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Effect of Weight Function in Nonlinear Part on Global Solvability of Cauchy Problem for Semi-Linear Hyperbolic Equations

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ABSTRACT

In this paper, we investigate the effect of weight function in the nonlinear part on global solvability of the Cauchy problem for a class of semi-linear hyperbolic equations with damping.

Keywords: Cauchy Problem; Wave Equation; Global Solvability; Weight Function; Semi-Linear Hyperbolic Equation

1. Introduction

Consider the Cauchy problem for the semi-linear wave equation with damping

$$u_{tt}-\Delta u+u_t=a(x)|u|^p,\ (t,x)\in[0,\infty)\times R^n, \tag{1}$$

$$u(0,x)=u_0(x),u_t(0,x)=u_1(x),\quad x\in R^n, \tag{2}$$

where $\Delta=\frac{\partial^2}{\partial x_1}+\cdots+\frac{\partial^2}{\partial x_n^2},\ a(x)\in L_q(R^n),\ q>1$

In the case when $a(.x)$ is independent of x, the existence and nonexistence of the global solutions was investigated in the papers [1-8]. The authors interests are focused on so called critical exponent $p_c(n)$, which is the number defined by the following property: if $p>p_c(n)$ then all small data solutions of corresponding Cauchy problem have a global solution, while $1<p\le p_c(n)$ all solutions with data positive on blow up in finite time regardless of the smallness of the data.

In the present paper we investigate the effect of the weight function $a(x)$ on global solvability of Cauchy problems (1) and (2).

2. Statement of Main Results

We consider the Cauchy problem for a class of semilinear hyperbolic equation

$$u_{tt}+(-1)^l\Delta^l u+u_t=f(t,x,u),\ (t,x)\in[0,\infty)\times R^n, \tag{3}$$

$$u(0,x)=u_0(x),u_t(0,x)=u_1(x),\ x\in R^n, \tag{4}$$

where $l=1,2,\cdots$

Throughout this paper, we assume that the nonlinear term $f(t,x,u)$ satisfies the following conditions:

1) $f(t,x,u)$ and $f_t(t,x,u)=\frac{\partial(t,x,u)}{\partial t}$ are continuous functions in the domain $[0,\infty)\times R^{n+1}$.

2) $f(t,x,0)=0$, and

$$\begin{aligned}&|f(t,x,u_1)-f(t,x,u_2)|\\&\le a(x)\left(|u_1|^{p-1}+|u_2|^{p-1}\right)|u_1-u_2|,\end{aligned} \tag{5}$$

where

$$a(x)\in L_q(R^n),\ q>1, \tag{6}$$

$$p\in\left(1+\frac{2l}{n}-\frac{1}{q},+\infty\right)\quad \text{for } n\le 2l, \tag{7}$$

$$p\in\left(2-\frac{1}{q},\frac{n(q-2)}{q(n-2l)}\right)\quad\text{for}\quad 2l<n<\frac{4lq}{q+1}. \tag{8}$$

In the sequel, by $\|.\|_q$, we denote the usual $L_q(\Omega)$-norm. For simplicity of notation, in particular, we write $\|.\|$ instead of $\|.\|_2$. The constants C, c used throughout this paper are positive generic constants, which may be different in various occurrences.

Theorem 1. *Suppose that the conditions* (5)-(8) *are satisfied. Then there exists a real number* $\delta_0>0$ *such that, if*

$$\begin{aligned}(u_0,u_1)\in\bigcup\nolimits_{\delta_0}=\Big\{(\varphi,\psi):\varphi\in W_2^l(R^n)\cap L_1(R^n),\\ \psi\in L_2(R^n)\cap L_1(R^n),\\ \|\varphi\|_{W_2^l(R^n)}+\|\varphi\|_{L_1(R^n)}+\|\psi\|_{L_2(R^n)}+\|\psi\|_{L_1(R^n)}<\delta_0\Big\}\end{aligned}$$

Then problem (3) *and* (4) *admit a unique solution*

$$u(t,x)\in C\left([0,\infty);W_2^l(R^n)\right)\cap C^1\left([0,\infty);L_2(R^n)\right)$$

satisfied the decay property

$$\sum_{|\alpha|=r}\left\|D^\alpha u(t,\cdot)\right\|\le c(d)(1+t)^{-\frac{n+2r}{4l}},\quad t\in[0,\infty),\ r=0,1,\cdots,l \tag{9}$$

$$\left\|u_t(t,\cdot)\right\|\le c(d)(1+t)^{-\eta},\ t>0, \tag{10}$$

where

$$\eta=\min\left\{1+\frac{n}{4l},\frac{n(p-1)}{4l}+\frac{n}{2lq}\right\},\quad c(\cdot)\in C(R_+,R_+).$$

3. Proof of Theorem 1

It is well known that if

$$\left\|u(t,\cdot)\right\|_{W_2^l(R^n)}+\left\|u_t(t,\cdot)\right\|_{L_2(R^n)}\le c,\quad t\in[0,T_{\max}), \tag{11}$$

then $T_{\max}=+\infty$, *i.e.* problem (3) and (4) have a global solution (see for example [9]).

Using the Fourier transformation, Plancherel theorem and the Hausdorff-Young inequality, for the solution $u(t,x)$ we have the following inequalities (see [1]):

$$\sum_{|\alpha|=l}\left\|D^\alpha u(t,\cdot)\right\|_{L_2(R^3)}\le c(1+t)^{-\frac{n+2l}{4l}}E(u_0,u_1)+c\int_0^t(1+t-\tau)^{-\frac{n+2l}{4l}}\Phi(\tau)\,\mathrm{d}\tau; \tag{12}$$

$$\left\|u(t,\cdot)\right\|_{L_2(R^3)}\le c(1+t)^{-\frac{n}{4l}}E(u_o,u_1)+c\int_0^t(1+t-\tau)^{-\frac{n}{4l}}\Phi(\tau)\,\mathrm{d}\tau, \tag{13}$$

$$\left\|u_t(t,\cdot)\right\|_{L_2(R^3)}\le c(1+t)^{-\frac{n}{4l}-1}E(u_o,u_1)+c\int_0^t(1+t-\tau)^{-\frac{n}{4l}-1}\Phi(\tau)\,\mathrm{d}\tau \tag{14}$$

where,

$$\begin{aligned}E(u_o,u_1)&=\|u_0\|_{L_1(R^3)}+\|u_1\|_{L_1(R^3)}+\|u_0\|_{W_2^{l-k}(R^3)}+\|u_1\|_{L_2(R^3)}\\ \Phi(\tau)&=\left\|f(\tau,x,u(\tau,x))\right\|_{L_1(R^3)}+\left\|f(\tau,x,u(\tau,x))\right\|_{L_2(R^3)}\end{aligned} \tag{15}$$

On the other hand, by virtue of condition 2°

$$\left\|f(t,x,u)\right\|_{L_1(R^n)}\le c\int_{R^n}a(x)|u(x)|^p\mathrm{d}x \tag{16}$$

and

$$\left\|f(t,x,u)\right\|\le c\int_{R^n}a^2(x)|u(x)|^{2p}\mathrm{d}x. \tag{17}$$

Using the Holder inequality, from (16) we have

$$\left\|f(t,x,u)\right\|_{L_1(R^n)}\le c\left(\int_{R^n}a^q(x)\mathrm{d}x\right)^{1/q}\left(\int_{R^n}|u(x)|^{\frac{pq}{q-1}}\mathrm{d}x\right)^{\frac{q-1}{q}}.$$

By virtue of condition (7), (8) and the multiplicative inequality of Gagliardo-Nirenberg type, we have

$$\left\|f(t,\cdot,u(\cdot))\right\|_{L_1(R^3)}\le c\left\|a(\cdot)\right\|_{L_q(R^n)}^q\|u\|^{p(1-\theta)}\cdot\left(\sum_{|\alpha|=l}\left\|D^\alpha u\right\|\right)^{p\theta}, \tag{18}$$

where

$$\theta=\frac{n}{l}\left(\frac{1}{2}-\frac{q-1}{pq}\right),\ \text{(see [10])}. \tag{19}$$

Analogously from (17) we have

$$\left\|f(t,\cdot,u(\cdot))\right\|\le c\left\|a(\cdot)\right\|_{L_q(R^n)}^2\|u\|^{2p(1-\theta')}\cdot\left(\sum_{|\alpha|=l}\left\|D^\alpha u\right\|\right)^{2p\theta'}, \tag{20}$$

where

$$\theta'=\frac{n}{2l}\left(1-\frac{q-2}{pq}\right). \tag{21}$$

From (12), (16) and (20) we have the following estimates

$$\begin{aligned}\sum_{|\alpha|=l}\left\|D^\alpha u(t,\cdot)\right\|&\le c(1+t)^{-\frac{n+2l}{4l}}E(u_0,u_1)\\&+c\int_0^t(1+t-\tau)^{-\frac{n+2l}{4l}}\left[\left\|u(\tau,.)\right\|^{p(1-\theta)}\cdot\left(\sum_{|\alpha|=l}\left\|D^\alpha u(\tau,.)\right\|\right)^{p\theta}\right.\\&\left.+\left\|u(\tau,.)\right\|^{2p(1-\theta')}\cdot\left(\sum_{|\alpha|=l}\left\|D^\alpha u(\tau,.)\right\|\right)^{2p\theta'}\right]\mathrm{d}\tau\end{aligned} \tag{22}$$

$$\begin{aligned}\left\|u(t,\cdot)\right\|&\le c(1+t)^{-\frac{n}{4l}}E(u_0,u_1)\\&+c\int_0^t(1+t-\tau)^{-\frac{n}{4l}}\left[\left\|u(\tau,.)\right\|^{p(1-\theta)}\cdot\left(\sum_{|\alpha|=l}\left\|D^\alpha u(\tau,.)\right\|\right)^{p\theta}\right.\\&\left.+\left\|u(\tau,.)\right\|^{2p(1-\theta')}\cdot\left(\sum_{|\alpha|=l}\left\|D^\alpha u(\tau,.)\right\|\right)^{2p\theta'}\right]\mathrm{d}\tau\end{aligned} \tag{23}$$

It follows from (22) and (23) that

$$G_1(t) \le cd + c(1-t)^{\frac{n}{4l}} \int_0^t (1+t-\tau)^{-\frac{n}{4l}} \times\Big[(1-\tau)^{-\gamma} G_1^{p(1-\theta)}(\tau) G_2^{p\theta}(\tau) + (1-\tau)^{-\gamma'} G_1^{2p(1-\theta')}(\tau) G_2^{2p\theta'}(\tau)\Big] \mathrm{d}\tau; \quad (24)$$

$$G_2(t) \le cd + c(1-t)^{\frac{n+2l}{4l}} \int_0^t (1+t-\tau)^{-\frac{n+2l}{4l}} \times\Big[\big((1-\tau)^{-\gamma} G_1^{p(1-\theta)}(\tau) G_2^{p\theta}(\tau)\big) + (1-\tau)^{-\gamma'} G_1^{2p(1-\theta')}(\tau) G_2^{2p\theta'}(\tau)\Big] \mathrm{d}\tau \quad (25)$$

where $G_1(t)$ and $G_2(t)$ are defined by

$$G_1(t) = (1-t)^{\frac{n}{4l}} \|u(t,\cdot)\|, \quad (26)$$

$$G_2(t) = (1-t)^{\frac{n+2l}{4l}} \sum_{|\alpha|=l} \|D^\alpha u(t,\cdot)\|, \quad (27)$$

and

$$\gamma = \frac{np}{4l} + \frac{p\theta}{2}, \gamma' = \frac{np}{2l} + p\theta'. \quad (28)$$

Then, we have from (19), (21) and (28) that

$$\gamma = \frac{np}{4l} + \frac{p}{2}.\frac{n}{l}\left(\frac{1}{2} - \frac{q-1}{pq}\right) = \frac{np}{2l} - \frac{n(q-1)}{2lq}, \quad (29)$$

$$\gamma' = \frac{np}{2l} + p \cdot \frac{n}{2l}\left(1 - \frac{q-2}{pq}\right) = \frac{np}{l} - \frac{n(q-2)}{2lq}. \quad (30)$$

It is clear from conditions (7), (8) and (29), (30) that

$$\gamma' > \gamma > 1.$$

Allowing for (24), (25) we obtain that

$$G_1(t) + G_2(t) \le c, t \in [0, T_{\max}). \quad (31)$$

Thus the a priori estimate (9) is satisfied, so $T = \infty$. From (14) and (31) we yield the inequality (10).

4. Nonexistence of Global Solutions

Next let us discus the counterpart of the conditions (7) and (8). To this end we considered the Cauchy problem for the semi-linear hyperbolic inequalities

$$u_{tt} + (-1)^l \Delta^l u + u_t \ge f(t,x,u), \quad t > 0, x \in R^n, \quad (32)$$

$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), \quad x \in R^n, \quad (33)$$

where

$$f(t,x,u) = \frac{1}{\left(1+|x|^2\right)^s} |u|^p.$$

The weak solution of inequality (32) with initial data (33) where

$$u_0(.) \in W_1^l(R^n), \quad u_1(.) \in L_1(R^n)$$

is called a function $u(t,x) \in L_1(R_+ \times R^n)$ which, and $u(t,x)$ satisfies the following inequality:

$$-\int_{R^n} [u_0(x) + u_1(x)] \zeta(0,x) \mathrm{d}x + \int_{R^n} u_0(x) \frac{\partial \zeta(0,x)}{\partial t} \mathrm{d}x + \int_0^\infty \int_{R^n} u(t,x)\big[\zeta_{tt}(t,x) - \zeta_t(t,x) + (-1)^l \Delta^l \zeta(t,x)\big] \mathrm{d}x\mathrm{d}t \ge \int_0^\infty \int_{R^n} f(t,x,u(t,x)) \zeta(t,x) \mathrm{d}x\mathrm{d}t,$$

for any function $\zeta(.) \in C_0^{2,2l}(R_+ \times R^n)$, where $\zeta(t,x) \ge 0$, $(t,x) \in R_+ \times R^n$.

From Theorem 1 it follows that if $n \le 2l$ and

$$p \in \left(1 + \frac{2l-2s}{n}, +\infty\right), \quad (34)$$

then there exists $\delta_0 > 0$ such that for any $(u_0(.), u_1(.)) \in U_{\delta_0}$, problems (30) and (31) have a unique solution

$$u(t,x) \in C\big([0,\infty); W_2^l(R^n)\big) \cap C^1\big([0,\infty); L_2(R^n)\big).$$

Theorem 2. Let

$$1 < p \le 1 + \frac{2l-2s}{n}, \quad (35)$$

and

$$\int_{R^3} [u_0(x) + u_1(x)] \mathrm{d}x \ge 0. \quad (36)$$

Then problems (32) and (33) *have no nontrivial solutions.*

5. Proof of Theorem 2

We assume that $u(t,x)$ is a global solution of (32) and (33). Let $\phi \in C^2(R;[0,1])$ be such that

$$\phi(r) = 1, r \le 1, \phi(r) = 0, r \ge 2$$

and, choose

$$\zeta(t,x) = \phi\left(\frac{t^2 + |x|^{2l}}{R^{2l}}\right), R > 0 \quad \text{(see [8])}.$$

Taking such a $\zeta(t,x)$ as the test function in Definition 1, we get that

$$\int_{R^n}\left[u_0(x)+u_1(x)\right]\zeta(0,x)\,\mathrm{d}x+\int_0^\infty\int_{R^n}\frac{1}{\left(1+|x|^2\right)^s}|u(t,x)|^p\zeta(t,x)\,\mathrm{d}x\mathrm{d}t\le\int_{R^n}u_0(x)\frac{\partial\zeta(0,x)}{\partial t}\mathrm{d}x+\int_0^\infty\int_{R^n}u(t,x)\left[\zeta_{tt}(t,x)-\zeta_t(t,x)+(-1)^l\Delta^l\zeta(t,x)\right]\mathrm{d}x\mathrm{d}t. \tag{37}$$

The choose of $\zeta(.)$ implies that

$$\int_{R^n}u_0(x)\frac{\partial\zeta(0,x)}{\partial t}\mathrm{d}x=0. \tag{38}$$

Define $\Omega=\left\{(t,x)\in[0,\infty)\times R^n,t^2+|x|^{2l}\le 2\right\}$. Again, by the choice of $\zeta(t,x)$, it is easy to show that

$$\int_0^\infty\int_{R^n}\left(1+|x|^2\right)^{\frac{sp'}{p}}\zeta^{-\frac{p'}{p}}|\zeta_t|^{p'}\mathrm{d}x\mathrm{d}t\le C_1<\infty,$$

$$\int_0^\infty\int_{R^n}\left(1+|x|^2\right)^{\frac{sp'}{p}}\zeta^{-\frac{p'}{p}}|\zeta_{tt}|^{p'}\mathrm{d}x\mathrm{d}t\le C_2<\infty,$$

$$\int_0^\infty\int_{R^n}\left(1+|x|^2\right)^{\frac{sp'}{p}}\zeta^{-\frac{p'}{p}}\left|\Delta^l\zeta\right|^{p'}\mathrm{d}x\mathrm{d}t\le C_3<\infty,$$

Take scaled variables $t=\lambda^{2l}\tau, x_i=\lambda y_i, i=1,\cdots,n$, then we have

$$\int_{R^n}\left[u_0(x)+u_1(x)\right]\zeta(0,x)\,\mathrm{d}x+\int_0^\infty\int_{R^n}\frac{1}{\left(1+|x|^2\right)^s}|u(t,x)|^p\zeta(t,x)\,\mathrm{d}x\mathrm{d}t\le\lambda^{\sigma_1}\eta_1+\lambda^{\sigma_2}\eta_2+\lambda^{\sigma_3}\eta_3, \tag{39}$$

where

$$\eta_1=c_2\iint_\Omega\left(\frac{1}{\lambda^{2/l\mu}}+|y|^2\right)^{\frac{sp'}{p}}(\phi\circ\rho)^{-\frac{p'}{p}}\left|(\phi\circ\rho)_{\tau\tau}\right|^{p'}\mathrm{d}y\mathrm{d}\tau\le c, \tag{40}$$

$$\eta_2=c_2\iint_\Omega\left(\frac{1}{\lambda^{2/l\mu}}+|y|^2\right)^{\frac{sp'}{p}}(\phi\circ\rho)^{-\frac{p'}{p}}\left|(\phi\circ\rho)_{\tau}\right|^{p'}\mathrm{d}y\mathrm{d}\tau\le c, \tag{41}$$

$$\eta_3=c_3\iint_\Omega\left(\frac{1}{\lambda^{2/l\mu}}+|y|^2\right)^{\frac{sp'}{p}}(\phi\circ\rho)^{-\frac{p'}{p}}\left|\Delta^l(\phi\circ\rho)\right|^{p'}\mathrm{d}y\mathrm{d}\tau\le c, \tag{42}$$

$$\sigma_1=\frac{2s}{p-1}-\frac{4lp}{p-1}+2l+n, \tag{43}$$

$$\sigma_2=\sigma_3=\frac{2s}{p-1}-\frac{2lp}{p-1}+n. \tag{44}$$

Letting $\lambda\to\infty$ in (39), owing to (35), (40), (41) we get

$$\int_{R^n}\left[u_0(x)+u_1(x)\right]\mathrm{d}x+\int_0^\infty\int_{R^n}\frac{1}{\left(1+|x|^2\right)^s}|u(t,x)|^p\,\mathrm{d}x\mathrm{d}t\le C<\infty. \tag{45}$$

Taking into account condition (36), from (45) it follows that

$$\int_0^\infty\int_{R^n}\frac{1}{\left(1+|x|^2\right)^s}|u(t,x)|^p\,\mathrm{d}x\mathrm{d}t\le C<\infty. \tag{46}$$

Further, by applying the Holder inequality, from (37) we obtain

$$\int_{|x|<\lambda}\left[u_0(x)+u_1(x)\right]h\left(\frac{|x|^{2l}}{\lambda^4}\right)\mathrm{d}x+\int_0^\infty\int_{R^n}\frac{1}{\left(1+|x|^2\right)^s}|u(t,x)|^p\varsigma(t,x)\,\mathrm{d}x\mathrm{d}t\le\left(\iint_{\lambda^2\le t^2+|x|^4\le 2\lambda^2}\frac{1}{\left(1+|x|^2\right)^s}|u(t,x)|^p\,\mathrm{d}x\mathrm{d}t\right)^{1/p}\times\left(\iint_{\lambda^2\le t^2+|x|^4\le 2\lambda^2}\frac{1}{\left(1+|x|^2\right)^{sp'}}\left|\zeta_{tt}+\zeta_t+\Delta^l\zeta\right|^{p'}\mathrm{d}x\mathrm{d}t\right). \tag{47}$$

Letting $\lambda\to\infty$ in (47), owing to (45), we get

$$\int_{R^n}\left[u_0(x)+u_1(x)\right]\mathrm{d}x+\int_0^\infty\int_{R^n}\frac{1}{\left(1+|x|^2\right)^s}|u(t,x)|^p\,\mathrm{d}x\mathrm{d}t\le 0.$$

Finally, taking into condition (36), we have that

$$u(t,x)=0.$$

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