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Inertial Manifolds for 2D Generalized MHD System

Zhaoqin Yuan, Liang Guo, Guoguang Lin*

Department of Mathematics, Yunnan University, Kunming, China
Email: 15925159599@163.com, *yuanzq091@163.com, *gglin@ynu.edu.cn.

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Abstract

In this paper, we prove the existence of inertial manifolds for 2D generalized MHD system under the spectral gap condition.

Keywords

MHD System, Spectral Gap, Inertial Manifolds

1. Introduction

In [1], Yuan, Guo and Lin prove the existence of global attractors and dimension estimation of a 2D generalized magnetohydrodynamic (MHD) system:

$$\begin{cases} \frac{\partial u}{\partial t}+(u\cdot\nabla)u-(v\cdot\nabla)v+\gamma(-\Delta)^{2\alpha}u=f(x) \\ \frac{\partial v}{\partial t}+(u\cdot\nabla)v-(v\cdot\nabla)u+\eta(-\Delta)^{2\beta}v=g(x) \\ \nabla u=\nabla v=0 \\ (u,v)(x,0)=(u_0,v_0)(x) \\ u(x,t)|_{\partial\Omega}=v(x,t)|_{\partial\Omega}=0. \end{cases} \tag{1.1}$$

where u is the fluid velocity field, v is the magnetic field, γ is the constant kinematic viscosity and η is constant magnetic diffusivity. $\Omega\subset R^n$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, $\gamma,\eta>0,\alpha,\beta>\frac{n}{2}$.

*Corresponding author.

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More results about inertial manifolds can be founded in [2]-[11].

In this paper, we consider the following 2D generalized MHD system:

$$\begin{cases} \dfrac{\partial u}{\partial t}+(u\cdot\nabla)u-(v\cdot\nabla)v+\gamma(-\Delta)^{2\alpha}u=f(x) \\ \dfrac{\partial v}{\partial t}+(u\cdot\nabla)v-(v\cdot\nabla)u+\gamma(-\Delta)^{2\alpha}v=g(x) \\ \nabla u=\nabla v=0 \\ (u,v)(x,0)=(u_0,v_0)(x) \\ u(x,t)\big|_{\partial\Omega}=v(x,t)\big|_{\partial\Omega}=0. \end{cases} \tag{1.2}$$

where u is the fluid velocity field, v is the magnetic field, γ is the constant kinematic viscosity and η is the constant magnetic diffusivity. $\Omega\subset R^n$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, $\gamma>0,\alpha>\dfrac{n}{2}$.

This paper is organized as follows. In Section 2, we introduce basic concepts concerning inertial manifolds. In Section 3, we obtain the existence of the inertial manifolds.

2. Preliminaries

We rewrite the problem (1.2) as a first order differential equation, the problem (1.2) is equivalent to:

$$\begin{cases} U_t+AU=F(U), \quad t>0, \\ U(0)=U_0, \end{cases} \tag{2.1}$$

where $U=\begin{pmatrix} u \\ v \end{pmatrix}$, $U_t=\begin{pmatrix} u_t \\ v_t \end{pmatrix}$, and

$$A=\begin{pmatrix} \gamma(-\Delta)^{2\alpha} & 0 \\ 0 & \gamma(-\Delta)^{2\alpha} \end{pmatrix},\quad F(U)=\begin{pmatrix} f(x)-(u\cdot\nabla)u+(v\cdot\nabla)v \\ g(x)-(u\cdot\nabla)v+(v\cdot\nabla)u \end{pmatrix}.$$

Let H is a Banach space, $H=L^2(\Omega)\times L^2(\Omega)$, $\|\cdot\|$ is norm of H, $(\cdot,\cdot)$ is inner product of H, $\|U\|^2=\|u\|^2+\|v\|^2$; $V_1=D\left((-\Delta)^\alpha\right)\times D\left((-\Delta)^\alpha\right)$, for any solution $U\in V_1$ of the problem (2.1), $\|U\|_{V_1}=\left(\left\|(-\Delta)^\alpha u\right\|^2+\left\|(-\Delta)^\alpha v\right\|^2\right)^{\frac{1}{2}}$, $\|\cdot\|_{V_1}$ is norm of V_1.

Definition 2.1. Suppose $S(t)$ denote the semi-group of solutions to the problem (2.l) in $V_1\times[0,T](T>0)$, subset M is an inertial manifolds of the problem (2.l), that is M satisfying the following properties:

1. M is a finite dimensional Lipshitz manifold;
2. M is positively invariant under $S(t)$, that is, $S(t)M\subset M$ for all $t\geq 0$;
3. M is attracts every trajectory exponentially, *i.e.*, for every $U_0\in V_1$,

$$dist\left(S(t)U_0,M\right)\to 0, t\to+\infty.$$

We now recall some notions. Let A is a closed linear operator on H satisfying the following **Standing Hypothesis 2.2**.

Standing Hypothesis 2.2. We suppose that A is a positive definite, self-adjoint operator with a discrete spectrum, A^{-1} compacts in H. Assume $w_j=\begin{pmatrix} u_j \\ v_j \end{pmatrix}$ is the orthonormal basis in H consisting of the corresponding eigenfunctions of the operator A. Say

$$Aw_j=\lambda_j w_j,\ j=1,2,\cdots, \tag{2.2}$$

$0 < \lambda_1 \le \lambda_2 \le \cdots$, each with finite multiplicity and $\lim_{j \to +\infty} \lambda_j = +\infty$.

Let now λ_N and λ_{N+1} be two successive and different eigenvalues with $\lambda_N < \lambda_{N+1}$, let further P be the orthogonal projection onto the first N eigenvectors of the operator A.

Let the bound absorbing set $B_\rho \subseteq V_1$, we define a smooth truncated function by setting $\theta : R^+ \to [0,1]$ is defined as

$$\begin{cases} \theta(\xi) = 1, \quad 0 \le \xi \le 1, \\ \theta(\xi) = 0, \quad \xi \ge 2, \\ |\theta'(\xi)| \le 2, \quad \xi \ge 0, \\ \theta_\rho(r) = \theta\left(\dfrac{r}{\rho}\right). \end{cases} \tag{2.3}$$

Suppose that $F_\theta(U) = \theta_\rho\left(\left|A^{\frac{1}{2}}U\right|\right)F(U)$, the problem (2.1) is equivalent to the following preliminary equation:

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} + AU = F_\theta(U), t > 0, \\ U(0) = U_0 \end{cases} \tag{2.4}$$

Denote by P_N is the orthogonal projection of H onto $H := span\{w_1, \cdots, w_N\}$, and $Q_N = I - P_N$. Set $p = P_N U, q = Q_N U$, then Equation (2.4) is equivalent to

$$\frac{\mathrm{d}p}{\mathrm{d}t} + Ap = P_N F_\theta(p+q), \tag{2.5}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} + Aq = Q_N F_\theta(p+q). \tag{2.6}$$

Lemma 2.3. Defined by $F(U)$ of the problem (2.1) on the bounded set of V_1 is a Lipschitz function, for every $U_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, U_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in V_1$, there exist a constant $C > 0$ such that

$$\|F(U_1) - F(U_2)\| \le C\left\|A^{\frac{1}{2}}(U_1 - U_2)\right\|, \tag{2.7}$$

where $C = C_3 k$.

Proof. Assume $U_1, U_2 \in V_1$, and let $U = U_1 - U_2 = \begin{pmatrix} u \\ v \end{pmatrix}$, use the fact that $\|U\|_{V_1} \le M_1$ and using Poincare inequality $\|U\| \le k\left\|A^{1/2}U\right\|$, we have

$$\begin{aligned} &|(F(U_1) - F(U_2), U)| \\ &\le |(-u_1\nabla u_1 + u_2\nabla u_2 + v_1\nabla v_1 - v_2\nabla v_2, u)| + |(-u_1\nabla v_1 + u_2\nabla v_2 + v_1\nabla u_1 - v_2\nabla u_2, v)| \\ &\le C_0 M_1\|u\|^2 + C_1 M_1\|u\|\|v\| + C_2 M_1\|v\|^2 \le (C_0 M_1 + C_1 M_1)\|u\|^2 + (C_1 M_1 + C_2 M_1)\|v\|^2 \\ &\le C_3\left(\|u\|^2 + \|v\|^2\right) = C_3\|U\|^2 \le C_3 k\left\|A^{\frac{1}{2}}U\right\|\|U\| = C\left\|A^{\frac{1}{2}}U\right\|\|U\|, \end{aligned} \tag{2.8}$$

where $C_3 = \max\{C_0 M_1 + C_1 M_1, C_1 M_1 + C_2 M_1\}$, so we can get

$$\left\| F(U_1) - F(U_2) \right\| \le C \left\| A^{\frac{1}{2}} U \right\|. \tag{2.9}$$

Lemma 2.3 is proved. □

Lemma 2.4. Let $T > 0$ be fixed, for any N and all $t \in [0,T]$, there exist $\zeta > 0$ such that

$$\left\| Q_N \left(U_1(t) - U_2(t) \right) \right\| \le \zeta \left\| P_N \left(U_1(t) - U_2(t) \right) \right\|, \tag{2.10}$$

otherwise, there exist constants $C_4 = \exp\left(C^2 T\right)$ and $C_5 = -\frac{\zeta^2}{\zeta^2 + 1} \exp\left(-C^2 T\right)$ are dependent on ζ, M_1, T such that

$$\left\| U_1(t) - U_2(t) \right\| \le C_4 \exp\left(-C_5 \lambda_{N+1} t\right) \left\| U_1(0) - U_2(0) \right\|, \tag{2.11}$$

and

$$\left\| U_1(t) - U_2(t) \right\| \le \exp\left(C^2 t\right) \left\| U_1(0) - U_2(0) \right\|, \tag{2.12}$$

for all $U_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, U_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in V_1$.

Proof. Let U_1, U_2 with initial values $U_1(0), U_2(0) \in V_1$, respectively, are two different solutions of the problem (2.1), we have the fact that $\|U\|_{V_1} \le M_1$, $\forall t \in [0,T]$. Put $U(t) = U_1(t) - U_2(t)$, so we obtain that

$$\frac{\mathrm{d}U}{\mathrm{d}t} + AU = F(U_1) - F(U_2). \tag{2.13}$$

Putting

$$p(t) = \frac{\left\| A^{\frac{1}{2}} U(t) \right\|^2}{\left\| U(t) \right\|^2} = \frac{\left(A^{\frac{1}{2}} U, A^{\frac{1}{2}} U \right)}{(U, U)}. \tag{2.14}$$

For $t \in [0,T]$, taking the derivative of Equation (2.14) with respect to t,we have

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{2}{\|U\|^4} \left(\|U\|^2 \left(A^{\frac{1}{2}} U', A^{\frac{1}{2}} U \right) - \left\| A^{\frac{1}{2}} U \right\|^2 (U', U) \right) \\ &= \frac{2}{\|U\|^2} \left((U', AU) - p(t)(U', U) \right). \end{aligned} \tag{2.15}$$

From Equation (2.13) and Equation (2.15), we have

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{-2}{\|U\|^2} \left(AU - \left(F(U_1) - F(U_2) \right), AU - p(t) U \right). \tag{2.16}$$

We notice that Equation (2.14)

$$(pU, AU - pU) = p \left(A^{\frac{1}{2}} U, A^{\frac{1}{2}} U \right) - p^2 (U, U) = 0,$$

so we have

$$\left(AU, AU - p(t)U \right) = \left(AU - p(t)U, AU - p(t)U \right) = \left\| AU - p(t)U \right\|^2. \tag{2.17}$$

By Equation (2.16) and Equation (2.17), and use the Cauchy-Schwarz inequality, we obtain

$$\frac{\mathrm{d}p}{\mathrm{d}t}+\frac{2}{\|U\|^2}\|AU-p(t)U\|^2=\frac{2}{\|U\|^2}\big(\big(F(U_1)-F(U_2)\big),AU-p(t)U\big)$$
$$\le\frac{2}{\|U\|^2}\|F(U_1)-F(U_2)\|\|AU-p(t)U\|$$
$$\le\frac{2}{\|U\|^2}\|AU-p(t)U\|^2+\frac{\|F(U_1)-F(U_2)\|^2}{\|U\|^2} \tag{2.18}$$
$$\le\frac{2}{\|U\|^2}\|AU-p(t)U\|^2+\frac{C^2\left\|A^{\frac{1}{2}}U\right\|^2}{\|U\|^2}.$$

Then using **Lemma 2.3**,we have

$$\frac{\mathrm{d}p}{\mathrm{d}t}\le C^2p.$$

For $0<\tau<t<T$, integrating the above inequality over $[\tau,t]$, we obtain

$$\frac{\left\|A^{\frac{1}{2}}U(t)\right\|^2}{\|U(t)\|^2}\le\frac{\left\|A^{\frac{1}{2}}U(\tau)\right\|^2}{\|U(\tau)\|^2}\exp\big(C^2(t-\tau)\big), \tag{2.19}$$

where C is given as in **Lemma 2.3**.

By multiplying (2.13) by U, using Cauchy-Schwarz inequality and **Lemma 2.3**, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2+\left\|A^{\frac{1}{2}}U\right\|^2=\big(F(U_1)-F(U_2),U\big)\le\|F(U_1)-F(U_2)\|\|U\|\le C\left\|A^{\frac{1}{2}}U\right\|\|U\|. \tag{2.20}$$

Using Holder inequality, from Equation (2.20) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2+\|U\|^2\left(\frac{\left\|A^{\frac{1}{2}}U\right\|^2}{\|U\|^2}-C^2\right)\le 0. \tag{2.21}$$

In Equation (2.19) setting $\tau=t,t=t_0$, we obtain

$$\frac{\left\|A^{\frac{1}{2}}U(t)\right\|^2}{\|U(t)\|^2}\ge\frac{\left\|A^{\frac{1}{2}}U(t_0)\right\|^2}{\|U(t_0)\|^2}\exp\big(-C^2(t_0-t)\big)\ge\varepsilon\exp\big(-C^2t_0\big), \tag{2.22}$$

where

$$\varepsilon=\frac{\left\|A^{\frac{1}{2}}U(t_0)\right\|^2}{\|U(t_0)\|^2}. \tag{2.23}$$

By Equation (2.21) and Equation (2.22), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2+\|U\|^2\left(\varepsilon\exp\big(-C^2t_0\big)-C^2\right)\le 0. \tag{2.24}$$

Integrating Equation (2.24) between 0 and t_0, we obtain

$$\|U(t_0)\|^2\le\|U(0)\|^2\exp\big(-\varepsilon t_0\exp\big(-C^2t_0\big)+C^2t_0\big). \tag{2.25}$$

To complete the proof of **Lemma 2.4**, we consider the following two cases,

$$\left\|Q_N U(t_0)\right\| > \zeta \left\|P_N U(t_0)\right\|. \tag{2.26}$$

and

$$\left\|Q_N U(t_0)\right\| \le \zeta \left\|P_N U(t_0)\right\|. \tag{2.27}$$

We only consider Equation (2.26), in this case,

$$\begin{aligned} \varepsilon &= \frac{\left\|A^{\frac{1}{2}}U(t_0)\right\|^2}{\left\|U(t_0)\right\|^2} = \frac{\left\|P_N A^{\frac{1}{2}}U(t_0)\right\|^2 + \left\|Q_N A^{\frac{1}{2}}U(t_0)\right\|^2}{\left\|P_N U(t_0)\right\|^2 + \left\|Q_N U(t_0)\right\|^2} \\ &\ge \frac{\left\|Q_N A^{\frac{1}{2}}U(t_0)\right\|^2}{\left(1+\frac{1}{\zeta^2}\right)\left\|Q_N U(t_0)\right\|^2} \ge \frac{\zeta^2}{\zeta^2+1}\lambda_{N+1}, \end{aligned} \tag{2.28}$$

where λ_{N+1} is $N+1$ eigenvector of the operator A. By Equation (2.25) and Equation (2.28), we obtain

$$\begin{aligned} \left\|U(t_0)\right\|^2 &\le \left\|U(0)\right\|^2 \exp\left(-\frac{\zeta^2}{\zeta^2+1}\lambda_{N+1}t_0 \exp\left(-C^2 t_0\right) + C^2 t_0\right) \\ &\le \left\|U(0)\right\|^2 \exp\left(-\frac{\zeta^2}{\zeta^2+1}\lambda_{N+1}T \exp\left(-C^2 T\right) + C^2 T\right), \end{aligned} \tag{2.29}$$

since $t_0 < T$, in Equation (2.29) setting $t = t_0$, which proves Equation (2.11), where $C_4 = \exp\left(C^2 T\right)$ and $C_5 = -\frac{\zeta^2}{\zeta^2+1}\exp\left(-C^2 T\right)$. Using again Equation (2.20), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|U\right\|^2 + 2\left\|A^{\frac{1}{2}}U\right\|^2 \le 2C\left\|A^{\frac{1}{2}}U\right\|\left\|U\right\| \le 2\left\|A^{\frac{1}{2}}U\right\|^2 + C^2\left\|U\right\|^2,$$

then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|U\right\|^2 \le C^2\left\|U\right\|^2. \tag{2.30}$$

Integrating Equation (2.30) between 0 and t_0, which proves Equation (2.12). **Lemma 2.4** is proved. □

3. Inertial Manifolds

In this section we will prove the existence of the inertial manifolds for solutions to the problem (2.1). We suppose that A satisfies **Standing Hypothesis 2.2** and recall that P is the orthogonal projection onto the first N orthonormal eigenvectors of A.

Let constants $b, l > 0$ be fixed, we define $F = F_{b,l}^{\frac{1}{2}}$ and denote the collection of all functions $\Phi : P_N V_1 \to Q_N V_1$ satisfies

$$\begin{cases} supp\Phi \subset \left\{ p \in P_N V_1, \left\|A^{\frac{1}{2}}p\right\| \le 2\rho \right\}, \\ \left\|A^{\frac{1}{2}}\Phi(p)\right\| \le b, \quad \forall p \in P_N V_1, \\ \left\|A^{\frac{1}{2}}\left(\Phi(p_1) - \Phi(p_2)\right)\right\| \le l\left\|A^{\frac{1}{2}}(p_1 - p_2)\right\|, \quad \forall p_1, p_2 \in V_1. \end{cases} \tag{3.1}$$

Note that

$$d\left(\Phi_1,\Phi_2\right)=\sup_{p\in P_N V_1}\left\|A^{\frac{1}{2}}\left(\Phi_1\left(p\right)-\Phi_2\left(p\right)\right)\right\|, \tag{3.2}$$

is the distance of $F=F_{b,l}^{\frac{1}{2}}$. So F is completely space.

For every $\Phi\in F_{b,l}^{\frac{1}{2}}$ and the initial data $p_0\in P_N V_1$, the initial value problem

$$\begin{cases}\dfrac{\mathrm{d}p}{\mathrm{d}t}+Ap=P_N F_\theta\left(p+\Phi\left(p\right)\right),\\ p\left(0\right)=p_0,\end{cases} \tag{3.3}$$

possesses a unique solution $p\left(t\right)=p\left(t;\Phi,p_0\right)$.

$$\frac{\mathrm{d}q}{\mathrm{d}t}+Aq=Q_N F_\theta\left(p+\Phi\left(p\right)\right), \tag{3.4}$$

where $Q_N F_\theta\left(p+\Phi\left(p\right)\right)\in L^\infty\left(R\times R;H\right)$ and the unique solution $q=q\left(t;\Phi,p_0\right)$ in Equation (3.4) is a successive bounded mapping acts from $R\times R$ into $Q_N V_1$. Particularly, the function

$$p_0\in P_N V_1\to q\left(0;\Phi,p_0\right)\in Q_N V_1. \tag{3.5}$$

by $\Phi\in F_{b,l}^{\frac{1}{2}}$, note that $T\Phi: p_0\to q\left(0;\Phi,p_0\right)$, we have

$$T\Phi\left(p_0\right)=\int_{-\infty}^{0}\mathrm{e}^{A\tau}Q_N F_\theta\left(p\left(\tau\right)+\Phi\left(p\left(\tau\right)\right)\right)\mathrm{d}\tau=q\left(0;\Phi,p_0\right). \tag{3.6}$$

We need to prove the following two conclusions:

1. For $\lambda_N^{\frac{1}{2}}$ and $\lambda_{N+1}^{\frac{1}{2}}-\lambda_N^{\frac{1}{2}}$ are sufficiently large, $T:F_{b,l}^{\frac{1}{2}}\to F_{b,l}^{\frac{1}{2}}$ is a contraction.
2. Φ_0 is a unique fixed point in *T*, $M=Graph\left(\Phi_0\right)$ is a inertial manifold of 2D generalized MHD system.

So we give the following Lemmas.

Lemma 3.1. Let $\forall\Phi\in F_{b,l}^{\frac{1}{2}}$, so we have

$$supp\Phi\subset\left\{p\in P_N V_1,\left\|A^{\frac{1}{2}}p\right\|\le 2\rho\right\}. \tag{3.7}$$

Proof. The proof is similar to Temam [3]. □

Lemma 3.2. Let $\forall\Phi\in F_{b,l}^{\frac{1}{2}}$, for $U_i=p_i+\Phi\left(p_i\right)\left(i=1,2\right)$, there exists constant $M_2,M_3>0$ such that

$$\left\|F_\theta\left(U_1\right)\right\|\le M_2, \tag{3.8}$$

and

$$\left\|F_\theta\left(U_1\right)-F_\theta\left(U_2\right)\right\|\le M_3\left(1+l\right)\left\|A^{\frac{1}{2}}\left(p_1-p_2\right)\right\|,\quad \forall p_1,p_2\in P_N V_1. \tag{3.9}$$

Proof. For any $\Phi\in F_{b,l}^{\frac{1}{2}}$, and $p_1,p_2\in P_N V_1$, we denote $U_i=p_i+\Phi\left(p_i\right)\left(i=1,2\right)$, using **Lemma 2.3** and see ([3], Chapter 8: Lemma 2.1 and Lemma 2.2), we derive that there exists constant $M_2,M_3>0$ such that

$$\left\|F_\theta\left(U_1\right)\right\|\le M_2, \tag{3.10}$$

and

$$\left\|F_\theta(U_1)-F_\theta(U_2)\right\|\le M_3\left\|A^{\frac{1}{2}}(U_1-U_2)\right\|, \tag{3.11}$$

which proves Equation (3.8). We now prove Equation (3.9), by the definition of $F_{b,l}^{\frac{1}{2}}$, we have

$$\left\|A^{\frac{1}{2}}\left(\Phi(p_1)-\Phi(p_2)\right)\right\|\le l\left\|A^{\frac{1}{2}}(p_1-p_2)\right\|. \tag{3.12}$$

And we have

$$\left\|A^{\frac{1}{2}}(U_1-U_2)\right\|\le\left\|A^{\frac{1}{2}}(p_1-p_2)\right\|+\left\|A^{\frac{1}{2}}\left(\Phi(p_1)-\Phi(p_2)\right)\right\|\le(1+l)\left\|A^{\frac{1}{2}}(p_1-p_2)\right\|. \tag{3.13}$$

Substituting Equation (3.13) into Equation (3.11) we obtain Equation (3.9). **Lemma 3.2** is proved. □

Lemma 3.3. Let $p_0\in P_NV_1$, one has $T\Phi(p_0)\in Q_NV_1$ and $\left\|A^{\frac{1}{2}}\left(T\Phi(p_0)\right)\right\|\le b_1$, where $b_1=6\mathrm{e}^{-\frac{1}{2}}M_2\lambda_{N+1}^{-\frac{1}{2}}$, for λ_{N+1} is sufficiently large one has $b_1<b$.

Proof. Let $p_0\in P_NV_1$, according to the definition of T, we have $T\Phi(p_0)\in Q_NV_1$, from Equation (3.6) and Equation (3.10), we have

$$\begin{aligned}\left\|A^{\frac{1}{2}}\left(T\Phi(p_0)\right)\right\|&\le\int_{-\infty}^{0}\left\|A^{\frac{1}{2}}\mathrm{e}^{A\tau}Q_NF_\theta\left(p(\tau)+\Phi(p(\tau))\right)\right\|\mathrm{d}\tau\\&\le\int_{-\infty}^{0}\left\|(AQ_N)^{\frac{1}{2}}\mathrm{e}^{A\tau}\right\|_{L(Q_NH)}\left\|F_\theta\left(p(\tau)+\Phi(p(\tau))\right)\right\|\mathrm{d}\tau\\&\le M_2\int_{-\infty}^{0}\left\|(AQ_N)^{\frac{1}{2}}\mathrm{e}^{A\tau}\right\|_{L(Q_NH)}\mathrm{d}\tau.\end{aligned} \tag{3.14}$$

Let $\delta\in R$ and $\tau<0$, suppose that $K_2(\delta)=\delta^\delta\mathrm{e}^{-\delta}$ and

$$K_3(\delta)=\begin{cases}1, & \delta<0,\\ \mathrm{e}^{-\delta}+\dfrac{K_2(\delta)}{1-\delta}\delta^{1-\delta}, & 0\le\delta<1.\end{cases}$$

So we obtain

$$\left\|(AQ_N)^\delta\mathrm{e}^{AQ_N\tau}\right\|_{L(Q_NH)}=\begin{cases}K_2(\delta)|\tau|^{-\delta}, & -\dfrac{\delta}{\lambda_{N+1}}\le\tau<0,\\ \lambda_{N+1}^\delta\mathrm{e}^{\tau\lambda_{N+1}}, & \tau<-\dfrac{\delta}{\lambda_{N+1}}.\end{cases} \tag{3.15}$$

Further more, for $\delta<1$, we have

$$\int_{-\infty}^{0}\left\|(AQ_N)^\delta\mathrm{e}^{AQ_N\tau}\right\|_{L(Q_NH)}\mathrm{d}\tau\le K_3(\delta)\lambda_{N+1}^{\delta-1}. \tag{3.16}$$

Setting $\delta=\frac{1}{2}$ in $K_2\left(\frac{1}{2}\right),K_3\left(\frac{1}{2}\right)$, then substituting $K_2\left(\frac{1}{2}\right),K_3\left(\frac{1}{2}\right)$ into Equation (3.15) and Equation (3.16), and from Equation (3.14) we can derive that

$$\left\|A^{\frac{1}{2}}\left(T\Phi(p_0)\right)\right\|\le3K_3\left(\frac{1}{2}\right)\lambda_{N+1}^{-\frac{1}{2}}M_2\le6\lambda_{N+1}^{-\frac{1}{2}}M_2\mathrm{e}^{-\frac{1}{2}}. \tag{3.17}$$

Lemma 3.3 is proved. □

Lemma 3.4. Let

$$\mu_N = (\lambda_{N+1} - \lambda_N) - M_3(1+l)\lambda_N^{\frac{1}{2}} > 0, \tag{3.18}$$

so for every $\Phi \in F_{b,l}^{\frac{1}{2}}$, one has

$$\left\| A^{\frac{1}{2}}\left(T\Phi(p_{01}) - T\Phi(p_{02})\right)\right\| \le l_1 \left\| A^{\frac{1}{2}}(p_{01} - p_{02})\right\|, \quad \forall p_{01}, p_{02} \in P_N V_1, \tag{3.19}$$

here

$$l_1 = M_3(1+l)\lambda_{N+1}^{-\frac{1}{2}}\left[\frac{1}{\sqrt{2}} + (1-\zeta_N\xi_N)^{-1}\right]\mathrm{e}^{-\frac{1}{2}}\exp\left(\frac{\zeta_N\xi_N}{2}\right), \tag{3.20}$$

$$\zeta_N = \frac{\lambda_N}{\lambda_{N+1}}, \tag{3.21}$$

$$\xi_N = 1 + M_3(1+l)\lambda_N^{-\frac{1}{2}}. \tag{3.22}$$

Proof. For any given $\Phi \in F_{b,l}^{\frac{1}{2}}$, let $p_1 = p_1(t), p_2 = p_2(t)$ are the solutions of the following initial value problem,

$$\begin{cases} \dfrac{\mathrm{d}p_1}{\mathrm{d}t} + Ap_1 = P_N F_\theta(U_1), \\ p_1(0) = p_{01}. \end{cases} \tag{3.23}$$

and

$$\begin{cases} \dfrac{\mathrm{d}p_2}{\mathrm{d}t} + Ap_2 = P_N F_\theta(U_2), \\ p_2(0) = p_{02}, \end{cases} \tag{3.24}$$

here $U_i = p_i + \Phi(p_i), i = 1, 2.$ Suppose that $p(t) = p_1(t) - p_2(t)$, so we have

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} + Ap = P_N\left(F_\theta(U_1) - F_\theta(U_2)\right), \\ p(0) = p_{01} - p_{02}. \end{cases} \tag{3.25}$$

Multiplying the first equation in Equation (3.25) by Ap, using Equation (3.9) in **Lemma 3.2**, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|A^{\frac{1}{2}}p\right\|^2 + \|Ap\|^2 \ge -\|F_\theta(U_1) - F_\theta(U_2)\|\|Ap\| \ge -M_3(1+l)\left\|A^{\frac{1}{2}}p\right\|\|Ap\|. \tag{3.26}$$

So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|A^{\frac{1}{2}}p\right\| + \left(\lambda_N + M_3(1+l)\lambda_N^{\frac{1}{2}}\right)\left\|A^{\frac{1}{2}}p\right\| \ge 0. \tag{3.27}$$

For $t \le 0$, from Equation (3.27) we have

$$\left\|A^{\frac{1}{2}}p(t)\right\| \le \left\|A^{\frac{1}{2}}p(0)\right\|\exp\left[-t\left(\lambda_N + M_3(1+l)\lambda_N^{\frac{1}{2}}\right)\right]. \tag{3.28}$$

By **Lemma 2.3**, to do the following estimate,using Equation (3.11) and Equation (3.28) we obtain

$$\begin{aligned}&\left\|A^{\frac{1}{2}}\left(T\Phi\left(p_{01}\right)-T\Phi\left(p_{02}\right)\right)\right\|\le\int_{-\infty}^{0}\left\|A^{\frac{1}{2}}\mathrm{e}^{At}Q_N\left(F_\theta\left(U_1\right)-F_\theta\left(U_2\right)\right)\right\|\mathrm{d}t\\&\le\int_{-\infty}^{0}\left\|\left(AQ_N\right)^{\frac{1}{2}}\mathrm{e}^{At}\right\|_{L(Q_NH)}\left\|F_\theta\left(U_1\right)-F_\theta\left(U_2\right)\right\|\mathrm{d}t\\&\le M_3\left(1+l\right)\int_{-\infty}^{0}\left\|\left(AQ_N\right)^{\frac{1}{2}}\mathrm{e}^{At}\right\|_{L(Q_NH)}\left\|A^{\frac{1}{2}}p\right\|\mathrm{d}t\\&\le M_3\left(1+l\right)\left\|A^{\frac{1}{2}}p\left(0\right)\right\|\int_{-\infty}^{0}\left\|\left(AQ_N\right)^{\frac{1}{2}}\mathrm{e}^{At}\right\|_{L(Q_NH)}\mathrm{e}^{-\lambda_N\xi_Nt}\mathrm{d}t,\end{aligned}\tag{3.29}$$

here $\xi_N=1+M_3\left(1+l\right)\lambda_N^{-\frac{1}{2}}$. From Equation (3.15), we have

$$\begin{aligned}&\int_{-\infty}^{-\frac{1}{2\lambda_{N+1}}}\left\|\left(AQ_N\right)^{\frac{1}{2}}\mathrm{e}^{At}\right\|_{L(Q_NH)}\mathrm{e}^{-\lambda_N\xi_Nt}\mathrm{d}t\le\int_{-\infty}^{-\frac{1}{2\lambda_{N+1}}}\lambda_{N+1}^{\frac{1}{2}}\mathrm{e}^{\lambda_{N+1}t}\mathrm{e}^{-\lambda_N\xi_Nt}\mathrm{d}t\\&\le\int_{-\infty}^{-\frac{1}{2\lambda_{N+1}}}\lambda_{N+1}^{\frac{1}{2}}\mathrm{e}^{\mu_Nt}\mathrm{d}t\le\lambda_{N+1}^{\frac{1}{2}}\frac{1}{\mu_N}\exp\left(-\frac{\mu_N}{2\lambda_{N+1}}\right),\end{aligned}\tag{3.30}$$

here $\mu_N=\lambda_{N+1}-\lambda_N\xi_N=\lambda_{N+1}\left(1-\zeta_N\xi_N\right),\zeta_N=\dfrac{\lambda_N}{\lambda_{N+1}}$.

Hence,

$$\int_{-\infty}^{-\frac{1}{2\lambda_{N+1}}}\left\|\left(AQ_N\right)^{\frac{1}{2}}\mathrm{e}^{At}\right\|_{L(Q_NH)}\mathrm{e}^{-\lambda_N\xi_Nt}\mathrm{d}t\le\lambda_{N+1}^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}}\left(1-\zeta_N\xi_N\right)^{-1}\exp\left(\frac{\zeta_N\xi_N}{2}\right).\tag{3.31}$$

Then from Equation (3.15) we have

$$\begin{aligned}&\int_{-\frac{1}{2\lambda_{N+1}}}^{0}\left\|\left(AQ_N\right)^{\frac{1}{2}}\mathrm{e}^{At}\right\|_{L(Q_NH)}\mathrm{e}^{-\lambda_N\xi_Nt}\mathrm{d}t\le\int_{-\frac{1}{2\lambda_{N+1}}}^{0}K_2\left(\frac{1}{2}\right)\left|t\right|^{-\frac{1}{2}}\mathrm{e}^{-\lambda_N\xi_Nt}\mathrm{d}t\\&\le\left(2\mathrm{e}\right)^{-\frac{1}{2}}\exp\left(\frac{\lambda_N\zeta_N}{2\lambda_{N+1}}\right)\int_{-\frac{1}{2\lambda_{N+1}}}^{0}\left|t\right|^{-\frac{1}{2}}\mathrm{d}t\le\left(2\mathrm{e}\right)^{-\frac{1}{2}}\lambda_{N+1}^{-\frac{1}{2}}\exp\left(\frac{\zeta_N\xi_N}{2}\right).\end{aligned}\tag{3.32}$$

Combining Equation (3.31) and Equation (3.32), we obtain

$$\int_{-\infty}^{0}\left\|\left(AQ_N\right)^{\frac{1}{2}}\mathrm{e}^{At}\right\|_{L(Q_NH)}\mathrm{e}^{-\lambda_N\xi_Nt}\mathrm{d}t\le\lambda_{N+1}^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}}\left[\left(1-\zeta_N\xi_N\right)^{-1}+2^{-\frac{1}{2}}\right]\exp\left(\frac{\zeta_N\xi_N}{2}\right).\tag{3.33}$$

Substituting Equation (3.33) into Equation (3.29), we obtain

$$\left\|A^{\frac{1}{2}}\left(T\Phi\left(p_{01}\right)-T\Phi\left(p_{02}\right)\right)\right\|\le l_1\left\|A^{\frac{1}{2}}\left(p_{01}-p_{02}\right)\right\|.$$

Lemma 3.4 is proved. □

Lemma 3.5. Let $\mu_N>0$ is defined as in **Lemma 3.4**, for all $\Phi_1,\Phi_2\in F_{b,l}^{\frac{1}{2}}$,

$$\left\|A^{\frac{1}{2}}\left(T\Phi_1\left(p_0\right)-T\Phi_2\left(p_0\right)\right)\right\|\le K_0d\left(\Phi_1,\Phi_2\right),\ \forall p_0\in P_NV_1,\tag{3.34}$$

here $K_0=M_3\left(6\lambda_{N+1}^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}}+\lambda_N^{-\frac{1}{2}}l_1\right)$, l_1 is defined by Equation (3.20), $d\left(\Phi_1,\Phi_2\right)$ is defined by Equation (3.2).

Proof. Let $p_i=p_i\left(t;\Phi_i,p_0\right),U_i=p_i+\Phi_i\left(p_i\right),i=1,2,$ and let $p=p_1-p_2$ is the solution of the initial value problem (3.25), then by the same way as in **Lemma 3.2** we can prove that

$$\begin{aligned}
&\left\|F_\theta(U_1)-F_\theta(U_2)\right\| \le M_3\left\|A^{\frac{1}{2}}(U_1-U_2)\right\| \\
&\le M_3\left(\left\|A^{\frac{1}{2}}(p_1-p_2)\right\|+\left\|A^{\frac{1}{2}}(\Phi_1(p_1)-\Phi_2(p_2))\right\|\right) \\
&\le M_3\left\|A^{\frac{1}{2}}(p_1-p_2)\right\|+M_3\left(\left\|A^{\frac{1}{2}}(\Phi_1(p_1)-\Phi_1(p_2))\right\|+\left\|A^{\frac{1}{2}}(\Phi_1(p_2)-\Phi_2(p_2))\right\|\right) \\
&\le M_3\left[(1+l)\left\|A^{\frac{1}{2}}(p_1-p_2)\right\|+d(\Phi_1,\Phi_2)\right].
\end{aligned} \tag{3.35}$$

From the first inequality of Equation (3.26) and the following estimate, we have

$$\|Ap\|=\left\|A^{\frac{1}{2}}A^{\frac{1}{2}}p\right\|\le\lambda_N^{\frac{1}{2}}\left\|A^{\frac{1}{2}}p\right\|,$$

then from the last inequality of Equation (3.35), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|A^{\frac{1}{2}}p\right\|^2+\lambda_N\left\|A^{\frac{1}{2}}p\right\|^2\ge -M_3(1+l)\lambda_N^{\frac{1}{2}}\left\|A^{\frac{1}{2}}p\right\|^2-M_3\lambda_N^{\frac{1}{2}}d(\Phi_1,\Phi_2)\left\|A^{\frac{1}{2}}p\right\|. \tag{3.36}$$

From Equation (3.36), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|A^{\frac{1}{2}}p\right\|+\left(\lambda_N+M_3(1+l)\lambda_N^{\frac{1}{2}}\right)\left\|A^{\frac{1}{2}}p\right\|\ge -M_3\lambda_N^{\frac{1}{2}}d(\Phi_1,\Phi_2). \tag{3.37}$$

Due to $p(0)=0$, integrating Equation (3.37) over $[0,t<0]$, we have

$$\left\|A^{\frac{1}{2}}p\right\|\le M_3\lambda_N^{\frac{1}{2}}(\lambda_N\xi_N)^{-1}\left(\exp(-t\lambda_N\xi_N)-1\right)\mathrm{d}(\Phi_1,\Phi_2). \tag{3.38}$$

From Equation (3.6), Equation (3.35) and Equation (3.38), we have

$$\begin{aligned}
&\left\|A^{\frac{1}{2}}(T\Phi_1(p_0)-T\Phi_2(p_0))\right\|\le\int_{-\infty}^0\left\|A^{\frac{1}{2}}\mathrm{e}^{At}Q_N(F_\theta(U_1)-F_\theta(U_2))\right\|\mathrm{d}t \\
&\le\int_{-\infty}^0\left\|(AQ_N)^{\frac{1}{2}}\mathrm{e}^{At}\right\|_{L(H)}\left\|F_\theta(U_1)-F_\theta(U_2)\right\|\mathrm{d}t \\
&\le M_3\int_{-\infty}^0\left\|(AQ_N)^{\frac{1}{2}}\mathrm{e}^{At}\right\|_{L(H)}\left[(1+l)\left\|A^{\frac{1}{2}}(p_1-p_2)\right\|+d(\Phi_1,\Phi_2)\right]\mathrm{d}t \\
&\le M_3d(\Phi_1,\Phi_2)\int_{-\infty}^0\left\|(AQ_N)^{\frac{1}{2}}\mathrm{e}^{At}\right\|_{L(H)}\left[1+(1+l)M_3\lambda_N^{-\frac{1}{2}}\mathrm{e}^{-t\lambda_N\xi_N}\right]\mathrm{d}t.
\end{aligned} \tag{3.39}$$

Then using Equation (3.16), Equation (3.33) and $\mu_N>0$, we have

$$\begin{aligned}
&\left\|A^{\frac{1}{2}}(T\Phi_1(p_0)-T\Phi_2(p_0))\right\| \\
&\le M_3\left[6\lambda_{N+1}^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}}+M_3(1+l)\lambda_{N+1}^{-\frac{1}{2}}\lambda_N^{-\frac{1}{2}}\left(\frac{1}{\sqrt{2}}+(1-\zeta_N\xi_N)^{-1}\right)\right]d(\Phi_1,\Phi_2) \\
&=M_3\left(6\lambda_{N+1}^{-\frac{1}{2}}e^{-\frac{1}{2}}+\lambda_N^{-\frac{1}{2}}l_1\right)d(\Phi_1,\Phi_2)=K_0d(\Phi_1,\Phi_2).
\end{aligned} \tag{3.40}$$

Lemma 3.5 is proved. □

Lemma 3.6. Suppose that $0<l<1$,

$$\lambda_{N+1}^{\frac{1}{2}}-\lambda_N^{\frac{1}{2}}\ge K_1, \tag{3.41}$$

$$\lambda_N^{\frac{1}{2}}\ge K_2, \tag{3.42}$$

we have $\mu_N>0, l_1<l$ and $K_0<\frac{1}{2}$, where K_0 is defined as in **Lemma 3.5**,

$$K_1=2M_3(1+l)l^{-1}, K_2=2M_3\left(6e^{-\frac{1}{2}}+l\right). \tag{3.43}$$

Proof. From $\mu_N=(\lambda_{N+1}-\lambda_N)-M_3(1+l)\lambda_N^{\frac{1}{2}}>0$ is equivalent to

$$1-\zeta_N\xi_N>0, \tag{3.44}$$

where ζ_N and ξ_N are defined as in **Lemma 3.4**. To find a sufficient condition of Equation (3.44), suppose that Equation (3.44) hold, so we have

$$\begin{aligned} l_1&=M_3(1+l)\lambda_{N+1}^{-\frac{1}{2}}e^{-\frac{1}{2}}e^{\frac{\zeta_N\xi_N}{2}}\left[\frac{1}{\sqrt{2}}+(1-\zeta_N\xi_N)^{-1}\right]\\ &\le M_3(1+l)\lambda_{N+1}^{-\frac{1}{2}}\left[\frac{1}{\sqrt{2}}+(1-\zeta_N\xi_N)^{-1}\right]. \end{aligned} \tag{3.45}$$

To make $l_1<l$, if and only if it satisfies

$$M_3(1+l)\lambda_{N+1}^{-\frac{1}{2}}\le\frac{l}{2}, \tag{3.46}$$

$$M_3(1+l)\lambda_{N+1}^{-\frac{1}{2}}\le\frac{l}{2}(1-\zeta_N\xi_N). \tag{3.47}$$

Equation (3.46) is equivalent to

$$K_1\le\lambda_{N+1}^{\frac{1}{2}}, K_1=2M_3(1+l)l^{-1}, \tag{3.48}$$

If Equation (3.48) is satisfied, so Equation (3.47) is equivalent to $K_1\lambda_{N+1}^{-\frac{1}{2}}\le 1-\zeta_N\xi_N$ or is equivalent to

$$K_1\lambda_{N+1}^{-\frac{1}{2}}-1+\zeta_N+M_2(1+l)\lambda_{N+1}^{-\frac{1}{2}}\lambda_N^{\frac{1}{2}}\le 0. \tag{3.49}$$

Suppose that Equation (3.41) is equivalent to

$$K_1\lambda_{N+1}^{-\frac{1}{2}}+\zeta_N^{\frac{1}{2}}\le 1. \tag{3.50}$$

Hence,

$$K_1\lambda_{N+1}^{-\frac{1}{2}}\zeta_N^{\frac{1}{2}}+\zeta_N\le\zeta_N^{\frac{1}{2}}. \tag{3.51}$$

Hence,

$$K_1\lambda_{N+1}^{-\frac{1}{2}}-1+\zeta_N+M_3(1+l)\lambda_{N+1}^{-\frac{1}{2}}\lambda_N^{\frac{1}{2}}\le K_1\lambda_{N+1}^{-\frac{1}{2}}+\zeta_N^{\frac{1}{2}}-1\le 0. \tag{3.52}$$

Therefore Equation (3.49) follows from Equation (3.52). From Equation (3.41) we conclude that $\mu_N>0$, Equation (3.48) follows from Equation (3.41), Equation (3.46) follows from Equation (3.48), Equation (3.46) follows from Equation (3.49), and from Equation (3.46) and Equation (3.47) we have $l_1 < l$. The last we need to prove is $K_0 < \frac{1}{2}$, from **Lemma 3.5**, we obtain

$$K_0 = M_3\left(6\lambda_{N+1}^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}} + \lambda_N^{-\frac{1}{2}}l_1\right) < \frac{1}{2}, \tag{3.53}$$

we notice that $l_1 < l, \lambda_{N+1}^{\frac{1}{2}} \ge \lambda_N^{\frac{1}{2}}, K_0 < M_3\left(6\mathrm{e}^{-\frac{1}{2}} + l\right)\lambda_N^{-\frac{1}{2}} < \frac{1}{2}$. **Lemma 3.6** is proved.

From **Lemma 3.1** to **Lemma 3.6**,we can obtain the following conclusions.

Theorem 3.1. Suppose that $F_{b,l}^{\frac{1}{2}}(b>0, l>0)$ is Lipschitz mapping space. $\Phi \in F_{b,l}^{\frac{1}{2}}$, $\Phi : P_N V_1 \to Q_N V_1$ satisfy Equation (3.1) and Equation (3.2), $p_0 \in P_N V_1$ and $q(0;\Phi, p_0) \in Q_N V_1$ is the unique solution of Equation (3.3) and Equation (3.4) for $t = 0$, respectively. Hence the transformation $T : F_{b,l}^{\frac{1}{2}} \to F_{b,l}^{\frac{1}{2}}$ is a contraction, and T exists a unique fixed point $\Phi_0 \in F_{b,l}^{\frac{1}{2}}$, $M = Graph(\Phi_0)$ is inertial manifolds of the problem (2.1).

Theorem 3.2. Suppose that $M = Graph(\Phi_0)$ is the mapping of Φ_0, for any $U_0 \in V_1$, there exists $t_0 > 0$ such that, for $t \ge t_0$,

$$dist(S(t)U_0, M) \le dist(U_0, M)\exp\left(-\frac{\ln 2}{2t_0}t\right), \tag{3.54}$$

where $t_0 = \min\left\{\frac{\ln 2}{C^2}, \frac{T}{2}\right\}$, C is defined as in **Lemma 2.3**.

Proof. Let U_1, U_2 with initial value $U_1(0), U_2(0) \in V_1$, respectively, be two solutions of the problem (2.1). For any arbitrary N and for $t \in [0, T]$, and use the fact $\|U_1\|_{V_1} \le M_1, \|U_2\|_{V_1} \le M_1$, there exists a constant $\zeta > 0$ such that Equation (2.10) or Equation (2.11) is satisfied. From Equation (2.12), we have

$$\|U_1(t) - U_2(t)\| \le 2\|U_1(0) - U_2(0)\|, \quad t < 2t_0. \tag{3.55}$$

Assume $\zeta = \frac{1}{8}$, and for $N > N_0, \lambda_{N_0+1} \ge \frac{\ln(2C_4)}{C_5 t_0}$, therefore Equation (2.10) and Equation (2.11) can rewrite

$$\|Q_N(U_1(t) - U_2(t))\| \le \frac{1}{8}\|P_N(U_1(t) - U_2(t))\|, \tag{3.56}$$

$$\|U_1(t) - U_2(t)\| \le \frac{1}{2}\|U_1(0) - U_2(0)\|, \tag{3.57}$$

Let $U_1(0), U_2(0) \in V_1, t_0 \le t \le 2t_0, B_\rho \subset V_1$ is absorbing set, the orbital solution $U(t)$ satisfies $\left\|A^{\frac{1}{2}}U(t)\right\| \le \rho, t \in [0, +\infty)$. Let $U_2(0) = U_{02} \in M, U_{02} = P_N U_{02} + \Phi_0(P_N U_{02})$ such that

$$dist(U_0, M) = \|U_1(0) - U_2(0)\|. \tag{3.58}$$

Substituting $S(t_1)U_1(0)$ and $S(t_1)U_2(0)$ into Equation (3.56) and Equation (3.57), we have

$$\begin{aligned} dist\left(S\left(t_1\right)U_0, M\right) &= \inf_{U_1 \in M}\left\|S\left(t_1\right)U_1\left(0\right) - U_2\right\| \le \left\|S\left(t_1\right)U_1\left(0\right) - S\left(t_1\right)U_2\left(0\right)\right\| \\ &\le \frac{1}{2}\left\|U_1\left(0\right) - U_2\left(0\right)\right\| = \frac{1}{2}dist\left(U_0, M\right). \end{aligned} \tag{3.59}$$

If Equation (3.56) is satisfied, assume $l = \frac{1}{8}, t_0 \le t_1 \le 2t_0$, so we have the cone property

$$\begin{aligned} dist\left(S\left(t_1\right)U_0, M\right) &= \inf_{U_1 \in M}\left\|S\left(t_1\right)U_1\left(0\right) - \left(P_N S\left(t_1\right)U_2\left(0\right) + \Phi\left(P_N S\left(t_1\right)U_2\left(0\right)\right)\right)\right\| \\ &\le \left\|Q_N S\left(t_1\right)U_1\left(0\right) - \Phi\left(P_N S\left(t_1\right)U_2\left(0\right)\right)\right\| \\ &\le \frac{1}{8}\left\|P_N\left(S\left(t_1\right)U_1\left(0\right) - S\left(t_1\right)U_2\left(0\right)\right)\right\| \\ &\le \frac{1}{2}\left\|U_1\left(0\right) - U_2\left(0\right)\right\| = \frac{1}{2}dist\left(U_0, M\right). \end{aligned} \tag{3.60}$$

In a word, for $t_0 \le t_1 \le 2t_0$, whenever $dist\left(S\left(t_1\right)U_0, M\right) \le \frac{1}{2}dist\left(U_0, M\right)$. By the properties of semigroups, for $t_0 \le t_1 \le 2t_0$, we have

$$\begin{aligned} dist\left(S\left(nt_1\right)U_0, M\right) &\le \left(\frac{1}{2}\right)^n dist\left(U_0, M\right) \le \exp\left(-\frac{t\ln 2}{t_1}\right)dist\left(U_0, M\right) \\ &\le \exp\left(-\frac{t\ln 2}{2t_0}\right)dist\left(U_0, M\right) \to 0\left(n \to \infty, t \ge t_0\right). \end{aligned} \tag{3.61}$$

Theorem 3.2 is proved. □

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