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Some Notes on the Paper "New Common Fixed Point Theorems for Maps on Cone Metric Spaces"

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ABSTRACT

In this paper, we show that Theorem 2.1 [1] (resp. Theorem 2.2 [1]) is a consequence of Corollary 2.1 [1] (resp. Corollary 2.2 [1]).

Keywords: Cone Metric; Weakly Compatible; Fixed Point

1. Introduction

In 2007, Huang and Zhang [2] initiated fixed point theory in cone metric spaces. On the other hand, in 2011, Haghi, Rezapour and Shahzad [3] gave a lemma and showed that some fixed point generalizations are not real generalizations. In this note, we show that Theorem 2.1 [1] and Theorem 2.2 [1] are so.

Following [2], let E be a real Banach space and θ be the zero vector in E, and $P \subseteq E$. P is called cone iff

1) P is closed, nonempty and $P \neq \{\theta\}$,

2) $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b,

3) $P \cap (-P) = \{\theta\}$.

For a given cone P, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ iff $y - x \in P$. $x \prec y$ (resp. $x \ll y$) stands for $x \preceq y$ and $x \neq y$ (resp. $y - x \in \operatorname{int}(P)$), where $\operatorname{int}(P)$ denotes the interior of P. In the paper we always assume that P is solid, *i.e.*, $\operatorname{int}(P) \neq \phi$. It is clear that $x \ll y$ leads to $x \preceq y$ but the reverse need not to be true.

The cone P is called normal if there exists a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K \|y\|$.

The least positive number satisfying above is called the normal constant of P.

Definition 1.1 [2]. Let X be a nonempty set. A function $d : X \times X \to E$ is called cone metric iff

(M_1) $\theta \preceq d(x, y)$,

(M_2) $d(x, y) = d(y, x) = \theta$ iff $x = y$,

(M_3) $d(x, y) = d(y, x)$,

(M_4) $d(x, y) \preceq d(x, z) + d(z, y)$,

for all $x, y, z \in X$. (X, d) is said to be a cone metric space.

Lemma 1.1 [3]. Let X be a nonempty and $f : X \to X$. Then there exists a subset $Y \subseteq X$ such that $f(Y) = f(X)$ and $f : Y \to X$ is one-to-one.

Definition 1.2 [4]. Let (X, d) be a cone metric space and $f, g : X \to X$ be mappings. Then, $z \in X$ is called a coincidence point of f and g iff $f(z) = g(z)$.

Definition 1.3 [4]. Let (X, d) be a cone metric space. The mappings $f, g : X \to X$ are weakly compatible iff for every coincidence point $z \in X$ of f and g, $f(g(x)) = g(g(x))$.

Theorem 1.1 (Theorem 2.1 [1]). Let (X, d) be a cone metric space and let $a_i \geq 0$ $(i = 1, 2, 3, 4, 5)$ be constants with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. Suppose that the mappings $f, g : X \to X$ satisfy the condition

$$\begin{aligned} d(f(x), f(y)) \preceq\ & a_1 d(g(x), g(y)) \\ & + a_2 d(f(x), g(x)) + a_3 d(f(y), g(y)) \\ & + a_4 d(g(x), f(y)) + a_5 d(f(x), g(y)) \end{aligned}$$

for all $x, y \in X$.

If the range of g contains the range of f and $g(X)$ is a complete subspace, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed point.

Theorem 1.2 (Corollary 2.1 [1]). Let (X, d) be a complete cone metric space and let $a_i \geq 0$ i = (1,2,3,4,5) be constants with $a_1+a_2+a_3+a_4+a_5<1$. Suppose that the mapping $f: X \to X$ satisfies the condition

$$d(f(x), f(y)) \preceq a_1 d(x,y) + a_2 d(x, f(x)) + a_3 d(y, f(y)) + a_4 d(x, f(y)) + a_5 d(y, f(x))$$

for all $x, y \in X$.

Then f has a unique fixed point x^* in X.

Theorem 1.3 (Theorem 2.2 [1]). Let (X,d) be a cone metric space and let the mappings $f, g: X \to X$ satisfy the condition

$$d(f(x), f(y)) \preceq \lambda \cdot u, \text{ for all } x, y \in X,$$

where

$$u \in \left\{ d(g(x), g(y)), d(f(x), g(x)), d(f(y), g(y)), \frac{1}{h}\left[d(f(x), g(y)) + d(f(y), g(x))\right] \right\},$$

$\lambda \in (0,1)$, $h > 2\lambda$.

If the range of g contains the range of f and $g(X)$ is a complete subspace, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed point.

Theorem 1.4 (Corollary 2.2 [1]). Let (X,d) be a complete cone metric space and let the mapping $f: X \to X$ satisfies the condition

$$d(f(x), f(y)) \preceq \lambda \cdot u, \text{ for all } x, y \in X,$$

where

$$u \in \left\{ d(x,y), d(f(x), x), d(f(y), y), \frac{1}{h}\left[d(f(x), y) + d(f(y), x)\right] \right\},$$

$\lambda \in (0,1)$, $h > 2\lambda$.

Then f has a unique fixed point x^* in X.

2. Main Result

In this section, we show that that Theorem 1.1 (resp. Theorem 1.3) is a consequence of Theorem 1.2 (resp. Theorem 1.4).

Theorem 2.1. Theorem 1.1 is a consequence of Theorem 1.2.

Proof. By Lemma 1.1, there exists $Y \subseteq X$ such that $g(Y) = g(X)$ and $g: Y \to X$ is one-to-one. Define a map $h: g(Y) \to g(Y)$ by $h(g(x)) = f(x)$ for each $x \in g(Y)$. Since g is one-to-one on Y, then h is well-defined. Also, for arbitrary $x, y \in X$,

$$d(h(g(x)), h(g(y))) \preceq a_1 d(g(x), g(y)) + a_2 d(h(g(x)), g(x)) + a_3 d(h(g(y)), g(y)) + a_4 d(g(x), h(g(y))) + a_5 d(h(g(x)), g(y))$$

where $a_i \geq 0$ $(i = 1,2,3,4,5)$ are constants with

$$a_1+a_2+a_3+a_4+a_5<1.$$

From the completeness of $g(Y) = g(X)$, there exists $x_0 \in X$ such that

$$h(g(x_0)) = g(x_0) = f(x_0)$$

by Theorem 1.2. Hence, f and g have a point of coincidence which is also unique. Since f and g are weakly compatible, then f and g have a unique common fixed point.

Theorem 2.2. Theorem 1.3 is a consequence of Theorem 1.4.

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