi(\tau Y),Y\rangle \mathrm{d}\tau = \int_0^1\int_0^1 \tau_1\langle J\Phi(\tau_1\tau_2 Y)Y,Y\rangle \mathrm{d}\tau_1\mathrm{d}\tau_2.$$

And

$$2\langle Y,H(X)\rangle = \int_0^1\int_0^1 \tau_1\langle Y,H(X)\rangle \mathrm{d}\tau_1\mathrm{d}\tau_2$$

we have that

$$\int_0^1 \langle \Phi(\tau Y),Y\rangle \mathrm{d}\tau + 2\langle Y,H(X)\rangle = \int_0^1\int_0^1 \tau_1\left\{\langle J\Phi(\tau_1\tau_2 Y)Y,Y\rangle + 2\langle Y,H(X)\rangle\right\}\mathrm{d}\tau_1\mathrm{d}\tau_2.$$

Since matrix $J\Phi$ is assumed symmetric and strictly positive definite. Consequently the square root $J\Phi^{\frac{1}{2}}$ exists which itself is symmetric and non-singular for all $Y \in \mathbb{R}^n$. Therefore, we have

$$\langle J\Phi Y,Y\rangle+2\langle Y,H(X)\rangle=\left\|J\Phi^{-\frac{1}{2}}Y+J\Phi^{-\frac{1}{2}}H(X)\right\|^2-\left\|J\Phi^{-\frac{1}{2}}H(X)\right\|^2 \tag{8}$$

where $J\Phi$ stands for $J\Phi(\tau_1\tau_2Y)$.

Thus,

$$\begin{aligned}2V_1&=2\int_0^1\langle H(\tau X),X\rangle \mathrm{d}\tau-\delta\left\|J\Phi^{\frac{1}{2}}H(X)\right\|^2+\delta\left\|Z+\delta^{-1}Y\right\|^2\\&+\int_0^1\left\langle \tau\left\{\Psi(X,\tau Y)-\delta^{-1}I\right\}Y,Y\right\rangle \mathrm{d}\tau+\int_0^1\int_0^1\delta\tau_1\left\|J\Phi^{\frac{1}{2}}Y+J\Phi^{-\frac{1}{2}}H(X)\right\|^2\mathrm{d}\tau_1\tau_2.\end{aligned} \tag{9}$$

From (9), the term

$$2\int_0^1\langle H(\tau X),X\rangle \mathrm{d}\tau-\delta\langle J\Phi^{-1}H(X),H(X)\rangle. \tag{10}$$

Since

$$\frac{\partial}{\partial\tau_3}\langle J\Phi^{-1}H(X),H(X)\rangle=2\langle J\Phi^{-1}JH(\tau_3X)X,H(\tau_3X)\rangle$$

by integrating both sides from $\tau_3=0$ to $\tau_3=1$ and because $H(0)=0$, then we obtain

$$\begin{aligned}\langle J\Phi^{-1}H(\tau_3X),H(\tau_3X)\rangle&=2\int_0^1\langle J\Phi^{-1}(\tau_1\tau_2Y)JH(\tau_3X)X,H(\tau_3X)\rangle\\&=2\int_0^1\int_0^1\int_0^1\tau_1\left\langle H(\tau_3X),\left\{I-J\Phi^{-1}(\tau_1\tau_2Y)JH(\tau_3X)\right\}X\right\rangle \mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3.\end{aligned}$$

But from

$$\begin{aligned}&\frac{\partial}{\partial\tau_4}\left\langle H(X\tau_3\tau_4X)X,\left\{I-J\Phi^{-1}(\tau_1\tau_2Y)JH(\tau_3X)\right\}X\right\rangle\\&=\left\langle \tau_3JH(X\tau_3\tau_4X)X,\left\{I-J\Phi^{-1}(\tau_1\tau_2Y)JH(\tau_3X)\right\}X\right\rangle\end{aligned}$$

integrating both sides from $\tau_4=0$ to $\tau_4=1$ and because $H(0)=0$, we find

$$\begin{aligned}&\left\langle H(X\tau_3X),\left\{I-J\Phi^{-1}(\tau_1\tau_2Y)JH(\tau_3X)\right\}X\right\rangle\\&=\int_0^1\tau_3\left\langle JH(X\tau_3\tau_4X)X,\left\{I-J\Phi^{-1}(\tau_1\tau_2Y)JH(\tau_3X)\right\}X\right\rangle \mathrm{d}\tau_4.\end{aligned}$$

Hence, (10) becomes

$$2\int_0^1\int_0^1\int_0^1\int_0^1\tau_1\tau_3\left\langle JH(X\tau_3\tau_4X)\left\{I-\delta J\Phi^{-1}(\tau_1\tau_2Y)JH(\tau_3X)\right\}X,X\right\rangle \mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3\mathrm{d}\tau_4$$

combining the estimate for V_1 in (9), we have

$$\begin{aligned}2V_1&=2\int_0^1\int_0^1\int_0^1\int_0^1\tau_1\tau_3\left\langle JH(X\tau_3\tau_4X)\left\{I-\delta J\Phi^{-1}(\tau_1\tau_2Y)JH(\tau_3X)\right\}X,X\right\rangle \mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3\mathrm{d}\tau_4\\&+2\int_0^1\left\langle \tau\left\{\Psi(X,\tau Y)-\delta^{-1}I\right\}Y,Y\right\rangle \mathrm{d}\tau+\delta\left\|Z+\delta^{-1}Y\right\|^2.\end{aligned}$$

By hypothesis (1) of Theorem 1 and lemmas 1 and 2, we have

$$2V_1\geq 2\gamma_o\left(1-\delta\gamma_1\beta_0^{-1}\right)\|X\|^2+2\left(\alpha_o-\delta^{-1}\right)\|Y\|^2+\delta\left\|Z+\delta^{-1}Y\right\|^2,$$

where $1-\delta\gamma_1\beta_0^{-1}>0$ and $\alpha_o-\delta^{-1}>0$ by (5).

Similarly, V_2 after re-arrangement becomes

$$\begin{aligned}2V_2 &= \|Z+\alpha_o Y+\mu\alpha_o\beta_o X\|^2+2\alpha_o\int_0^1\langle\tau\Psi(X,\tau Y)Y,Y\rangle\mathrm{d}\tau-2\alpha_o^2\langle Y,Y\rangle\\ &+2\int_0^1\langle\Phi(\tau Y),Y\rangle\mathrm{d}\tau-\beta_o\langle Y,Y\rangle+\mu\alpha_o\beta_o^2(1-\mu\alpha_o)\langle X,X\rangle\\ &+2\alpha_o\int_0^1\langle H(\tau X),X\rangle\mathrm{d}\tau-\beta_o^{-1}\|H(X)\|^2+\alpha_o(\alpha_o-\mu\beta_o)\langle Y,Y\rangle\\ &+\beta_o\|Y+\beta_o^{-1}H(X)\|^2.\end{aligned} \tag{11}$$

It is obvious that

$$\begin{aligned}&2\alpha_o\int_0^1\langle\tau\Psi(X,\tau Y)Y,Y\rangle\mathrm{d}\tau-2\alpha_o^2\langle Y,Y\rangle\\ &=2\alpha_o\int_0^1\tau\langle\{\Psi(X,\tau Y)-\alpha_o I\}Y,Y\rangle\mathrm{d}\tau\ge 0,\end{aligned}$$

also,

$$\begin{aligned}&2\int_0^1\langle\Phi(\tau Y),Y\rangle\mathrm{d}\tau-\beta_o\langle Y,Y\rangle\\ &=2\alpha_o\int_0^1\int_0^1\tau_1\langle\{J\Phi(\tau_1\tau_2 Y)-\beta_o I\}Y,Y\rangle\mathrm{d}\tau_1\mathrm{d}\tau_2\ge 0\end{aligned}$$

and

$$\begin{aligned}&2\alpha_o\int_0^1\langle H(\tau X),X\rangle\mathrm{d}\tau-\beta_o^{-1}\|H(X)\|^2\\ &=2\alpha_o\int_0^1\langle H(\tau X),X\rangle\mathrm{d}\tau-2\beta_o^{-1}\int_0^1\langle JH(\tau X)H(\tau X),X\rangle\mathrm{d}\tau\\ &=2\int_0^1\int_0^1\{\alpha_o-\beta_o^{-1}JH(\tau_1 X)\}JH(\tau_1\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2\\ &\ge 2(\alpha_o-\beta_o^{-1}\gamma_1)\gamma_o\|X\|^2.\end{aligned}$$

Combining all the estimates of V_2 and (11), we have

$$\begin{aligned}2V_2 &\ge \|Z+\alpha_o Y+\mu\alpha_o\beta_o X\|^2+2(\alpha_o-\beta_o^{-1}\gamma_1)\gamma_o\|X\|^2\\ &+\mu\alpha_o\beta_o^2(1-\mu\alpha_o)\|X\|^2+\alpha_o(\alpha_o-\mu\beta_o)\|Y\|^2.\end{aligned}$$

Now, combining V_1 and V_2 we must have

$$V=V_1+V_2,$$

that is,

$$\begin{aligned}V &\ge \{\gamma_o(1-\delta\gamma_1\beta_o^{-1})+2(\alpha_o-\beta_o^{-1}\gamma_1)\gamma_o+\mu\alpha_o\beta_o^2(1-\mu\alpha_o)\}\|X\|^2\\ &+\{2(\alpha_o-\delta^{-1})+\alpha_o(\alpha_o-\mu\beta_o)\}\|Y\|^2+\delta\|Z+\delta^{-1}Y\|^2.\end{aligned} \tag{12}$$

Thus, it is evident from the terms contained in (12) that there exists sufficiently small positive constants D_3 such that

$$V\ge D_3\left(\|X\|^2+\|Y\|^2+\|Z\|^2\right),$$

where

$$\begin{aligned}D_3=\min\{&\gamma_o(1-\delta\gamma_1\beta_o^{-1})+2(\alpha_o-\beta_o^{-1}\gamma_1)\gamma_o+\mu\alpha_o\beta_o^2(1-\mu\alpha_o);\\ &2(\alpha_o-\delta^{-1})+\alpha_o(\alpha_o-\mu\beta_o)\}.\end{aligned}$$

The right half inequality in lemma 6 follows from lemma 1 and 2.
Thus,

$$V \le D_4\left(\|X\|^2+\|Y\|^2+\|Z\|^2\right),$$

where

$$D_4 = \max\{2\gamma_1+\delta\gamma_1+\mu\alpha_o^2\beta_o+2\alpha_o\gamma_1+\mu\alpha_o^2\beta_o+\mu\alpha_o\beta_o+\gamma_1;$$
$$\alpha_1+2\delta\beta_1+2\delta\gamma_1+\alpha_o\alpha_1+2\beta_1+\mu\alpha_o^2\beta_o+\alpha_o+\gamma_1-\mu\alpha_o\beta_o;$$
$$\delta+2+\mu\alpha_o\beta_o+\alpha_o\}.$$

Hence,

$$D_3\left(\|X\|^2+\|Y\|^2+\|Z\|^2\right) \le V \le D_4\left(\|X\|^2+\|Y\|^2+\|Z\|^2\right). \tag{13}$$

4. Proof of Theorem 1

Let $X_1(t)$, $X_2(t)$ be any two solutions of (2), we define

$$W = W(t).$$

By

$$W(t) = V\left(X_2(t)-X_1(t), Y_2(t)-Y_1(t), Z_2(t)-Z_1(t)\right),$$

where V is the function defined in (4) with X, Y, Z replaced by $(X_2-X_1), (Y_2-Y_1), (Z_2-Z_1)$ respectively.

By lemma 6, (13) becomes

$$D_3\left(\|X_2-X_1\|^2+\|Y_2-Y_1\|^2+\|Z_2-Z_1\|^2\right) \le W \le D_4\left(\|X_2-X_1\|^2+\|Y_2-Y_1\|^2+\|Z_2-Z_1\|^2\right) \tag{14}$$

for $D_3 > 0$ and $D_4 > 0$.

The derivative of $W(t)$ with respect to t along the solution path and using Lemma 3, 4 and 5, after simplification yields

$$\begin{aligned}
\dot{W} = &-\frac{1}{2}\mu\alpha_o\beta_o\int_0^1\left\langle (X_2-X_1), JH\tau(X_2-X_1)(X_2-X_1)\right\rangle \mathrm{d}\tau \\
&-\left\langle\left\{(1+\alpha_o)J\Phi-(1+\delta)JH-\mu\alpha_o^2\beta_o I\right\}(Y_2-Y_1),(Y_2-Y_1)\right\rangle \\
&-\left\langle\left\{(1+\delta)\Psi-(1+\alpha_o)I\right\}(Z_2-Z_1),(Z_2-Z_1)\right\rangle \\
&-\frac{1}{4}\mu\alpha_o\beta_o\int_0^1\left\{\left\langle JH(X_2-X_1),(X_2-X_1)\right\rangle+4\left\langle\left[\Psi-\alpha_o I\right](X_2-X_1),(Z_2-Z_1)\right\rangle\right\}\mathrm{d}\tau \\
&-\frac{1}{4}\mu\alpha_o\beta_o\int_0^1\left\{\left\langle JH(X_2-X_1),(X_2-X_1)\right\rangle+4\left\langle\left[J\Phi-\beta_o I\right](X_2-X_1),(Y_2-Y_1)\right\rangle\right\}\mathrm{d}\tau \\
&+\left\langle \mu\alpha_o\beta_o(X_2-X_1)+(1+\alpha_o)(Y_2-Y_1)+(1+\delta)(Z_2-Z_1),\theta\right\rangle
\end{aligned}$$

where $\theta = P(t,X_2,Y_2,Z_2)-P(t,X_1,Y_1,Z_1)$, $JH = JH\tau(X_2-X_1)$, $J\Phi = J\Phi(Y_2-Y_1)$ and $\Psi = \Psi\left[(X_2-X_1),(Y_2-Y_1)\right]$.

Using the fact that

$$\Phi(Y_2)-\Phi(Y_1) = \int_0^1 J\Phi(\eta)(Y_2-Y_1)\mathrm{d}u$$

and

$$H(X_2)-H(X_1) = \int_0^1 JH(\xi)(X_2-X_1)\mathrm{d}v$$

where

$$\eta = uY_2 + (1-u)Y_1,\ \ 0 \le u \le 1$$
$$\text{and}\ \ \xi = vX_2 + (1-v)X_1,\ \ 0 \le v \le 1.$$

Following (8),

$$\left\langle JH(X_2 - X_1),(X_2 - X_1)\right\rangle + 4\left\langle [\Psi - \alpha_o I](X_2 - X_1),(Z_2 - Z_1)\right\rangle$$
$$= \left\| JH^{-\frac{1}{2}}(X_2 - X_1) + 2JH^{-\frac{1}{2}}[\Psi - \alpha_o I](Z_2 - Z_1)\right\|^2 - \left\| 2[\Psi - \alpha_o I]JH^{-\frac{1}{2}}(Z_2 - Z_1)\right\|^2$$

and

$$\left\langle JH(X_2 - X_1),(X_2 - X_1)\right\rangle + 4\left\langle [J\Phi - \beta_o I](X_2 - X_1)(Y_2 - Y_1)\right\rangle$$
$$= \left\| JH^{-\frac{1}{2}}(X_2 - X_1) + 2JH^{-\frac{1}{2}}[J\Phi - \beta_o I](Y_2 - Y_1)\right\|^2 - \left\| 2[J\Phi - \beta_o I]JH^{-\frac{1}{2}}(Y_2 - Y_1)\right\|^2.$$

Thus,

$$\begin{aligned}\dot{W} = &-\frac{1}{2}\mu\alpha_o\beta_o\int_0^1\left\langle (X_2 - X_1), JH\tau(X_2 - X_1)(X_2 - X_1)\right\rangle \mathrm{d}\tau \\ &-\left\langle \left\{(1+\alpha_o)J\Phi - (1+\delta)JH - \mu\alpha_o^2\beta_o I\right\}(Y_2 - Y_1),(Y_2 - Y_1)\right\rangle \\ &-\left\langle \left\{(1+\delta)\Psi - (1+\alpha_o)I\right\}(Z_2 - Z_1),(Z_2 - Z_1)\right\rangle \\ &-\frac{1}{4}\mu\alpha_o\beta_o\int_0^1\left\| 2[\Psi - \alpha_o I]JH^{-\frac{1}{2}}(Z_2 - Z_1)^2\right\| \mathrm{d}\tau \\ &-\frac{1}{4}\mu\alpha_o\beta_o\int_0^1\left\| 2[J\Phi - \beta_o I]JH^{-\frac{1}{2}}(Y_2 - Y_1)\right\|^2 \mathrm{d}\tau\mathrm{d}\tau \\ &+\left\langle \mu\alpha_o\beta_o(X_2 - X_1) + (1+\alpha_o)(Y_2 - Y_1) + (1+\delta)(Z_2 - Z_1),\theta\right\rangle.\end{aligned}$$

Note that

$$\int_0^1\left\| 2[\Psi - \alpha_o I]JH^{-\frac{1}{2}}(Z_2 - Z_1)\right\|^2 \mathrm{d}\tau$$
$$= 4\int_0^1\left\langle JH^{-1}[\Psi - \alpha_o I](Z_2 - Z_1),[\Psi - \alpha_o I](Z_2 - Z_1)\right\rangle \mathrm{d}\tau$$

and

$$\int_0^1\left\| 2[J\Phi - \beta_o I]JH^{-\frac{1}{2}}(Y_2 - Y_1)\right\|^2 \mathrm{d}\tau$$
$$= 4\int_0^1\left\langle JH^{-1}[J\Phi - \beta_o I](Y_2 - Y_1),[J\Phi - \beta_o I](Y_2 - Y_1)\right\rangle \mathrm{d}\tau.$$

We have;

$$\begin{aligned}\dot{W} \le &-\frac{1}{2}\mu\alpha_o\beta_o\int_0^1\left\langle (X_2 - X_1), JH\tau(X_2 - X_1)(X_2 - X_1)\right\rangle \mathrm{d}\tau \\ &-\int_0^1\left\langle \left\{(1+\alpha_o)J\Phi - (1+\delta)JH + \mu\alpha_o^2\beta_o I + \mu\alpha_o^2\beta_o JH^{-1}[J\Phi - \beta_o I]^2(Y_2 - Y_1),(Y_2 - Y_1)\right\}\right\rangle \mathrm{d}\tau \\ &-\int_0^1\left\langle \left\{(1+\delta)\Psi - (1+\alpha_o)I - \mu\alpha_o\beta_o J_H^{-1}[\Psi - \alpha_o I]^2(Z_2 - Z_1),(Z_2 - Z_1)\right\}\right\rangle \mathrm{d}\tau \\ &+\left\langle \mu\alpha_o^2\beta_o(X_2 - X_1) + (1+\alpha_o)(Y_2 - Y_1) + (1+\delta)(Z_2 - Z_1),\theta\right\rangle.\end{aligned}$$

On applying Lemma 1 and 2, we have

$$
\begin{aligned}
\dot{W} \le & -\frac{1}{2}\mu\alpha_o\beta_o\gamma_o\|X_2-X_1\|^2 \\
& -\left\{(1+\alpha_o)\beta_o-(1+\delta)\gamma_1-\mu\alpha_o^2\beta_o-\mu\alpha_o\beta_o\gamma_o^{-1}[\beta_1-\beta_o]^2\right\}\|Y_2-Y_1\|^2 \\
& -\left\{(1+\delta)\alpha_o-(1+\alpha_o)-\mu\alpha_o\beta_o\gamma_o^{-1}[\alpha_1-\alpha_o]^2\right\}\|Z_2-Z_1\|^2 \\
& +\left\{\mu\alpha_o\beta_o\|X_2-X_1\|+(1+\alpha_o)\|Y_2-Y_1\|+(1+\delta)\|Z_2-Z_1\|\right\}\|\theta\|.
\end{aligned}
$$

If we choose μ, such that it satisfies (6), and using (3), we obtain

$$
\begin{aligned}
\dot{W} \le & -\delta_1\|X_2-X_1\|^2-\delta_2\|Y_2-Y_1\|^2+\delta_3\|Z_2-Z_1\|^2 \\
& +3^{\frac{1}{2}}\delta_4\left\{\|X_2-X_1\|^2+\|Y_2-Y_1\|^2+\|Z_2-Z_1\|^2\right\}^{\frac{1}{2}} \\
& \cdot\left\{\Delta_o\|X_2-X_1\|^2+\|Y_2-Y_1\|^2+\|Z_2-Z_1\|^2\right\}^{\frac{1}{2}},
\end{aligned}
$$

where

$$\delta_1=\frac{1}{2}\mu\alpha_o\beta_o\gamma_o,$$

$$\delta_2=\alpha_o\beta_o-\gamma_1+\beta_o-\delta\gamma_1-\mu\alpha_o\beta_o\left\{\alpha_o+\gamma_o^{-1}[\beta_1-\beta_o]^2\right\},$$

$$\delta_3=\alpha_o\delta-1-\mu\alpha_o\beta_o\gamma_o^{-1}[\alpha_1-\alpha_o]^2,$$

$$\delta_4=\max\left\{\mu\alpha_o\beta_o;1+\alpha_o;1+\delta\right\}.$$

Thus,

$$\dot{W}(t)\le-\left(\delta_5-3^{\frac{1}{2}}\delta_4\Delta_o\right)\left\{\|X_2-X_1\|^2+\|Y_2-Y_1\|^2+\|Z_2-Z_1\|^2\right\}$$

with $\delta_5=\min\{\delta_1,\delta_2,\delta_3\}$.

There exists a constants δ_6 such that

$$\dot{W}(t)\le-\delta_6\left\{\|X_2-X_1\|^2+\|Y_2-Y_1\|^2+\|Z_2-Z_1\|^2\right\}.$$

In view of (14), the above inequality implies

$$\dot{W}(t)\le-\delta_6 W(t). \tag{15}$$

Let ϵ be now fixed as $\epsilon=3^{-\frac{1}{2}}\delta_4^{-1}\delta_5$. Thus, last part of the theorem is immediate, provided $\Delta_o\le\epsilon$ and on integrating (15) between t_o and t, we have

$$W(t)\le W(t_o)\exp\left\{-\delta_6(t-t_o)\right\},\quad t\ge t_o,$$

which implies that

$$W(t)\to 0\quad \text{as } t\to\infty.$$

Thus, by (14), it shows that

$$\|X_2(t)-X_1(t)\|\to 0,\ \|Y_2(t)-Y_1(t)\|\to 0\ \text{ and }\ \|Z_2(t)-X_1(t)\|\to 0,\ \text{ as }\ t\to\infty.$$

From system (1) this implies that

$$\|X_2(t)-X_1(t)\|\to 0,\ \|\dot{X}_2(t)-\dot{X}_1(t)\|\to 0\ \text{ and }\ \|\ddot{X}_2(t)-\ddot{X}_1(t)\|\to 0,\ \text{ as }\ t\to\infty.$$

This completes the proof of Theorem 1.

5. Conclusions

Analysis of nonlinear systems literary shows that Lyapunov's theory in convergence of solutions is rarely scarce. The second Lyapunov's method allows predicting the convergence behavior of solutions of sufficiently complicated nonlinear physical system.

Example 4.0.1. *As a special case of system* (2), let us take for $n=2$ such that $P \neq 0$ is a function of *t* only and

$$\Psi(X,Y)=\begin{pmatrix} x^2+y^2+8 & 0 \\ 0 & x^2+y^2+10 \end{pmatrix}$$

$$\Phi(Y)=\begin{pmatrix} \tan^{-1} y_1 + 0.004 \\ y_2 \end{pmatrix}, \quad H(X)=\begin{pmatrix} \tan^{-1} x_1 + 2x_1 \\ \tan^{-1} x_2 + 3x_2 \end{pmatrix}$$

$$X=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ and } P(t)=\begin{pmatrix} \frac{1}{1+t^2} \\ \frac{1}{1+t^2} \end{pmatrix}.$$

Thus,

$$J\Phi(Y)=\begin{pmatrix} \frac{1}{1+y_1^2}+0.004 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } JH(X)=\begin{pmatrix} 2+\frac{1}{1+x_1^2} & 0 \\ 0 & 3+\frac{1}{1+x_2^2} \end{pmatrix}.$$

Clearly, $\Psi, J\Phi$ and JH are symmetric and commute pairwise. That is,

$$\Psi(X,Y)J\Phi(Y)=J\Phi(Y)\Psi(X,Y),$$

$$\Psi(X,Y)JH(X)=JH(X)\Psi(X,Y)$$

and

$$J\Phi(Y)JH(X)=JH(X)J\Phi(Y).$$

Then, by easy calculation, we obtain eigenvalues of the matrices $\Psi, J\Phi$ and JH as follows

$$\lambda_1(\Psi(X,Y))=8+x^2+y^2, \quad \lambda_2(\Psi(X,Y))=10+x^2+y^2$$

$$\lambda_1(J\Phi(Y))=1, \quad \lambda_2(J\Phi(Y))=\frac{1.004}{1+y_1^2}+\frac{y_1^2}{250(1+y_1^2)}$$

$$\lambda_1(JH(X))=2+\frac{1}{1+x_1^2}, \quad \lambda_2(JH(X))=3+\frac{1}{1+x_1^2}.$$

It is obvious that $\alpha_o=8$, $\alpha_1=10$, $\beta_o=1$, $\beta_1=1.004$, $\gamma_o=2$ and $\gamma_1=4$.

If we choose $\delta=\frac{1}{6}$, we must have that

$$\mu<\min\left\{\frac{1}{8},8,\frac{9}{64},\frac{1}{48}\right\} \text{ and } \|P(t)\|=\frac{1}{1+t^2}\leq 2.$$

Thus, all the conditions of Theorem 1 are satisfied. Therefore, all solutions of (1) converge since (5) and (6) hold.

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