International Journal of Modern Nonlinear Theory and Application, 2013, 2, 164-166
http://dx.doi.org/10.4236/ijmnta.2013.23022 Published Online September 2013 (http://www.scirp.org/journal/ijmnta)

Scientific Research

α-Times Integrated *C* Semigroups and Strong Solution of Abstract Cauchy Problem

Man Liu[1], Shuling Wang[1], Qingzhi Yu[2], Zhengming Wang[3]
[1]Basic Department, Air Force Logistics College, Xuzhou, China
[2]Department of Aviation Ammunition, Air Force Logistics College, Xuzhou, China
[3]Department of Training, Air Force Logistics College, Xuzhou, China
Email: liuman8866@163.com

Received June 13, 2013; revised July 14, 2013; accepted July 25, 2013

ABSTRACT

The infinite generator of *α*-times Integrated *C* semigroups and the properties of resolvent are given. At the same time, we discuss the relationship between the existence of strong solution of a class of nonhomogeneous abstract Cauchy problem and *α*-times integrated *C* semigroups, and a sufficient and necessary condition is obtained.

Keywords: *α*-Times Integrated *C* Semigroups; Abstract Cauchy Problem; Strong Solution

1. Introduction

Integrated semigroups are more general than strongly continuous semigroups (*i.e.*, C_0 semigroups, cosine operator functions and exponentially bounded distribution semigroups). Frank Neubrander made use of the theory of operator semigroup to solve the existence of classical solution of homogeneous Cauchy problem where the infinite generator generated one time integrated semigroup in paper [1]. H. R. Thieme discussed the existence of integral solution for the same problem in paper [2]. Hermann Kellerman and Matthias Hieber studied the existence of the solution of nonhomogeneous Cauchy problem where the infinite generator generated one time integrated semigroup in paper [3], for a class of integral and differential function of infinte generator where the main operator was m-times integrated semigroup. N. U. Ahemed introduced the classical solution and generalized solution, and at the same time, he discussed the existence of solution in papers [4,5]. In this paper, we will introduce and study *α*-times integrated *C* semigroups, $\alpha \in R^+$. In theorem 3.1 we give a necessary and sufficient condition for *α*-times integrated *C* semigroups and the existence of strong solution of a class of nonhomogeneous abstract Cauchy problem.

Throughout this paper, *X* is a Banach space, $B(X)$ is the space of bounded linear operators from *X* into *X*, $D(A)$, $R(A)$, and $K(A)$ denote the domain, range and core of operator *A* respectively, $C \in B(X)$.

2. Definitions and Properties of *α*-Times Integrated C Semigroups

DEFINITION 2.1 Let $\alpha \in R^+$, a strongly continuous family $\{S(t)\}_{t\ge 0} \in B(X)$ is called *α*-times integrated *C* semigroups, if

(V$_1$) $S(t)C = CS(t)$, and $S(0)=0$;

(V$_2$) $S(t)S(s)x =$

$$\frac{1}{\Gamma(\alpha)}\left[\int_t^{s+t}(t+s-r)^{\alpha-1}S(r)Cx\mathrm{d}r - \int_0^s (t+s-r)^{\alpha-1}S(r)Cx\mathrm{d}r\right], \quad \forall t,s\ge 0 \tag{2.1}$$

If $\alpha > 0$, $S(t)x = 0$ $(t\ge 0)$ implies $x=0$, then *α*-times integrated *C* semigroups $\{S(t)\}_{t\ge 0}$ is non-degenerated.

If there exist $M>0$ and $\omega \in R$, such that $\|S(t)\| \le M\mathrm{e}^{\omega t}, t\ge 0$, then $\{S(t)\}_{t\ge 0}$ is called exponentially bounded.

For convenience, $(A, S(t)) \in G(\alpha, M, \omega)$ denote *A* generating *α*-times integrated integrated *C* semigroups $\{S(t)\}_{t\ge 0}$, and $\{S(t)\}_{t\ge 0}$ is exponentially bounded.

DEFINITION 2.2 Let $\alpha \ge 0$, a strongly continuous family $\{S(t)\}_{t\ge 0} \in B(X)$ is called *α*-times exponentially bounded integrated *C* semigroups generated by *A*, if $S(0)=0$, and there exists $M>0, \omega>0$, such that $(\omega,\infty) \subset \rho(A)$, $\|S(t)\| \le M\mathrm{e}^{\omega t}, t\ge 0$, and for arbitrary $\lambda > \omega$, $x\in X$, we have

$$R_C(\lambda, A)x = (\lambda - A)^{-1} Cx = \lambda^{\alpha} \int_0^{\infty} e^{-\lambda t} S(t) x dt. \quad (2.2)$$

THEOREM 2.1 [1] Let A be the generator of an α-times integrated C semigroups $\{S(t)\}_{t\geq 0}$, for $\alpha \geq 0$. Then

1) For all $x \in D(A)$ and $t \geq 0$,

$$S(t)x \in D(A),\ AS(t)x = S(t)Ax \quad (2.3)$$

$$S(t)x = \frac{t^{\alpha}}{\Gamma(\alpha+1)} Cx + \int_0^t S(s) Ax ds \quad (2.4)$$

2) $\int_0^t S(s) x ds \in D(A)$, for all $x \in X$, and $t \geq 0$, such that

$$A\int_0^t S(s) x ds = S(t)x - \frac{t^{\alpha}}{\Gamma(\alpha+1)} Cx \quad (2.5)$$

Proof referring to papers [1,2].

COROLLARY 2.2 Let $\alpha \in R^+$. Then $S(t)x \in \overline{D(A)}$ for all $x \in X$ and $t \geq 0$. Then $S(\cdot)x$ is right differentiable in $t \geq 0$ if and only if $S(t)x \in D(A)$.

In that case

$$\frac{d}{dt} S(t)x = AS(t)x + \frac{t^{\alpha-1}}{\Gamma(\alpha)} Cx, t \geq 0, x \in X.$$

At first we introduce the fractional differential and integral of function.

For $\alpha \geq 0$, $[\alpha]$, (α) denote the integral part and decimal part of α respectively. $\Gamma(\cdot)$ is well known Gamma function, and $\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$, $s\Gamma(s) = \Gamma(s+1)$.

For $\beta \geq -1$, we definite the function $j_{\beta}: (0, \infty) \to R$, and $j_{\beta}(t) = \frac{t^{\beta}}{\Gamma(\beta+1)}$, j_{-1} denotes 0-point Dirac measure δ_0.

For continuous function $f(\cdot)$, $\beta \geq -1$, the definition of convolution product is as following

$$(j_{\beta} * f)(t) = \begin{cases} \int_0^t \frac{(t-s)^{\beta}}{\Gamma(\beta+1)} f(s) ds, & \beta > -1 \\ f(t), & \beta = -1 \end{cases}.$$

For arbitrary $\alpha > 0$, α-order differential of function u denotes $(D_{\alpha} u)(t_0) = \omega^{(n-1)}(t_0)$.

For arbitrary $\alpha > 0$, α-times cumulative integral of function u denotes $(I_{\alpha} u) = (j_{\alpha-1} * u)(t)$.

LEMMA 2.3 [3]

1) If $f \in C^{\alpha}$, then

$$(D_{\alpha} f)(t) = \left(\frac{d}{dt}\right)^n \left(j_{n-\alpha-1} * [f - f(0)]\right)(t);$$

2) If $f \in C$, then $D_{\beta} I_{\alpha} f = I_{\alpha-\beta} f$, for all $\beta \in [0, \alpha]$, especially $D_{\alpha} I_{\alpha} f = f$;

3) If $f, g \in C$, then

$$D_{\alpha}(I_{\alpha} f * g) = f * g;$$

4) If f is constant, Then $(D_{\alpha} f)(t) = 0$;

5) If $g(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}$, then $g(t) \in C^{\alpha}[0, T]$ and $(D_{\alpha} g)(t) = 1$.

3. Main Results

We consider the following nonhomogeneous abstract Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + f(t),\ t \in R^+ = [0, +\infty) \\ x(0) = x_0 \end{cases} \quad (3.1)$$

DEFINITON 3.1 Let $A \in G(\alpha, M, \omega)$, $x(t) \in C(R^+, X)$, if

2) $x(t)$ is almost everywhere differential at R^+ and

$$\frac{d}{dt} x(t) \in L^1(R^+, X);$$

2) $x(t)$ satisfies (3.1) almost everywhere at R^+, then $x(t)$ is called the strong solution of (3.1).

THEOREM 3.1 Let $A \in G(\alpha, M, \omega)$, $\{S(t)\}_{t\geq 0}$ is α-times integrated C semigroups, $f(t) \in L^1(R^+, X)$, defining $Cy(t) = S(t)x_0 + (S * f)(t)$, then the sufficient and necessary condition of (3.1) exists strong solution is $y(t) \in C^{\alpha}(R^+, X)$, $(D_{\alpha} y)(t)$ is almost everywhere differential at R^+, and $d/dt(D_{\alpha} y)(t) \in L^1(R^+, X)$, where $x(t) = (D_{\alpha} y)(t)$, $t \geq 0$.

3.1. Proof Necessary

Let $x(t)$ be the strong solution of (3.1), definite

$$\alpha(s) = S(t-s)x(s),\ t > s \geq 0$$

Then by definition 3.1 and theorem 2.1, we know, for almost all $s \in R^+$

$$\frac{d}{ds}\alpha(s) = -\left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} C + AS(t-s)\right] x(s) + S(t-s)[Ax(s) + f(s)] = -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Cx(s) + S(t-s) f(s)$$

For the above equation integrating s from 0 to t, then

$$\text{Left} = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\alpha(s)\mathrm{d}s = \alpha(t) - \alpha(0) = -S(t)x_0$$

$$\text{Right} = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Cx(s)\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s$$

So

$$S(t)x_0 + \int_0^t S(t-s)f(s)\mathrm{d}s = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Cx(s)\mathrm{d}s$$

i.e. $y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\mathrm{d}s$, then $y(t) \in C^{\alpha}(R^+, X)$, $(D_\alpha y)(t)$ are almost everywhere differential at R^+, and $\frac{\mathrm{d}}{\mathrm{d}t}(D_\alpha y)(t) \in L^1(R^+, X)$.

Therefore $(Dy)(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)\mathrm{d}s$, $(D_{\alpha-1}y)(t) = \int_0^t x(s)\mathrm{d}s$, by integrating again we have $(D_\alpha y)(t) = x(t)$ since $(D_\alpha y)(t)$ is the strong solution of (3.1).

3.2. Sufficient

Let $\beta(t) = \int_0^t Cy(r)\mathrm{d}r$, then

$$\beta(t) = \int_0^t \left[S(r)x_0 + \int_0^r S(r-u)f(u)\mathrm{d}u \right] dr$$
$$= \int_0^t S(r)x_0\mathrm{d}r + \int_0^t \left[\int_u^t S(r-u)f(u)\mathrm{d}r \right]\mathrm{d}u$$
$$= \int_0^t S(r)x_0\mathrm{d}r + \int_0^t f(u)\left[\int_0^{t-u} S(s)\mathrm{d}s \right]\mathrm{d}u$$
$$= \int_0^t S(r)x_0\mathrm{d}r + \int_0^t \left[\int_0^{t-s} S(u)f(s)\mathrm{d}u \right]\mathrm{d}s$$

By Theorem 2.1 we obtain $\int_0^t S(r)x_0\mathrm{d}r \in D(A)$, $\int_0^t \left[\int_0^{t-s} S(u)f(s)\mathrm{d}u \right]\mathrm{d}s \in D(A)$, therefore $\beta(t) \in D(A)$.

$$A\beta(t) = A\int_0^t S(r)x_0\mathrm{d}r + \int_0^t \left[\int_0^{t-s} S(u)Af(s)\mathrm{d}u \right]\mathrm{d}s$$
$$= S(t)x_0 - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx_0 + \int_0^t \left[S(t-s)f(s) - \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}Cf(s) \right]\mathrm{d}s$$
$$= Cy(t) - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx_0 - \int_0^t \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}Cf(s)\mathrm{d}s$$

Combining

$$Cy(t) = A\int_0^t Cy(r)\mathrm{d}r + \frac{t^\alpha}{\Gamma(\alpha+1)}Cx_0 + \int_0^t \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}Cf(s)\mathrm{d}s$$

Above equation integrating *t*, then

$$\frac{\mathrm{d}}{\mathrm{d}t}Cy(t) = ACy(t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)}Cx_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}Cf(s)\mathrm{d}s \qquad (3.2)$$

Integrating (3.2) α times, then

$$C(D_\alpha y)(t) = C(D_{\alpha-1}y)(t) + Cx_0 + \int_0^t Cf(s)\mathrm{d}s$$

Integrating *t* once again, then $C(D_{\alpha+1}y)(t) = C(D_\alpha y)(t) + Cf(t)$, C is dense. Since $(D_{\alpha+1}y)(t) = (D_\alpha y)(t) + f(t)$, so $(D_\alpha y)(t)$ is the solution of (3.1).

4. Acknowledgements

We would like to thank the referees for the helpful suggestions, which greatly improved the original manuscript.

REFERENCES

[1] F. M. Neubrander, "Integrated Semigroups and Their Applications to the Abstract Cauchy Problem," *Pacific Journal of Mathematics*, Vol. 35, No. 1, 1988, pp. 111-155. doi:10.2140/pjm.1988.135.111

[2] H. R. Thieme, "'Integrated Semigroup' and Integrated Solution to Abstract Cauchy Problem," *Journal of Mathematical Analysis and Applications*, Vol. 152, No. 2, 1990, pp. 416-447. doi:10.1016/0022-247X(90)90074-P

[3] H. Kelleman and M. Heber, "Integrated Semigroups," *Journal of Functional Analysis*, Vol. 84, No. 1, 1989, pp. 160-180. doi:10.1016/0022-1236(89)90116-X

[4] N. U. Ahmed, "Semigroup Theory with Application to System and Control," Longman Scientific and Technical Press, New York, 1991, p. 58.

[5] N. U. Ahmed, "Generalized Solution for Linear System Governed by Operators beyond Hille- Yosida Type," *Publication Mathematical Debrecen*, Vol. 48, No. 1, 1996, pp. 45-64.