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Relative Continuity and New Decompositions of Continuity in Bitopological Spaces

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Abstract

The aim of this work is to introduce some weak forms of continuity in bitopological spaces. Then we use these new forms of weak continuity to give many decompositions of i-continuity and pairwise continuity.

Keywords

Relative Continuity, Decompositions of Continuity, Bitopological Spaces, i-Continuity, Pairwise Continuity

1. Introduction

The concept of bitopological spaces has been introduced by Kelly [1]. Functions and continuous functions stand among the most important notions in mathematical science. Many different weak forms of continuity in bitopological spaces have been introduced in the literature. For instance, we have pairwise almost and pairwise weakly continuity [2], pairwise semi-continuity [3], pairwise pre continuity [4], pairwise ρ-continuity [5], pairwise α-continuity [5] and many others, see ([6] [7]). N. Levine, in [8] introduced decomposition of continuity in topological spaces. In 2004 [9] Tong introduced twenty weak forms of continuity in topological spaces. In this paper, we generalize the results obtained by Tong to the setting of bitopological spaces.

Throughout this paper (X,τ_1,τ_2) and (Y,σ_1,σ_2) (or briefly, X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of X, by $i\text{-}cl(A)$ (resp. $i\text{-}\operatorname{int}(A)$) we denote the closure (resp. interior) of A with respect to τ_i (or σ_i) and $X \setminus A = A^c$ will denote the complement of A. Here $i, j = 1,2$ and $i \neq j$.

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2. Preliminaries

We recall some known definitions

Definition 1 ([3]) *A subset* A *of a bitopological space* (X,τ_1,τ_2) *is called* ij*-semi open if there is an* i*-open set U in X such that* $U\subset A\subset j\text{-}cl(U)$.

Definition 2 ([3]) *A function* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is called* ij*-semi continuous if* $f^{-1}(V)$ *is* ij*-semi open in X for each* i*-open set V of Y.*

Definition 3 ([2]) *A function* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is called* ij*-weakly (resp.* ij*-almost) continuous if for each point* $x\in X$ *and each* i*-open set V of Y containing* $f(x)$, *there exists an* i*-open set U of X containing* x *such that* $f(U)\subset j\text{-}cl(V)$ *(resp.* $f(U)\subset i\text{-}\operatorname{int}(j\text{-}cl(V))$.

Definition 4 ([5]) *A subset A of a bitopological space* (X,τ_1,τ_2) *is called ij-α-open if*

$A\subset i\text{-}\operatorname{int}(j\text{-}cl(i\text{-}\operatorname{int}(A)))$.

Definition 5 ([5]) *A function* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is called* $ij\text{-}\alpha$*-continuous if* $f^{-1}(V)$ *is* $ij\text{-}\alpha$*-open in X for each i-open set V of Y.*

Definition 6 ([4]) *A subset A of a bitopological space* (X,τ_1,τ_2) *is called ij-pre open if* $A\subset i\text{-}\operatorname{int}(j\text{-}cl(A))$.

Definition 7 ([4]) *A function* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is called* ij *pre continuous if* $f^{-1}(V)$ *is* ij*-pre open in X for each i-open set V of Y.*

The relations of the above weak forms of continuity are as follows:

ij-semi continuity
⇑
i-continuity ⇒ ij-α-continuity ⇒ ij-pre continuity
⇓
ij-almos continuity ⇒ ij-weak continuity

[Diagram 1]

3. Classification of *ij*-Weak Continuity

Lemma 1 *For a subset* A *of a bitopological space* (X,τ_1,τ_2), *we have*

1) $i\text{-}\operatorname{int}(i\text{-}\operatorname{int}(A))=i\text{-}\operatorname{int}(A)$;
2) $j\text{-}cl(j\text{-}cl(A))=j\text{-}cl(A)$;
3) $i\text{-}\operatorname{int}(j\text{-}cl(i\text{-}\operatorname{int}(j\text{-}cl(A))))=i\text{-}\operatorname{int}(j\text{-}cl(A))$;
4) $j\text{-}cl(i\text{-}\operatorname{int}(j\text{-}cl(i\text{-}\operatorname{int}(A))))=j\text{-}cl(i\text{-}\operatorname{int}(A))$.

Proof (1) *and* (2) *are obvious.* (3) *Since* $i\text{-}\operatorname{int}(j\text{-}cl(A))\subset j\text{-}cl(A)$, *then* $j\text{-}cl(i\text{-}\operatorname{int}(j\text{-}cl(A)))\subset j\text{-}cl(A)$. *Therefore,* $i\text{-}\operatorname{int}(j\text{-}cl(i\text{-}\operatorname{int}(j\text{-}cl(A))))\subset j\text{-}\operatorname{int}(j\text{-}cl(A))$. *On the other hand,* $i\text{-}\operatorname{int}(j\text{-}cl(A))\subset j\text{-}cl(i\text{-}\operatorname{int}(j\text{-}cl(A)))$, *then* $i\text{-}\operatorname{int}(j\text{-}cl(A))\subset i\text{-}\operatorname{int}(j\text{-}cl(i\text{-}\operatorname{int}(j\text{-}cl(A))))$ (4) *Similar to* (3).

Proposition 1 *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a function. Then*

1) f *is* i*-continuous if and only if* $f^{-1}(V)\subset i\text{-}\operatorname{int}(f^{-1}(j\text{-}cl(V)))$ *for each* i*-open set* V *in* Y;

2) f *is* ij*-pre continuous if and only if* $f^{-1}(V)\subset i\text{-}\operatorname{int}(j\text{-}cl(f^{-1}(V)))$ *for each* i*-open set* V *in* Y;

3) *f is* $ij\text{-}\alpha$ *-continuous if and only if* $f^{-1}(V)\subset i\text{-}\operatorname{int}(j\text{-}cl(i\text{-}\operatorname{int}(f^{-1}(V))))$ *for each* i*-open set* V *in* Y.

It is known [2] that a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is ij*-weakly continuous* if and only if for each i-open set V of Y, $f^{-1}(V)\subset i\text{-}\operatorname{int}(f^{-1}(j\text{-}cl(V)))$. From this we define the following.

Definition 8 *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a function. Then*

1) f *is* ij*-pre weakly continuous if and only if* $f^{-1}(V)\subset i\text{-}\operatorname{int}(j\text{-}cl(f^{-1}(j\text{-}cl(V))))$ *for each* i*-open set V in Y;*

2) f *is* $ij\text{-}\alpha$*-weakly continuous if and only if* $f^{-1}(V)\subset i\text{-}\operatorname{int}(j\text{-}cl(i\text{-}\operatorname{int}(f^{-1}(j\text{-}cl(V)))))$ *for each* i*-*

open set V in Y.

It is well known [2] that $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is ij-almost continuous if and only if $f^{-1}(V)\subset i\text{-int}\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)$. From this we define the following group of definitions.

Definition 9 *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a function. Then*

1) f *is* ij *-pre-almost continuous if and only if* $f^{-1}(V)\subset i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)\right)$ *for each* i *-open set V in Y;*

2) *f is ij-α-almost continuous if and only if* $f^{-1}(V)\subset i\text{-int}\left(j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)\right)\right)$ *for each i-mopen set V in Y.*

Lemma 2 *A function* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is* ij *-semi continuous if and only if for each* i *-open set* V *of* Y, $f^{-1}(V)\subset j\text{-}cl\left(i\text{-int}\left(f^{-1}(V)\right)\right)$.

Proof *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be an* ij *-semi continuous function. Then* $f^{-1}(V)$ *is* ij *-semi open in* X *for each* i *-open set* V *of* Y. *Since* $f^{-1}(V)$ *is a* ij *-semi open set in* X, *there exist an* i *-open set* $U\subset X$ *such that* $U\subset f^{-1}(V)\subset j\text{-}cl(U)$.

Since $U=i\text{-int}(U)$ *we have* $U\subset i\text{-int}\left(f^{-1}(V)\right)$. *Hence,* $j\text{-}cl(U)\subset j\text{-}cl\left(i\text{-int}\left(f^{-1}(V)\right)\right)$, *and therefore,* $f^{-1}(V)\subset j\text{-}cl\left(i\text{-int}\left(f^{-1}(V)\right)\right)$.

Conversely, assume that $f^{-1}(V)\subset j\text{-}cl\left(i\text{-int}\left(f^{-1}(V)\right)\right)$ *for each* i *-open set* $V\subset Y$. *Now* $i\text{-int}\left(f^{-1}(V)\right)\subset f^{-1}(V)\, j\text{-}cl\left(i\text{-int}\left(f^{-1}(V)\right)\right)$. *Put* $i\text{-int}\left(f^{-1}(V)\right)=U$. *Then there exists an* i *-open set* $U\subset X$ *such that* $U\subset f^{-1}(V)\subset j\text{-}cl(U)$. *It means* $f^{-1}(V)$ *is* ij *-semi open in* X *for each* i *-open set* V *of* Y. *Hence,* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is an* ij *-semi continuous function.*

In view of the above lemma we define the following:

Definition 10 *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a function. Then*

1) f *is* ij *-weak semi continuous if and only if* $f^{-1}(V)\subset j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)$ *for each* i *-open set V in Y;*

2) f is ij *-almost semi continuous if and only if* $f^{-1}(V)\subset j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)\right)$ *for each* i *-open set V in Y.*

Definition 11 *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a function. Then*:

1) f *is* ij *-pre semi continuous if and only if* $f^{-1}(V)\subset j\text{-}cl\left(i\text{-int}\left(j\text{-}cl\left(f^{-1}(V)\right)\right)\right)$ *for each* i *-open set V in Y;*

2) f *is* ij *-pre weak semi continuous if and only if* $f^{-1}(V)\subset j\text{-}cl\left(i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)\right)$ *for each i-open set V in Y;*

3) f *is* ij *-pre almost semi continuous if and only if* $f^{-1}(V)\subset j\text{-}cl\left(i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)\right)\right)$ *for each* i *-open set V in Y.*

The following diagram gives the relations between all the weak forms of continuity

ij-weak continuity	⇒	ij-α-weak continuity	⇒	ij-pre weak continuity
⇑		⇑		
ij-almost continuity	⇒	ij-α-almost continuity	⇒	ij-pre almost continuity
⇑		⇑		
i-continuity	⇒	ij-α-continuity		ij-pre continuity
		⇓		⇓
		ij-semi continuity	⇒	ij-pre semi continuity
		⇓		⇓
		ij-almost semi continuity	⇒	ij-pre almostsemi continuity
		⇓		⇓
		ij-weak semi continuity	⇒	ij-pre weak semi continuity

[Diagram 2]

Proof (*Proof of some relations in Diagram* 2).

1) *ij -weak continuity* $\Rightarrow$ *ij-α -weak continuity*

Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be an ij -weak continuous function. Then* $f^{-1}(V)\subset i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)$ *for each i -open set V of Y. Since* $i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\subset i\text{-int}\left(j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)\right)$, $f^{-1}(V)\subset i\text{-int}\left(j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)\right)$. *Hence, f is ij-$\alpha$ -weak continuous;*

2) *ij-α -weak continuity* $\Rightarrow$ *ij -pre weak continuity.*

Let $i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\subset f^{-1}\left(j\text{-}cl(V)\right)$. *This implies* $j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)\subset j\text{-}cl\left(f^{-1}\left(j\text{-}cl(V)\right)\right)$ *hence* $i\text{-int}\left(j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)\right)\subset i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)$. *Assume* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is ij-α -weak continuous. Then* $f^{-1}(V)\subset i\text{-int}\left(j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)\right)$ *for each i -open set V of Y. Since* $i\text{-int}\left(j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)\right)\subset i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)$, $f^{-1}(V)\subset i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)$.

Hence, f is ij-pre weak continuous.

We could use similar ways to prove other relations in Diagram 2.

4. Classification of Relative Continuity

Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a function. Then f is i -continuous if and only if $f^{-1}(V)$ is an i -open set in X for each i -open set V in Y. If we change the requirement on $f^{-1}(V)$ from being i -open in X to being i -open in a subspace, then we can obtain many new weak forms of continuity.

Definition 12 *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a function. Then*

1) *f is* $i^{\#}$ *-continuous if and only if* $f^{-1}(V)$ *is an i -open set in the subspace* $f^{-1}(V)$ *for each i -open set V in Y;*

2) *f is ij-pre*$^{\#}$ *continuous if and only if* $f^{-1}(V)$ *is an i -open set in the subspace* $j\text{-}cl\left(f^{-1}(V)\right)$ *for each i-open set V in Y;*

3) *f is ij-*$\alpha^{\#}$ *-continuous if and only if* $f^{-1}(V)$ *is an i -open set in the subspace* $j\text{-}cl\left(i\text{-int}\left(f^{-1}(V)\right)\right)$ *for each i -open set V in Y.*

Proposition 2 *Any function f is an* $i^{\#}$*-continuous function.*

Proof *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a function. For each i-open set V in Y we have* $f^{-1}(V)=f^{-1}(V)\cap X$, *then* $f^{-1}(V)$ *is an i-open set in the subspace* $f^{-1}(V)$. *Hence, f is* $i^{\#}$ *-continuous function.*

Definition 13 *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a function. Then*

1) *f is ij -weak*$^{\#}$ *continuous if and only if* $f^{-1}(V)$ *is an i-open set in the subspace* $f^{-1}\left(j\text{-}cl(V)\right)$ *for each i-open set V in Y;*

2) *f is ij-pre weak*$^{\#}$ *continuous if and only if* $f^{-1}(V)$ *is an i-open set in the subspace* $j\text{-}cl\left(f^{-1}\left(j\text{-}cl(V)\right)\right)$ *for each i -open set V in Y;*

3) *f is ij-α -weak*$^{\#}$ *continuous if and only if* $f^{-1}(V)$ *is an i-open set in the subspace* $j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)$ *for each i -open set V in Y.*

Definition 14 *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a function. Then*

1) *f is ij -almost*$^{\#}$ *continuous if and only if* $f^{-1}(V)$ *is an i-open set in the subspace* $f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)$ *for each i -open set V in Y;*

2) *f is ij -pre almost*$^{\#}$ *continuous if and only if* $f^{-1}(V)$ *is an i -open set in the subspace* $j\text{-}cl\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)$ *for each i -open set V in Y;*

3) *f is ij-α -almost*$^{\#}$ *continuous if and only if* $f^{-1}(V)$ *is an i -open set in the subspace* $j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)\right)$ *for each i -open set V in Y.*

Definition 15 *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a function. Then*

1) f *is a* ij *-pre-semi*$^{\#}$ *continuous if and only if* $f^{-1}(V)$ *is an* i *-open set in the subspace* $j\text{-}cl\left(i\text{-int}\left(j\text{-}cl\left(f^{-1}(V)\right)\right)\right)$ *for each* i *-open set* V *in* Y;

2) f *is a* ij *-pre weak semi*$^{\#}$ *continuous if and only if* $f^{-1}(V)$ *is an* i *-open set in the subspace* $j\text{-}cl\left(i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)\right)$ *for each* i *-open set* V *in* Y;

3) f *is a* ij *-pre-almost semi*$^{\#}$ *continuous if and only if* $f^{-1}(V)$ *is an* i *-open set in the subspace* $j\text{-}cl\left(i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)\right)\right)$ *for each* i *-open set* V *in* Y.

Lemma 3 *Let* $Z \subset Y \subset X$ *and* (X,τ_1,τ_2) *be a bitopological space. Then* $\left((\tau_i)_Y\right)_Z = (\tau_i)_Z$ *for* $i = 1,2$.

Proof *Let* $U \in \left((\tau_i)_Y\right)_Z$. *Then there exists an i-open set V in the subspace* Y *such that* $U = V \cap Z$. *We can write* $V = O \cap Y$, *where* O *is an* i *-open set in* X. *Therefore,* $U = O \cap Y \cap Z = O \cap Z$. *Hence,* U *is an* i *-open set in the subspace Z.*

Conversely, assume that $G \in (\tau_i)_Z$. *Then there exists an* i *-open* H *in* H *such that* $G = H \cap Z$. *Since* $Z \subset Y \subset X$, $G = H \cap Y \cap Z = C \cap Z$ *where* C *is an* i *-open set in the subspace* Y. *Hence* $G \in \left((\tau_i)_Y\right)_Z$.

Lemma 4 *If* $V \subset Y \subset X$ *and* V *is an* i *-open set in* (X,τ_1,τ_2) *then* V *is also* i *-open relative to* Y *for* $i = 1,2$.

Proof *The proof follows immediately from* $V = V \cap Y$ *where* V *is an* i *-open in* X.

The following diagram gives the relations between all the weak forms of continuity

ij-weak$^{\#}$ continuity
⇑
ij-α-weak$^{\#}$ continuity ⇐ ij-pre weak$^{\#}$ continuity
⇓
ij-α-almost$^{\#}$ continuity ⇐ ij-pre almost$^{\#}$ continuity ⇒ ij-almost$^{\#}$ continuity
⇓
ij-$\alpha^{\#}$ continuity ⇐ ij-pre$^{\#}$ continuity
⇓
ij-pre semi$^{\#}$ continuity
⇑
ij-pre almost semi$^{\#}$ continuity
⇑
ij-pre weak semi$^{\#}$ continuity

[Diagram 3]

Proof (*Proof of some relations in Diagram* 2).

1) ij *-pre weak semi*$^{\#}$ *continuity* $\Rightarrow ij$ *-pre almost semi*$^{\#}$ *weak continuity;*

Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be an* ij *-pre weak semi*$^{\#}$ *continuous. Then* $f^{-1}(V)$ *is an i-open set in the subspace* $j\text{-}cl\left(i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)\right)$. *Now*

$j\text{-}cl\left(i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)\right)\right) \subset j\text{-}cl\left(i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)\right) \subset X$. *By Lemmas* 4.6 *and* 4.7, *we obtain* $f^{-1}(V)$ *is an* i *-open in the subspace* $j\text{-}cl\left(i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)\right)\right)$. *Hence,* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is* ij *pre almost semi continuous.*

2) ij *-pre almost semi*$^{\#}$ *continuity* $\Rightarrow ij$ *-pre semi*$^{\#}$ *continuity;*

Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be* ij *-pre almost semi*$^{\#}$ *continuous. Then* $f^{-1}(V)$ *is an* i *-open set in*

$j\text{-}cl\left(i\text{-}\mathrm{int}\left(j\text{-}cl\left(f^{-1}\left(i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)\right)\right)\right)\right)$. *Since* $V = i\text{-}\mathrm{int}(V) \subset i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)$,
$f^{-1}(V) \subset f^{-1}\left(i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)\right)$. *Therefore,*
$j\text{-}cl\left(i\text{-}\mathrm{int}\left(j\text{-}cl\left(f^{-1}(V)\right)\right)\right) \subset j\text{-}cl\left(i\text{-}\mathrm{int}\left(j\text{-}cl\left(f^{-1}\left(i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)\right)\right)\right)\right) \subset X$. *By Lemmas* 4.6 *and* 4.7, *we obtain* $f^{-1}(V)$ *is an i-open in the subspace* $j\text{-}cl\left(i\text{-}\mathrm{int}\left(j\text{-}cl\left(f^{-1}\left(i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)\right)\right)\right)\right)$. *Hence,* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is* ij *-pre semi*$^{\#}$ *continuous.*

3) *ij -pre*$^{\#}$ *continuity* $\Rightarrow$ *ij pre semi*$^{\#}$ *continuity;*

Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be* ij *-pre*$^{\#}$ *continuous function. Then* $f^{-1}(V)$ *is* i *-open set in the subspace* $j\text{-}cl\left(f^{-1}(V)\right)$. *Since* $j\text{-}cl\left(i\text{-}\mathrm{int}\left(j\text{-}cl\left(f^{-1}(V)\right)\right)\right) \subset j\text{-}cl\left(f^{-1}(V)\right) \subset X$, *then by using Lemma* 4.6 *and Lemma* 4.7, *we obtain* $f^{-1}(V)$ *is an* i *-open in the subspace* $j\text{-}cl\left(i\text{-}\mathrm{int}\left(j\text{-}cl\left(f^{-1}(V)\right)\right)\right)$. *So* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is* ij *-pre semi*$^{\#}$ *continuous.*

4) *ij -pre almost*$^{\#}$ *continuity* $\Rightarrow$ *ij -pre*$^{\#}$ *continuity;*

Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be* ij *-pre almost*$^{\#}$ *continuous function. Then* $f^{-1}(V)$ *is* i *-open set in* $j\text{-}cl\left(f^{-1}\left(i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)\right)\right)$. *Since* $V = i\text{-}\mathrm{int}(V) \subset i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)$, $f^{-1}(V) \subset f^{-1}\left(i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)\right)$. *So* $j\text{-}cl\left(f^{-1}(V)\right) \subset j\text{-}cl\left(f^{-1}\left(i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)\right)\right) \subset X$, *by using Lemmas* 4.6 *and* 4.7, *we obtain* $f^{-1}(V)$ *is* i *-open in the subspace* $j\text{-}cl\left(f^{-1}(V)\right)$. *Then* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is an* ij *-pre*$^{\#}$ *continuous.*

5) *ij -pre almost*$^{\#}$ *continuity* $\Rightarrow$ *ij -pre semi*$^{\#}$ *continuity;*

Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be* ij *pre almost*$^{\#}$ *continuous function. Then* $f^{-1}(V)$ *is* i *-open set in* $j\text{-}cl\left(i\text{-}\mathrm{int}\left(j\text{-}cl\left(f^{-1}\left(i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)\right)\right)\right)\right)$. *Since* $V \subset j\text{-}cl(V)$, *then* $V = i\text{-}\mathrm{int}(V) \subset i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)$, *therefore* $f^{-1}(V) \subset f^{-1}\left(i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)\right)$. *This implies* $j\text{-}cl\left(f^{-1}(V)\right) \subset j\text{-}cl\left(f^{-1}\left(i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)\right)\right)$, *so*
$i\text{-}\mathrm{int}\left(j\text{-}cl\left(f^{-1}(V)\right)\right) \subset i\text{-}\mathrm{int}\left(j\text{-}cl\left(f^{-1}\left(i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)\right)\right)\right) \subset j\text{-}cl\left(f^{-1}\left(i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)\right)\right)$. *Then*
$j\text{-}cl\left(i\text{-}\mathrm{int}\left(j\text{-}cl\left(f^{-1}(V)\right)\right)\right) \subset j\text{-}cl\left(f^{-1}\left(i\text{-}\mathrm{int}\left(j\text{-}cl(V)\right)\right)\right)$. *By using Lemmas* 4.6 *and* 4.7, *we obtain* $f^{-1}(V)$ *is* i *-open in the subspace* $j\text{-}cl\left(i\text{-}\mathrm{int}\left(j\text{-}cl\left(f^{-1}(V)\right)\right)\right)$. *So* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is an* ij *-pre semi*$^{\#}$ *continuous.*

We could also use the similar ways to prove other relations in Diagram 3.

The following examples show that the reverse implications of Diagram 3 is not true.

Example 1 *Let* $X=\{a,b,c,d\}$, $\tau_1=\{\phi,X,\{a\}\}$, $\tau_2=\{\phi,X,\{a,b\}\}$, $Y=\{a,b,c\}$, $\sigma_1=\{\phi,Y,\{a\}\}$ *and* $\sigma_2=\{\phi,Y,\{a\},\{b,c\}\}$ *Define a map* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *by* $f(a)=b$, $f(b)=c$, $f(c)=f(d)=a$. *The map* f *is* 12*-pre weak*$^{\#}$ *continuous but not* 12*-* α *-weak*$^{\#}$ *continuous because* $f^{-1}(\{a\})=\{c,d\}$ *which is not* 1*-open in the subspace* $2\text{-}cl\left(1\text{-}\mathrm{int}\left(2\text{-}cl\left(f^{-1}\left(2\text{-}cl(\{c\})\right)\right)\right)\right)$.

Example 2 *Let* $X=\{a,b,c.d,e\}$, $\tau_1=\{\phi,X\}$, $\tau_2=\{\phi,X,\{b,e\}\}$, $Y=\{a,b,c,d\}$, $\sigma_1=\{\phi,Y,\{c\}\}$ *and* $\sigma_2=\{\phi,Y,\{b,d\}\}$. *Define a map* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *by* $f(a)=f(c)=f(d)=c$, $f(b)=f(e)=b$ *Then the map f is* 12*-pre weak*$^{\#}$ *continuous but not* 12*-pre weak semi*$^{\#}$ *continuous, because* $f^{-1}(\{c\})=\{a,c,d\}$ *which is not* 1*-open set in the subspace* $2\text{-}cl\left(1\text{-}\mathrm{int}\left(2\text{-}cl\left(f^{-1}\left(2\text{-}cl(\{c\})\right)\right)\right)\right)$.

Example 3 *Let* $X=Y=\{a,b,c.d,e\}$, $\tau_1=\{\phi,X,\{d,e\},\{b,c,d,e\}\}$, $\tau_2=\{\phi,Y,\{a\}\}$, $\sigma_1=\{\phi,Y,\{c,d,e\}\}$ *and* $\sigma_2=\{\phi,Y,\{a,b\}\}$ *Define a map* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *by* $f(a)=f(c)=b$, $f(b)=a$, $f(d)=d$ *and* $f(e)=e$. *The map f is* 21*-pre almost*$^{\#}$ *continuous but not* 21*-pre almost semi*$^{\#}$ *continuous because* $f^{-1}(\{a,b\})=\{a,b,c\}$ *is not* 2*-open in the subspace* $1\text{-}cl\left(2\text{-}\mathrm{int}\left(1\text{-}cl\left(f^{-1}\left(2\text{-}\mathrm{int}\left(1\text{-}cl(\{a,b\})\right)\right)\right)\right)\right)$.

5. Decompositions of *i*-Continuity and Pairwise Continuity

For a property Q of a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, we say that f is pairwise Q if f is 12-Q and 21-Q. For example, f is called pairwise weakly continuous if it is 12-weakly continuous and 21-weakly continuous. f is pairwise continuous if $f:(X,\tau_1)\to(Y,\sigma_1)$ and $f:(X,\tau_2)\to(Y,\sigma_2)$ are continuous.

In this section we will give eight decompositions of i-continuity and pairwaise continuity.

Lemma 5 *Let $\alpha:2^X\to 2^X$ be a mapping with $\alpha(A\cap B)\subset\alpha A\cap\alpha B$ and let $\beta:2^X\to 2^X$ be another mapping with $U\subset\beta U$ for each i-open set U of X. Let $f:X\to Y$ be a function such that for each i-open set V in Y,*

1) $f^{-1}(V)\subset i\text{-int}\left(\alpha f^{-1}(\beta V)\right)$;

2) *There is an i-open set G of X such that $f^{-1}(V)=\alpha f^{-1}(\beta V)\cap G$.*

Then f is i-continuous.

Proof *Since $f^{-1}(V)=\alpha f^{-1}(\beta V)\cap G$, then $f^{-1}(V)\subset G$. Therefore,* $\text{int}\left(f^{-1}(V)\right)=i\text{-int}\left(\alpha f^{-1}(\beta V)\right)\cap i\text{-int}(G)=i\text{-int}\left(\alpha f^{-1}(\beta V)\right)\cap G\supset f^{-1}(V)\cap f^{-1}(V)=f^{-1}(V)$. *We have proved that $f^{-1}(V)$ is an i-open set and hence f is i-continuous.*

Now we turn to the decomposition of i-continuity and pairwise continuity.

Theorem 1 *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a function. Then each of the following conditions implies that f is i-continuous.*

1) *f is ij-pre continuous and ij-pre$^{\#}$-continuous;*
2) *f is ij-α-continuous and ij-$\alpha^{\#}$-continuous;*
3) *f is ij-weakly continuous and ij-weak$^{\#}$-continuous;*
4) *f is ij-pre weakly continuous and ij-pre weak$^{\#}$-continuous;*
5) *f is ij-α-weakly continuous and ij-α-weak$^{\#}$-continuous;*
6) *f is ij-almost continuous and ij-almost$^{\#}$-continuous;*
7) *f is ij-it pre-almost continuous and ij-pre-almost$^{\#}$-continuous;*
8) *f is ij-α-almost continuous and ij-α-almost$^{\#}$-continuous.*

Proof

1) *Since f is ij-pre continuous,* $f^{-1}(V)\subset i\text{-int}\left(j\text{-}cl(V)\right)$. *Since f is ij-pre#-continuous,* $f^{-1}(V)=j\text{-}cl(V)\cap O$, *where O is i-open set in X. By Lemma* 5.1, *f is continuous, where* $\alpha=j\text{-}cl:2^X\to 2^X$ *and* $\beta=i:2^X\to 2^X$;

2) *Since f is ij-α-continuous,* $f^{-1}(V)\subset i\text{-int}\left(j\text{-}cl(V)\right)$. *Since f is ij-$\alpha^{\#}$-continuous,* $f^{-1}(V)=j\text{-}cl\left(i\text{-int}(V)\right)\cap O$, *where O is i-open set in X. By Lemma* 5.1, *f is continuous, where* $\alpha=j\text{-}cl\text{-int}:2^X\to 2^X$ *and* $\beta=i:2^X\to 2^X$;

3) *Since f is ij-weakly continuous,* $f^{-1}(V)\subset i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)$. *Since f is ij-weak$^{\#}$-continuous,* $f^{-1}(V)=f^{-1}\left(j\text{-}cl(V)\right)\cap O$, *where O is i-open set in X. By Lemma* 5.1, *f is continuous, where* $\alpha=i:2^X\to 2^X$ *and* $\beta=j\text{-}cl:2^X\to 2^X$;

4) *Since f is ij-pre weakly continuous,* $f^{-1}(V)\subset i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)$. *Since f is ij-pre weak$^{\#}$-continuous,* $f^{-1}(V)=j\text{-}cl\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\cap O$ *where O is i-open set in X. By Lemma* 5.1, *f is continuous, where* $\alpha=j\text{-}cl:2^X\to 2^X$ *and* $\beta=j\text{-}cl:2^X\to 2^X$;

5) *Since f is ij-α-weakly continuous,* $f^{-1}(V)\subset i\text{-int}\left(j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)\right)$. *Since f is ij-α-weak#-continuous,* $f^{-1}(V)=j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(j\text{-}cl(V)\right)\right)\right)\cap O$, *where O is i-open set in X. By Lemma* 5.1, *f is continuous, where* $\alpha=j\text{-}cl\text{-int}:2^X\to 2^X$ *and* $\beta=j\text{-}cl:2^X\to 2^X$;

6) *Since f is ij-almost continuous,* $f^{-1}(V)\subset i\text{-int}\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)$. *Since f is ij-almost$^{\#}$-continuous,* $f^{-1}(V)=f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\cap O$, *where O is i-open set in X. By Lemma* 5.1, *f is continuous, where* $\alpha=i:2^X\to 2^X$ *and* $\beta=i\text{-int}\ j\text{-}cl:2^X\to 2^X$;

7) *Since f is ij-pre-almost continuous,* $f^{-1}(V)\subset i\text{-int}\left(j\text{-}cl\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)\right)$. *Since f is ij-pre-*

almost$^{\#}$*-continuous*, $f^{-1}(V) = j\text{-}cl\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right) \cap O$, *where* O *is* i*-open set in* X. *By Lemma* 5.1, f *is continuous, where* $\alpha = j\text{-}cl : 2^X \to 2^X$ *and* $\beta = i\text{-int}\, j\text{-}cl : 2^X \to 2^X$;

8) *Since* f *is ij-*α*-almost continuous*, $f^{-1}(V) \subset i\text{-int}\left(j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)\right)\right)$. *Since* f *is ij-α-almost*$^{\#}$*-continuous, so* $f^{-1}(V) = j\text{-}cl\left(i\text{-int}\left(f^{-1}\left(i\text{-int}\left(j\text{-}cl(V)\right)\right)\right)\right) \cap O$, *where* O *is* i*-open set in X*. *By Lemma* 5.1, f *is continuous, where* $\alpha = j\text{-}cl\,i\text{-int} : 2^X \to 2^X$ *and* $\beta = i\text{-int}\, j\text{-}cl : 2^X \to 2^X$.

Corollary 1 *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a function. Then each of the following conditions implies that* f *is pairwise continuous.*

1) f *is pairwise pre continuous and pairwise pre*$^{\#}$*-continuous*;
2) f *is pairwise* α*-continuous and pairwise* $\alpha^{\#}$*-continuous*;
3) f *is pairwise weakly continuous and pairwise weakly*$^{\#}$*-continuous*;
4) f *is pairwise pre weakly continuous and pairwise pre weak*$^{\#}$*-continuous*;
5) f *is pairwise* α*-weakly continuous and pairwise* α*-weak*$^{\#}$*-continuous*;
6) f *is pairwise almost continuous and pairwise almost*$^{\#}$*-continuous*;
7) f *is pairwise pre-almost continuous and pairwise pre-almost*$^{\#}$*-continuous*;
8) f *is pairwise* α*-almost continuous and pairwise* α*-almost*$^{\#}$*-continuous*.

Proof *The proof follows immediately from Theorem* 5.3.

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