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Nonlinear General Integral Control Design via Equal Ratio Gain Technique

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Abstract

This paper proposes two kinds of nonlinear general integral controllers, that is, one is generic and another is practical, for a class of uncertain nonlinear system. By extending equal ratio gain technique to a canonical interval system matrix and using Lyapunov method, theorems to ensure regionally as well as semi-globally asymptotic stability are established in terms of some bounded information. Moreover, for the practical nonlinear integral controller, a real time method to evaluate the equal ratio coefficient is proposed such that its value can be chosen moderately. Theoretical analysis and simulation results demonstrated that not only nonlinear general integral control can effectively deal with the uncertain nonlinear system but also equal ratio gain technique is a powerful and practical tool to solve the control design problem of dynamics with the nonlinear and uncertain actions.

Keywords

General Integral Control, Nonlinear Control, Robust Control, Generic Controller, Equal Ratio Gain Technique, Output Regulation

1. Introduction

Integral control [1] plays an important role in practice because it ensures asymptotic tracking and disturbance rejection when exogenous signals are constants or planting parametric uncertainties appears. However, nonlinear general integral control design is not trivial matter because it depends on not only the uncertain nonlinear actions and disturbances but also the nonlinear control actions. Therefore, it is of important significance to develop the design method for nonlinear general integral control.

For general integral control design, there were various design methods, such as general integral control design based on linear system theory, sliding mode technique, feedback linearization technique and singular perturba-

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tion technique and so on, were presented by [2]-[5], respectively. In addition, general concave integral control [6], general convex integral control [7], constructive general bounded integral control [8] and the generalization of the integrator and integral control action [9] were all developed by using Lyapunov method and resorting to a known stable control law. Equal ratio gain technique firstly was proposed by [10], and was used to address the linear general integral control design for a class of uncertain nonlinear system.

All these general integral controllers above constitute only a minute portion of general integral control, and therefore lack generalization. Moreover, in consideration of the complexity of nonlinear system, it is clear that we can not expect that a particular integral controller has the high control performance for all nonlinear system. Thus, the generalization of general integral controller naturally appears since for all nonlinear system, we can not enumerate all the categories of integral controllers with high control performance. It is not hard to know that this is a very valuable and challenging problem, and equal ratio gain technique can be used to deal with this trouble since it is a powerful and practical tool to solve the nonlinear control design problem.

Motivated by the cognition above, this paper proposes a generic nonlinear integral controller and a practical nonlinear integral controller for a class of uncertain nonlinear system. The main contributions are that: 1) By defining two function sets, the generalization of general integral controller is achieved; 2) A canonical interval system matrix can be designed to be Hurwitz as any row controller gains, or controller and its integrator gains increase with the same ratio; 3) Theorems to ensure regionally as well as semi-globally asymptotic stability is established in terms of some bounded information. Moreover, for the practical nonlinear integral controller, a real time method to evaluate the equal ratio coefficient is proposed such that its value can be chosen moderately. Theoretical analysis and simulation results demonstrated that not only nonlinear general integral control can effectively deal with the uncertain nonlinear system but also equal ratio gain technique is a powerful and practical tool to solve the control design problem of dynamics with the nonlinear and uncertain actions.

Throughout this paper, we use the notation $\lambda_m(A)$ and $\lambda_M(A)$ to indicate the smallest and largest eigenvalues, respectively, of a symmetric positive define bounded matrix $A(x)$, for any $x \in R^n$. The norm of vector *x* is defined as $\|x\| = \sqrt{x^{\mathrm{T}} x}$, and that of matrix *A* is defined as the corresponding induced norm $\|A\| = \sqrt{\lambda_M\left(A^{\mathrm{T}} A\right)}$.

The remainder of the paper is organized as follows: Section 2 describes the system under consideration, assumption and definition. Sections 3 and 4 present the generic and practical nonlinear integral controllers along with their design method, respectively. Example and simulation are provided in Section 5. Conclusions are presented in Section 6.

2. Problem Formulation

Consider the following controllable nonlinear system,

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \quad \vdots \\ \dot{x}_n = f(x,w) + g(x,w)u \end{cases} \tag{1}$$

where $x \in R^n$ is the state; $u \in R$ is the control input; $w \in R^l$ is a vector of unknown constant parameters and disturbances. The function $f(x,w)$ is the uncertain nonlinear action, and the uncertain nonlinear function $g(x,w)$ is continuous in (x,w) on the control domain $D_x \times D_w \subset R^n \times R^l$. We want to design a control law u such that $x(t) \to 0$ as $t \to \infty$.

Assumption 1: There is a unique pair $(0,u_0)$ that satisfies the equation,

$$0 = f(0,w) + g(0,w)u_0 \tag{2}$$

so that $x = 0$ is the desired equilibrium point and u_0 is the steady-state control that is needed to maintain equilibrium at $x = 0$.

Assumption 2: Suppose that the functions $f(x,w)$ and $g(x,w)$ satisfy the following inequalities,

$$\|f(x,w) - f(0,w)\| \le l_f^x \|x\| \tag{3}$$

$$0 < g_m < g(x,w) < g_M \tag{4}$$

$$\|g(x,w)-g(0,w)\| \le l_g^x \|x\| \tag{5}$$

$$\left\| f(0,w) g^{-1}(0,w) \right\| \le \gamma_g^f \tag{6}$$

for all $x \in D_x$ and $w \in D_w$. where l_f^x, l_g^x, g_m, g_M and γ_g^f are all positive constants.

Definition 1: $F_u(a_u, b_u, x, y)$ with $0 < a_u < b_u$, $x \in D_x \subset R^n$ and $y, y_0 \in R$ denotes the set of all continuous functions, $u(x,y)$ such that

$$u(x,y)-u(0,y_0)=\left.\frac{\partial u(x,y)}{\partial x}\right|_{(x,y)=(z_x,z_y)} x+\left.\frac{\partial u(x,y)}{\partial y}\right|_{(x,y)=(z_x,z_y)} (y-y_0)$$

and

$$0<a_u<\frac{\partial u(x,y)}{\partial x_i}, \frac{\partial u(x,y)}{\partial y}<b_u \quad (i=1,2,\cdots,n)$$

hold for all $x \in D_x$ and $y, y_0 \in R$. Where (z_x, z_y) is a point on the line segment connecting (x,y) to $(0,y_0)$.

Definition 2: $F_v(a_v, b_v, x)$ with $0 < a_v < b_v$, and $x \in D_x \subset R^n$ denotes the set of all integrable functions, $v(x)$ such that

$$v(x)=\left.\frac{\partial v(x)}{\partial x}\right|_{x=z} x$$

and

$$0<a_v<\frac{\partial v(x)}{\partial x_i}<b_v \quad (i=1,2,\cdots,n)$$

hold for all $x \in D_x$. Where z is a point on the line segment connecting x to the origin.

3. Generic Nonlinear Integral Control

The generic nonlinear integral controller is given as,

$$\begin{cases} u=-\varepsilon_\alpha^{-1} u(x,\sigma) \\ \dot{\sigma}=\varepsilon_\beta^{-1} \mu(\sigma) v(x) \end{cases} \tag{7}$$

where $u(x,\sigma)$ and $v(x)$ belong to the function sets F_u and F_v, respectively, $0<\mu_m<\mu(\sigma)<\mu_M$, ε_α and ε_β are all positive constants.

Thus, substituting (7) into (1), obtain the augmented system,

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \quad \vdots \\ \dot{x}_n = f(x,w) - g(x,w)\varepsilon_\alpha^{-1} u(x,\sigma) \\ \dot{\sigma} = \varepsilon_\beta^{-1} \mu(\sigma) v(x) \end{cases} \tag{8}$$

By Assumption 1 and choosing ε_α^{-1} to be large enough, and then setting $\dot{x}=0$ and $x=0$ of the system (8), obtain,

$$g(0,w)\varepsilon_\alpha^{-1} u(0,\sigma_0) = f(0,w) \tag{9}$$

Therefore, we ensure that there is a unique solution σ_0, and then $(0,\sigma_0)$ is a unique equilibrium point of the closed-loop system (8) in the domain of interest. At the equilibrium point, $x=0$, irrespective of the value of w.

Now, by Definition 1, 2 and $0<\mu_m<\mu(\sigma)<\mu_M$, $u(x,\sigma)$ and $\mu(\sigma)v(x)$ can be written as,

$$u(x,\sigma)-u(0,\sigma_0)=\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n+\alpha_{n+1}(\sigma-\sigma_0) \tag{10}$$

$$\mu(\sigma)v(x)=\beta_1x_1+\beta_2x_2+\cdots+\beta_nx_n \tag{11}$$

where $0<\alpha_i^m\le\alpha_i\le\alpha_i^M$ $(i=1,2,\cdots,n+1)$ and $0<\beta_j^m\le\beta_j\le\beta_j^M$ $(j=1,2,\cdots,n)$.

Thus, substituting (9)-(11) into (8), obtain,

$$\dot{\eta}=A\eta+F(x,w) \tag{12}$$

where $\eta=\begin{bmatrix}x^{\mathrm{T}} & \sigma-\sigma_0\end{bmatrix}^{\mathrm{T}}$

$$A=\begin{bmatrix}0 & 1 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 1 & 0\\ -\varepsilon_\alpha^{-1}\alpha_1 & -\varepsilon_\alpha^{-1}\alpha_2 & \cdots & -\varepsilon_\alpha^{-1}\alpha_n & -\varepsilon_\alpha^{-1}\alpha_{n+1}\\ \varepsilon_\beta^{-1}\beta_1 & \varepsilon_\beta^{-1}\beta_2 & \cdots & \varepsilon_\beta^{-1}\beta_n & 0\end{bmatrix}$$

And $F(x,w)$ is an $n+1\times1$ matrix, all its elements are equal to zero except for

$$f_{n1}=f(x,w)-f(0,w)-\big(g(x,w)-g(0,w)\big)f(0,w)g^{-1}(0,w)$$

Moreover, it is worthy to note that the function $g(x,w)$ is integrated into ε_α via a change of variable. This has not influence on the results if the inequality (4) holds and it can be taken as g_m in the design. Therefore, it is omitted in all the following demonstrations.

For analyzing the stability of closed-loop system (12), we must ensure that the matrix A is Hurwitz for all $0<\alpha_i^m\le\alpha_i\le\alpha_i^M$, $0<\beta_j^m\le\beta_j\le\beta_j^M$, $0<\varepsilon_\alpha<\varepsilon_\alpha^*$ and $0<\varepsilon_\beta<\varepsilon_\beta^*$. This can be achieved by Routh's stability criterion.

3.1. Hurwitz Stability

Hurwitz stability of the matrix A can be achieved by Routh's stability criterion, which is motivated by the idea [10], as follows:

Step 1: the polynomial of the matrix A with $\varepsilon_\alpha=\varepsilon_\beta=1$ is,

$$s^{n+1}+\alpha_ns^n+(\alpha_{n+1}\beta_n+\alpha_{n-1})s^{n-1}+\cdots+(\alpha_{n+1}\beta_2+\alpha_1)s+\alpha_{n+1}\beta_1=0 \tag{13}$$

By Routh's stability criterion, α_i and β_j can be chosen such that the polynomial (13) is Hurwitz for all $0<\alpha_i^m\le\alpha_i\le\alpha_i^M$ and $0<\beta_j^m\le\beta_j\le\beta_j^M$. Obviously, if α_i and β_j are all large to zero, and then the necessary condition, that is, the coefficients of the polynomial (13) are all positive, is naturally satisfied.

Step 2: based on the gains α_i, β_j and Hurwitz stability condition to be obtained by Step 1, the maximums of ε_α and ε_β, that is, ε_α^* and ε_β^*, can be obtained, respectively. Since ε_α and ε_β interact, there exist innumerable ε_α^* and ε_β^*. Thus, two kinds of typical cases are interesting, that is, one is that ε_α^* is evaluated with $\varepsilon_\beta=1$; another is that let $\varepsilon_\alpha=\varepsilon_\beta$, and then $\varepsilon_\alpha^*=\varepsilon_\beta^*$ can be obtained together.

Step 3: by ε_α^* and ε_β^* obtained by Step 2, check Hurwitz stability of the matrix A for all $0<\varepsilon_\alpha<\varepsilon_\alpha^*$ and $0<\varepsilon_\beta<\varepsilon_\beta^*$. If it does not hold, redesign α_i and β_j and repeat the previous steps until the matrix A is Hurwitz for all $0<\varepsilon_\alpha<\varepsilon_\alpha^*$ and $0<\varepsilon_\beta<\varepsilon_\beta^*$.

It is well known that Hurwitz stability condition is more and more complex as the order of the matrix A increases. Thus, for clearly illustrating the design method above, we consider two kinds of cases, that is, $n=2$ and $n=3$, respectively, as follows:

Case 1: for $n=2$, the polynomial (13) is,

$$s^3+\alpha_2s^2+(\alpha_3\beta_2+\alpha_1)s+\alpha_3\beta_1=0 \tag{14}$$

By Routh's stability criterion, if α_3, α_2, α_1, β_2, and β_1 are all positive numbers, and the following inequality,

$$\alpha_2^m\left(\alpha_3^m\beta_2^m+\alpha_1^m\right)>\alpha_3^M\beta_1^M \tag{15}$$

holds, and then the polynomial (14) is Hurwitz for all $0<\alpha_i^m\le\alpha_i\le\alpha_i^M$ and $0<\beta_j^m\le\beta_j\le\beta_j^M$.

***Sub-class* 1:** α_3, α_2 and α_1 are multiplied by ε_α^{-1}, and then substituting them into (15), obtain,

$$\alpha_2^m\left(\alpha_3^m\beta_2^m+\alpha_1^m\right)>\varepsilon_\alpha\alpha_3^M\beta_1^M \tag{16}$$

By the inequality (16), obtain,

$$\varepsilon_\alpha^*=\frac{\alpha_2^m\left(\alpha_3^m\beta_2^m+\alpha_1^m\right)}{\alpha_3^M\beta_1^M}$$

***Sub-class* 2:** α_3, α_2, α_1, β_2 and β_1 are multiplied by ε_α^{-1}, and then substituting them into (15), obtain,

$$\alpha_2^m\alpha_3^m\beta_2^m>\varepsilon_\alpha\left(\alpha_3^M\beta_1^M-\alpha_2^m\alpha_1^m\right) \tag{17}$$

For this sub-class, there are two kinds of cases:

1) if $\alpha_3^M\beta_1^M-\alpha_2^m\alpha_1^m>0$, and then by the inequality (17), obtain,

$$\varepsilon_\alpha^*=\frac{\alpha_2^m\alpha_3^m\beta_2^m}{\alpha_3^M\beta_1^M-\alpha_2^m\alpha_1^m}$$

2) if $\alpha_3^M\beta_1^M-\alpha_2^m\alpha_1^m\le 0$, and then by the inequality (17), obtain,

$$\varepsilon_\alpha^*=\infty$$

Case 2: for $n=3$, the polynomial (13) is,

$$s^4+\alpha_3s^3+\left(\alpha_4\beta_3+\alpha_2\right)s^2+\left(\alpha_4\beta_2+\alpha_1\right)s+\alpha_4\beta_1=0 \tag{18}$$

By Routh's stability criterion, if α_4, α_3, α_2, α_1, β_3, β_2 and β_1 are all positive numbers, and the following inequality,

$$\alpha_3^m\left(\alpha_4^m\beta_3^m+\alpha_2^m\right)\left(\alpha_4^m\beta_2^m+\alpha_1^m\right)>\left(\alpha_4^M\beta_2^M+\alpha_1^M\right)^2+\alpha_3^M\alpha_3^M\alpha_4^M\beta_1^M \tag{19}$$

holds, and then the polynomial (18) is Hurwitz for all $0<\alpha_i^m\le\alpha_i\le\alpha_i^M$ and $0<\beta_j^m\le\beta_j\le\beta_j^M$.

***Sub-class* 1:** α_4, α_3, α_2 and α_1 are multiplied by ε_α^{-1}, and then substituting them into (19), obtain,

$$\alpha_3^m\left(\alpha_4^m\beta_3^m+\alpha_2^m\right)\left(\alpha_4^m\beta_2^m+\alpha_1^m\right)-\alpha_3^M\alpha_3^M\alpha_4^M\beta_1^M>\varepsilon_\alpha\left(\alpha_4^M\beta_2^M+\alpha_1^M\right)^2 \tag{20}$$

By the inequality (20), obtain,

$$\varepsilon_\alpha^*=\frac{\alpha_3^m\left(\alpha_4^m\beta_3^m+\alpha_2^m\right)\left(\alpha_4^m\beta_2^m+\alpha_1^m\right)-\alpha_3^M\alpha_3^M\alpha_4^M\beta_1^M}{\left(\alpha_4^M\beta_2^M+\alpha_1^M\right)^2}$$

***Sub-class* 2:** α_4, α_3, α_2, α_1, β_3, β_2 and β_1 are multiplied by ε_α^{-1}, and then substituting them into (19), obtain,

$$\alpha_3^m\left(\alpha_4^m\beta_3^m+\varepsilon_\alpha\alpha_2^m\right)\left(\alpha_4^m\beta_2^m+\varepsilon_\alpha\alpha_1^m\right)>\varepsilon_\alpha\left(\alpha_4^M\beta_2^M+\varepsilon_\alpha\alpha_1^M\right)^2+\varepsilon_\alpha\alpha_3^M\alpha_3^M\alpha_4^M\beta_1^M$$

For this sub-class, although the situation is complex, a moderate solution can still be obtained, that is,

$$\varepsilon_\alpha^*=\frac{\alpha_3^m\alpha_4^m\beta_3^m\alpha_4^m\beta_2^m}{\left(\alpha_4^M\beta_2^M+\alpha_1^M\right)^2+\alpha_3^M\alpha_3^M\alpha_4^M\beta_1^M}$$

From the demonstration above, it is obvious that for $n=2$, $n=3$ and $\varepsilon_\beta=1$ or $\varepsilon_\beta=\varepsilon_\alpha$ of the matrix A, there all exist ε_α^* such that the matrix A is Hurwitz for all $\alpha_i^m\le\alpha_i\le\alpha_i^M$, $\beta_j^m\le\beta_j\le\beta_j^M$ and $0<\varepsilon_\alpha<\varepsilon_\alpha^*$. Therefore, for the high order matrix A, the same result can be still obtained with the help of computer. Thus, we can conclude that the n+1-order matrix A can be designed to be Hurwitz for all $\alpha_i^m\le\alpha_i\le\alpha_i^M$, $\beta_j^m\le\beta_j\le\beta_j^M$, $0<\varepsilon_\alpha<\varepsilon_\alpha^*$ and $0<\varepsilon_\beta<\varepsilon_\beta^*$.

Theorem 1: There exist $0<\alpha_i^m \le \alpha_i \le \alpha_i^M$ and $0<\beta_j^m \le \beta_j \le \beta_j^M$ such that the system matrix A for $\varepsilon_\alpha = \varepsilon_\beta = 1$ is Hurwitz, and then it is still Hurwitz for all $0<\varepsilon_\alpha<\varepsilon_\alpha^*$ and $0<\varepsilon_\beta<\varepsilon_\beta^*$.

Discussion 1: From the statements above, it is easy to see that: 1) the system matrix A is an interval matrix; 2) in consideration of the controllable canonical form of linear system, the system matrix A can be called as the controllable canonical interval system matrix; 3) although Theorem 1 is demonstrated by the single variable system matrix A, it is easy to extend Theorem 1 to the multiple variable case since Routh's stability criterion applies to not only the single variable system matrix but also the multiple variable one. Thus, the following proposition can be established.

Proposition 1: A canonical interval system matrix can be designed to be Hurwitz as any row controller gains, or controller and its integrator gains increase with the same ratio.

3.2. Closed-Loop Stability Analysis

The matrix A can be designed to be Hurwitz for all $0<\alpha_i^m \le \alpha_i \le \alpha_i^M$, $0<\beta_j^m \le \beta_j \le \beta_j^M$, $0<\varepsilon_\alpha<\varepsilon_\alpha^*$ and $0<\varepsilon_\beta<\varepsilon_\beta^*$. Thus, by linear system theory, if the matrix A is Hurwitz, and then for any given positive define symmetric matrix Q, there exists positive define symmetric matrix P that satisfies Lyapunov equation $PA + A^{\mathrm{T}}P = -Q$. Therefore, the solution of Lyapunov equation [11] is,

$$P = 0.5(S-Q)A^{-1} \tag{21}$$

where

$$S = PA - A^{\mathrm{T}}P,\quad S^{\mathrm{T}} = -S \quad \text{and} \quad A^{\mathrm{T}}S + SA = A^{\mathrm{T}}Q - QA$$

Thus, using $V(\eta) = \eta^{\mathrm{T}}P\eta$ as Lyapunov function candidate, and then its time derivative along the trajectories of the closed-loop systems (12) is,

$$\dot{V}(\eta) = \eta^{\mathrm{T}}\left(PA + A^{\mathrm{T}}P\right)\eta + \frac{\partial V(\eta)}{\partial \eta}F(x,w) = -\eta^{\mathrm{T}}Q\eta + 2P_n\eta f_{n1} \tag{22}$$

where $P_n = \left[p_{n1} \quad p_{n2} \quad \cdots \quad p_{n,n+1}\right]$.

Now, using the inequalities (3), (5) and (6), obtain,

$$\|f_{n1}\| \le \kappa_f^x \|x\| \tag{23}$$

where κ_f^x is a positive constant.

Substituting (23) into (22), and using $\|x\| \le \|\eta\|$, obtain,

$$\dot{V}(\eta) \le -\left(\lambda_m(Q) - 2\kappa_f^x\|P_n\|\right)\|\eta\|^2 \tag{24}$$

By proposition proposed by [10], that is, as any row controller gains, or controller and its integrator gains of a canonical system matrix tend to infinity with the same ratio, if it is always Hurwitz, and then the same row solutions of Lyapunov equation all tend to zero, we have,

1) $\|P_n\| = \left\|P_n^{\varepsilon\alpha}\right\|\varepsilon_\alpha \to 0 \quad \forall \varepsilon_\beta \in \left(0, \varepsilon_\beta^*\right)$ as $\varepsilon_\alpha \to 0$

2) $\|P_n\| = \left\|P_n^{\varepsilon\alpha\beta}\right\|\varepsilon_\alpha \to 0$ as $\varepsilon_\beta = \varepsilon_\alpha \to 0$

where $P_n = P_n^{\varepsilon\alpha}\varepsilon_\alpha = P_n^{\varepsilon\alpha\beta}\varepsilon_\alpha = \left[p_{n1} \quad p_{n2} \quad \cdots \quad p_{n,n+1}\right]$.

Although there is innumerable P_n, the maximum $\|P_n\|_M$ exists and $\|P_n\|_M \to 0$ as $\varepsilon_\alpha \to 0$. Thus, there exist ε_α^{**} with $0<\varepsilon_\beta<\varepsilon_\beta^*$, or $\varepsilon_\alpha^{**} = \varepsilon_\beta^{**}$ such that the following inequality,

$$\lambda_m(Q) > 2\kappa_f^x\|P_n\|_M \tag{25}$$

holds for all $0<\varepsilon_\alpha<\varepsilon_\alpha^{**}$ with $0<\varepsilon_\beta<\varepsilon_\beta^*$, or $0<\varepsilon_\beta=\varepsilon_\alpha<\varepsilon_\alpha^{**}=\varepsilon_\beta^{**}$. Therefore, we have $\dot{V}(\eta) \le 0$.

Using the fact that Lyapunov function $V(\eta)$ is a positive define function and its time derivative is a negative define function if the inequality (25) holds, we conclude that the closed-loop system (12) is stable. In fact, $\dot{V}(\eta) = 0$ means $x = 0$ and $\sigma = \sigma_0$. By invoking LaSalle's invariance principle, it is easy to know that the closed-loop system (12) is exponentially stable. As a result, the following theorem can be established.

Theorem 2: Under Assumptions 1 and 2, if the system matrix A is Hurwitz for all

$$0<\alpha_i^m \le \alpha_i \le \alpha_i^M,\quad 0<\beta_j^m \le \beta_j \le \beta_j^M,$$

$$0<\varepsilon_\alpha<\varepsilon_\alpha^* \text{ and } 0<\varepsilon_\beta<\varepsilon_\beta^*,$$

and then the equilibrium points $x=0$ and $\sigma=\sigma_0$ of the closed-loop system (12) is an exponentially stable for all

$$0<\varepsilon_\alpha<\varepsilon_\alpha^{**}\le\varepsilon_\alpha^* \text{ with } 0<\varepsilon_\beta<\varepsilon_\beta^*, \text{ or } 0<\varepsilon_\beta=\varepsilon_\alpha<\varepsilon_\alpha^{**}=\varepsilon_\beta^{**}\le\varepsilon_\alpha^*=\varepsilon_\beta^*.$$

Moreover, if all assumptions hold globally, then it is globally exponentially stable.

Remark 1: From the statements of Subsections 3.1 and 3.2, it is to see that: by extending equal ratio gain technique to a canonical interval system matrix and using Lyapunov method, the asymptotic stability of the uncertain nonlinear system with generic nonlinear integral control can be ensured in terms of some bounded information. This shows that not only nonlinear general integral control can effectively deal with the uncertain nonlinear system but also equal ratio gain technique is a powerful tool to solve the control design problem of dynamics with the nonlinear and uncertain actions.

Discussion 2: From the statements above, it is obvious that: although the generalization of nonlinear general integral control is achieved by defining two function sets, there are two unavoidable drawbacks, that is, one is that the controller (7) is too generic such that it is shortage of pertinence; another is that it is difficulty to obtain the less conservative ε_α^{**} or $\varepsilon_\alpha^{**}=\varepsilon_\beta^{**}$ such that it is shortage of practicability. All these mean that Theorem 2 has only theoretical significance and not practical significance. Therefore, a practical nonlinear integral controller along with a new design method is proposed to solve these troubles in the next Section.

4. Practical Nonlinear Integral Control

For making up the shortage indicated by Discussion 2, a practical nonlinear integral controller is given as,

$$\begin{cases} u=-\varepsilon_\alpha^{-1}\left(u_1(x_1)+u_2(x_2)+\cdots+u_n(x_n)+\alpha_\sigma\sigma\right)-\phi(x)-\varphi(\sigma) \\ \dot{\sigma}=\varepsilon_\beta^{-1}\mu(\sigma)\left(v_1(x_1)+v_2(x_2)+\cdots+v_n(x_n)\right) \end{cases} \tag{26}$$

where $u_i(x_i)=\alpha_i(x_i)x_i\ \left(0<\alpha_i^m\le\alpha_i(x_i)\le\alpha_i^M\right)$, $v_i(x_i)=\beta_i(x_i)x_i\ \left(0<\beta_i^m\le\beta_i(x_i)\le\beta_i^M\right)$, $\alpha_i(x_i)$ and $\beta_i(x_i)$ are the slopes of the line segment connecting x_i to the origin $(i=1,2,\cdots,n)$, and they are utilized to harmonize the actions of x_i in the controller and integrator, respectively. $\phi(x)$ $(\phi(0)=0)$ is used to attenuate the uncertain nonlinear action of $f(x,w)-f(0,w)$. $\varphi(\sigma)$ $(\varphi(0)=0)$ is applied to improve the performance of integral control action. $\mu(\sigma)$ $\left(0<\mu_m<\mu(\sigma)<\mu_M\right)$ is used to reorganize the integrator output. α_σ, ε_α and ε_β are all positive constants, and $0<\alpha_\sigma^m<\alpha_\sigma+\mathrm{d}\varphi(\sigma)/\mathrm{d}\sigma\le\alpha_\sigma^M$.

Assumptions 3: By the definition of the controller (26), it is convenient to suppose that the following inequalities,

$$\left\|f(x,w)-f(0,w)-g(x,w)\varphi(x)\right\|\le l_{f\varphi}^x\|x\| \tag{27}$$

$$\left\|\varphi(\sigma)-\varphi(\sigma_0)\right\|\le l_\varphi^\sigma\|\sigma-\sigma_0\| \tag{28}$$

hold for all $x\in D_x$, $w\in D_w$ and $\sigma,\sigma_0\in R$, where $l_{f\varphi}^x$ and l_φ^σ are all positive constants.

By the same way as Section 3, we have,

$$\dot{\eta}=A\eta+F(x,\sigma-\sigma_0,w) \tag{29}$$

where $\eta=\left[x^{\mathrm{T}}\quad \sigma-\sigma_0\right]^{\mathrm{T}}$,

$$A=\begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ -\varepsilon_\alpha^{-1}\alpha_1 & -\varepsilon_\alpha^{-1}\alpha_2 & \cdots & -\varepsilon_\alpha^{-1}\alpha_n & -\varepsilon_\alpha^{-1}\alpha_\sigma \\ \varepsilon_\beta^{-1}\beta_1 & \varepsilon_\beta^{-1}\beta_2 & \cdots & \varepsilon_\beta^{-1}\beta_n & 0 \end{bmatrix}$$

$F(x,\sigma-\sigma_0,w)$ is an $n+1\times 1$ matrix, all its elements are equal to zero except for

$$f_{n1} = f(x,w) - f(0,w) - g(x,w)\phi(x) - (g(x,w) - g(0,w)) \times f(0,w) g^{-1}(0,w) - g(x,w)(\varphi(\sigma) - \varphi(\sigma_0))$$

and the functions $g(x,w)$ and $\mu(\sigma)$ are integrated into ε_α and β_i, respectively.

By the design method proposed by Subsection 3.1, the system matrix A can be designed to be Hurwitz for all $0<\alpha_i^m \le \alpha_i(x_i) \le \alpha_i^M$, $0<\beta_i^m \le \beta_i(x_i) \le \beta_i^M$, α_σ, $0<\varepsilon_\alpha<\varepsilon_\alpha^*$ and $0<\varepsilon_\beta<\varepsilon_\beta^*$ $(i=1,2,\cdots,n)$. Thus, by linear system theory, there exists positive define symmetric matrix P that satisfies Lyapunov equation $PA+A^{\mathrm{T}}P=-Q$ for any given positive define symmetric matrix Q. Therefore, we can utilize $V(\eta)=\eta^{\mathrm{T}}P\eta$ as Lyapunov function candidate, and then its time derivative along the trajectories of the closed-loop system (29) is,

$$\dot{V}(\eta) = \eta^{\mathrm{T}}(PA+A^{\mathrm{T}}P)\eta + \frac{\partial V(\eta)}{\partial \eta}F(x,\sigma-\sigma_0,w) = -\eta^{\mathrm{T}}Q\eta + 2P_n\eta f_{n1} \tag{30}$$

where $P_n = \begin{bmatrix} p_{n1} & p_{n2} & \cdots & p_{n,n+1}\end{bmatrix}$.

Now, using the inequalities (4), (5), (6), (27) and (28), obtain,

$$\| f_{n1} \| \le \kappa_f^\eta \|\eta\| \tag{31}$$

where κ_f^η is a positive constant.

Substituting (31) into (30), obtain,

$$\dot{V}(\eta) \le -(\lambda_m(Q) - 2\kappa_f^\eta \|P_n\|)\|\eta\|^2 \tag{32}$$

By proposition proposed by [10] (details see Subsection 3.2), for any moment t, there exists $\varepsilon_\alpha^{**}(t)$ with $0<\varepsilon_\beta<\varepsilon_\beta^*$, or $\varepsilon_\beta^{**}(t)=\varepsilon_\alpha^{**}(t)$ such that the inequality,

$$\lambda_m(Q) > 2\kappa_f^\eta \|P_n(t)\| \tag{33}$$

holds for all $0<\varepsilon_\alpha(t)<\varepsilon_\alpha^{**}(t)$ with $0<\varepsilon_\beta<\varepsilon_\beta^*$, or $0<\varepsilon_\beta(t)=\varepsilon_\alpha(t)<\varepsilon_\alpha^{**}(t)=\varepsilon_\beta^{**}(t)$. Consequently, if the inequality (33) holds for all $t\in[0,\infty)$, and then we conclude that $\dot{V}(\eta)\le 0$ holds uniformly in t.

Using the fact that Lyapunov function $V(\eta)$ is a positive define function and its time derivative is a negative define function if the inequality (33) holds for all $t\in[0,\infty)$, we conclude that the closed-loop system (29) is stable. In fact, $\dot{V}(\eta)=0$ means $x=0$ and $\sigma=\sigma_0$. By invoking LaSalle's invariance principle, it is easy to know that the closed-loop system (29) is uniformly exponentially stable. As a result, we have the following theorem.

Theorem 3: Under Assumptions 1, 2 and 3, if the system matrix A is Hurwitz for all

$$0<\alpha_i^m \le \alpha_i(x_i) \le \alpha_i^M,\quad 0<\beta_i^m \le \beta_i(x_i) \le \beta_i^M,\quad \alpha_\sigma,\quad 0<\varepsilon_\alpha<\varepsilon_\alpha^* \text{ and } 0<\varepsilon_\beta<\varepsilon_\beta^*,$$

and then the equilibrium point $x=0$ and $\sigma=\sigma_0$ of the closed-loop system (29) is uniformly exponentially stable for all

$$0<\varepsilon_\alpha(t)<\varepsilon_\alpha^{**}(t)\le\varepsilon_\alpha^* \text{ with } 0<\varepsilon_\beta<\varepsilon_\beta^*, \text{ or } 0<\varepsilon_\beta(t)=\varepsilon_\alpha(t)<\varepsilon_\alpha^{**}(t)=\varepsilon_\beta^{**}(t)\le\varepsilon_\alpha^*=\varepsilon_\beta^*.$$

Moreover, if all assumptions hold globally, and then it is globally uniformly exponentially stable.

Now, the design task is to provide a method to evaluate the instantaneous value $\varepsilon_\alpha^{**}(t)$ with $0<\varepsilon_\beta<\varepsilon_\beta^*$, or $\varepsilon_\beta^{**}(t)=\varepsilon_\alpha^{**}(t)$. To achieve this objective, the procedure can be summarized as follows:

Firstly, by the definitions of $\alpha_i(x_i)$ and $\beta_i(x_i)$, the instantaneous values $\alpha_i(t)$ and $\beta_i(t)$ can be given as,

$$\begin{cases} \alpha_i(t) = \dfrac{u_i(x_i(t))}{x_i(t)} & \text{if } x_i(t)\ne 0 \\ \alpha_i(t) = \left.\dfrac{\mathrm{d}u_i(x_i(t))}{\mathrm{d}x_i(t)}\right|_{x_i(t)=0} & \text{if } x_i(t)=0 \end{cases}$$

and

$$\begin{cases} \beta_i(t) = \dfrac{v_i(x_i(t))}{x_i(t)} & \text{if } x_i(t) \neq 0 \\ \beta_i(t) = \left.\dfrac{\mathrm{d}v_i(x_i(t))}{\mathrm{d}x_i(t)}\right|_{x_i(t)=0} & \text{if } x_i(t) = 0 \end{cases}$$

Secondly, by the inequality (33), the impermissible minimum of $\|P_n(t)\|$ is,

$$\|P_n(t)\| = \frac{\lambda_m(Q)}{2\kappa_f^\eta}$$

Finally, by the limitation conditions,

$$0 < \varepsilon_\alpha^{**}(t) \le \varepsilon_\alpha^* \text{ with } 0 < \varepsilon_\beta < \varepsilon_\beta^*, \text{ or } 0 < \varepsilon_\alpha^{**}(t) = \varepsilon_\beta^{**}(t) \le \varepsilon_\alpha^* = \varepsilon_\beta^*$$

and the iterative method to solve Lyapunov equation,

$$P(t)A(t) + A^\mathrm{T}(t)P(t) = -Q$$

$\varepsilon_\alpha^{**}(t)$ with $0 < \varepsilon_\beta < \varepsilon_\beta^*$, or $\varepsilon_\beta^{**}(t) = \varepsilon_\alpha^{**}(t)$ can be obtained.

Discussion 3: From the statements above, it is easy to see that: 1) all the component of the nonlinear integral controller (26) have the clear actions; 2) κ_f^η is not greater than κ_f^x since $\phi(x)$ can be used to attenuate the uncertain nonlinear action of $f(x,w) - f(0,w)$ and $\varphi(\sigma)$ can be designed moderately; 3) $\varepsilon_\alpha^{**}(t) = \varepsilon_\beta^{**}(t)$ or $\varepsilon_\alpha^{**}(t)$ is less conservative since they are all evaluated by the instantaneous values $\alpha_i(t)$ and $\beta_i(t)$. All these not only solve the problem indicated by Discussion 2 but also mean that equal ratio gain technique is a powerful and practical tool to solve the control design problem of dynamics with the nonlinear and uncertain actions.

5. Example and Simulation

Consider the pendulum system [1] described by,

$$\ddot{\theta} = -a\sin(\theta) - b\dot{\theta} + cT$$

where $a, b, c > 0$, θ is the angle subtended by the rod and the vertical axis, and T is the torque applied to the pendulum. View T as the control input and suppose we want to regulate θ to δ. Now, taking $x_1 = \theta - \delta$, $\dot{x}_2 = \dot{\theta}$, the pendulum system can be written as,

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a\sin(x_1 + \delta) - bx_2 + cu \end{cases}$$

and then it can be verified that $u_0 = a\sin(\delta)/c$ is the steady-state control that is needed to maintain equilibrium at the origin.

By the practical nonlinear integral controller (26), the control law can be given as,

$$\begin{cases} u = -\varepsilon_\alpha^{-1}(2x_1 + 3\sinh(x_1) + 3x_2 + 4\tanh(x_2) + 8\sigma) - 0.3\tanh(\sigma) + \dfrac{4}{3}\sin(x_1) \\ \dot{\sigma} = \varepsilon_\beta^{-1}(3x_1 + \sinh(x_1) + x_2 + 2\tanh(x_2)) \end{cases}$$

Thus, it is easy to obtain $5 \le \alpha_1 < 11.1$, $3 \le \alpha_2 \le 7$, $\alpha_\sigma = 8$, $4 \le \beta_1 < 6.68$ and $1 < \beta_2 \le 3$, and then the closed-loop system can be written as,

$$\dot{\eta} = A\eta + F(x, \sigma - \sigma_0, w)$$

where $\eta = [x_1 \quad x_2 \quad \sigma - \sigma_0]^\mathrm{T}$,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -\varepsilon_\alpha^{-1}c\alpha_1 & -\varepsilon_\alpha^{-1}c(\alpha_2 + \varepsilon_\alpha c^{-1}b) & -\varepsilon_\alpha^{-1}c\alpha_\sigma \\ \varepsilon_\beta^{-1}\beta_1 & \varepsilon_\beta^{-1}\beta_2 & 0 \end{bmatrix}$$

and

$$F\left(x,\sigma-\sigma_0,w\right)=\left[0\quad a\sin(\delta)-a\sin(x_1+\delta)+\frac{4c}{3}\sin(x_1)\quad 0.3\left(\tanh(\sigma)-\tanh(\sigma_0)\right)\right]^{\mathrm{T}}$$

The normal parameters are $a=c=10$ and $b=2$, and in the perturbed case, b and c are reduced to 1 and 5, respectively, corresponding to double the mass. Thus, we have $\left\|F\left(x,\sigma-\sigma_0,w\right)\right\|\le 4.5\|\eta\|$.

Now, with $\alpha_1^m=5$, $\alpha_2^m=3$, $\alpha_\sigma=8$, $\beta_1^M=6.68$, $\beta_2^m=1$, $c=5$ and $b=1$, the following inequality,

$$c\alpha_2^m\alpha_\sigma\beta_2^m+\varepsilon_\alpha\varepsilon_\alpha b\alpha_1^m+\varepsilon_\alpha\left(c\alpha_2^m\alpha_1^m+b\alpha_\sigma\beta_2^m-\alpha_\sigma\beta_1^M\right)>0$$

holds for all $0<\varepsilon_\beta=\varepsilon_\alpha<\infty$, and then the matrix A is Hurwitz for all $0<\varepsilon_\beta=\varepsilon_\alpha<\infty$. Consequently, taking $\varepsilon_\beta=\varepsilon_\alpha=1.2$ as the initial value, the simulation is implemented under the normal and perturbed cases, respectively. Moreover, in the perturbed case, we consider an additive impulse-like disturbance $d(t)$ of magnitude 60 acting on the system input between 15 s and 16 s.

Figure 1 and **Figure 2** showed the simulation results under normal (solid line) and perturbed (dashed line) cases. The following observations can be made: 1) The instantaneous value $9\left\|P_2(t)\right\|<1$ holds for all $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$, $\beta_2(t)$ $(t>0)$, $\alpha_\sigma=8$, $c=5$, $b=1$ and $\varepsilon_\beta=\varepsilon_\alpha=1.2$. This shows that the closed-loop system is uniformly asymptotic stable and the equal ratio coefficient can be used to improve the conservatism. 2) The system responses are almost identical before the additive impulse-like disturbance appears. This means that by equal ratio gain technique, we can tune a nonlinear general integral controller with good robustness and high control performance. All these demonstrate that not only nonlinear general integral control can effectively deal with the uncertain nonlinear system but also equal ratio gain technique is a powerful and practical tool to solve the control design problem of dynamics with the nonlinear and uncertain actions.

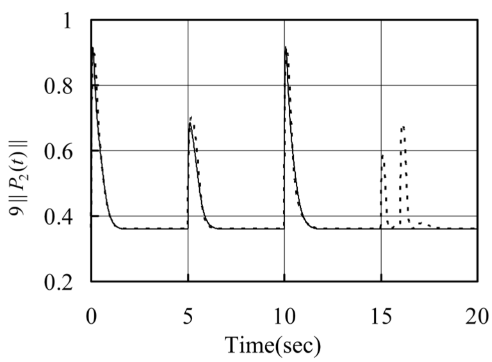

Figure 1. The values of $9\left\|P_2(t)\right\|$ under normal (solid line) and perturbed case (dashed line).

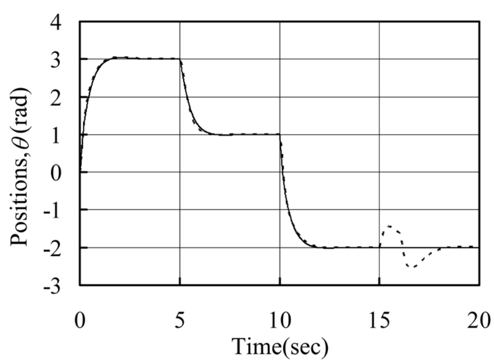

Figure 2. System output under normal (solid line) and perturbed case (dashed line).

6. Conclusions

This paper proposes a generic nonlinear integral controller and a practical nonlinear integral controller for a class of uncertain nonlinear system. The main contributions are that: 1) By defining two function sets, the generalization of general integral controller is achieved; 2) A canonical interval system matrix can be designed to be Hurwitz as any row controller gains, or controller and its integrator gains increase with the same ratio; 3) Theorems to ensure regionally as well as semi-globally asymptotic stability are established in terms of some bounded information. Moreover, for the practical nonlinear integral controller, a real time method to evaluate the equal ratio coefficient is proposed such that its value can be chosen moderately.

Theoretical analysis and simulation results demonstrated that not only nonlinear general integral control can effectively deal with the uncertain nonlinear system but also equal ratio gain technique is a powerful and practical tool to solve the control design problem of dynamics with the nonlinear and uncertain actions.

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