International Journal of Modern Nonlinear Theory and Application, **2012, 1, 97-101**
http://dx.doi.org/10.4236/ijmnta.2012.13014 Published Online September 2012 (http://www.SciRP.org/journal/ijmnta

Boundary Stabilization of a More General Kirchhoff-Type Beam Equation*

Jianwen Zhang, Danxia Wang#
Department of Mathematics, Taiyuan University of Technology, Taiyuan, China
Email: jianwen.z2008@163.com, #danxia.wang@163.com

Received July 12, 2012; revised August 12, 2012; accepted September 12, 2012

ABSTRACT

Simultaneously, considering the viscous effect of material, damping of medium, geometrical nonlinearity, physical nonlinearity, we set up a more general equation of beam subjected to axial force and external load. We prove the existence and uniqueness of global solutions under non-linear boundary conditions which the model is added one damping mechanism at l end. What is more, we also prove the exponential decay property of the energy of above mentioned system.

Keywords: Kirchhoff-Type Beam; Non-Linear Boundary; Global Solutions; Exponential Decay

1. Introduction

The problem is based on the equation

$$u_{tt} + u_{xxxx} - \left(\alpha + \beta \int_0^l \left|u_x(s,t)\right|^2 \mathrm{d}s\right) u_{xx} = 0$$

which was proposed by Woinowsky-krieger [1], as a model for vibrating beams with hinged ends. One of the first mathematical analysis for the equation

$$u_{tt} + u_{xxxx} - M\left(\int_0^l \left|u_x\right|^2 \mathrm{d}x\right) u_{xx} = 0$$

was done by Ball [2], which was later extended to an abstract setting by defining a linear operator A by Medeiros [3]. In [4], Tucsnak considered the above beam equation which clamped boundary and obtained the exponential decay of the energy when a damping of the type $a(x)u_t$ is effective near the boundary. In the same direction, Kouemon Patchen [5] obtained the exponential decay of the energy for above-equation when a nonlinear damping $g(u_t)$ was effective in Ω. To [6] considered the above kirchhoff-type beam equation under non-linear boundary conditions

$$u(0,t) = u_x(0,t) = u_{xx}(l,t) = 0$$

$$u_{xxx}(l,t) - M\left(\int_0^l \left|u_x\right|^2\right) u_x(l,t) = f\left(u(l,t)\right) + g\left(u_t(l,t)\right)$$

which the model is clamped at $x = 0$ and is supported $x = l$. He proved the existence and decay rates of the solutions. A rather general kirchhoff-type beam equation

$$\begin{aligned} &u_{tt} + \alpha u_{xxxx} + \gamma u_{xxxxt} \\ &-\left(\beta + k\int_0^l u_x^2 \mathrm{d}x + \sigma \int_0^l u_x u_{xt} \mathrm{d}x)\right) u_{xx} + \delta u_t = 0 \end{aligned}$$

was set up by Ball [7], who presented the existence and uniqueness of solution under linear boundary conditions. However the global solution and exponential decay for the more general beam equation is open under nonlinear boundary conditions. In the present work, we are concerned with the existence and uniqueness of solutions and the exponential decay property of energy on the nonlinear beam equation with external load

$$\begin{aligned} &u_{tt} + u_{xxxx} + \mu u_{xxxxt} + \eta u_t \\ &-\left(M\left(\left\|u_x\right\|^2\right) + N\left(\int_0^l u_{xt} u_x \mathrm{d}x\right)\right) u_{xx} = q(x,t) \end{aligned} \quad (1)$$

with nonlinear boundary conditions

$$u(0,t) = u_{xx}(0,t) = u_x(l,t) = 0 \quad (2)$$

$$u_{xxx}(l,t) + \mu u_{xxxt}(l,t) = f\left(u(l,t)\right) + g\left(u_t(l,t)\right) \quad (3)$$

and initial conditions

$$u(x,0) = u^0(x) \text{ and } u_t(x,0) = u^1(x) \quad (4)$$

2. Definition and Assumptions

In this paper, our analysis is based on the Sobolev spaces

$$V = \left\{u \in H^2(0,l) \middle| u(0) = u_x(l) = 0\right\},$$

---

*This work is supported by Natural Science Foundation of Shanxi Provice, and the Natural Science Foundation for Young Scientists of Shanxi Province (2010011008, 2011021002-2).

#Corresponding author.

Copyright © 2012 SciRes.

$$W=\left\{u\in V\cap H^4(0,l)\middle|u_{xx}(0)=0\right\}$$

espectively equipped with the norm $\|u\|_V=\|u_{xx}\|$ and $\|u\|_W=\|u_{xx}\|+\|u_{xxxx}\|$. We assume that *f*, *g*:*R* → *R* are continuously differentiable functions such that

$$f(s)s\ge 0$$

and $$f(s)s-2\hat{f}(s)\ge 0,\forall s\in R \tag{5}$$

where $$\hat{f}(s)=\int_0^s f(z)\mathrm{d}z \text{ and}$$

$$g(0)=0$$

and $$\big(g(r)-g(s)\big)(r-s)\ge\rho|r-s|^2,\forall r,s\in R \tag{6}$$

for some $\rho>0$.

Assume that the functions $M(\cdot),N(\cdot)\in C^1[0,\infty)$ are non-negative functions and respectively satisfy

$$M(0)=0$$

and $$\hat{M}(s)=\int_0^l M(z)\mathrm{d}z \tag{7}$$

$$N(0)=0$$

and $$N(s)s\ge 0 \tag{8}$$

3. Existence and Uniqueness of Global Solutions

Now we come to the following conclusions of the existence and uniqueness of global solutions.

Theorem 1. Assume that the assumptions of (5)-(8) and $q(x,t)\in L^2\left([0,\infty);L^2(0,l)\right)$ hold. Then for any $u^0,u^1\in W$ satisfying the compatibility condition

$$u^0_{xxx}(l)+\mu u^1_{xxx}(l)=f\left(u^0(l)\right)+g\left(u^1(l)\right) \tag{9}$$

There exists a function *u* satisfying (1)-(4) such that

$$u\in L^2(0,\infty;W)\cap C^0\left([0,\infty);V\right)\cap W^{2,\infty}\left(0,\infty;L^2(0,l)\right)$$

Proof. Let us solve the variational problem associated with (1)-(4), which is given by: find $u(t)\in W$ such that

$$\begin{aligned}&\int_0^l u_{tt}\omega\mathrm{d}x+\int_0^l u_{xx}\omega_{xx}\mathrm{d}x+\mu\int_0^l u_{xxt}\omega_{xx}\mathrm{d}x+\eta\int_0^l u_t\omega\mathrm{d}x\\&+\left[M\left(\|u_x\|^2\right)+N\left(\int_0^l u_xu_{xt}\mathrm{d}x\right)\right]\int_0^l u_x\omega_x\mathrm{d}x\\&+\left[f\left(u(l,t)\right)+g\left(u_t(l,t)\right)\right]\omega(l)\\&=\int_0^l q(x,t)\omega\mathrm{d}x\end{aligned} \tag{10}$$

for all $\omega\in V$. Let $\left\{\omega^j\right\}$ be a complete orthogonal system of *W*. For each $m\in N$, let us put

$$W^m=\operatorname{span}\left\{\omega^1,\omega^2,\cdots,\ \omega^m\right\}.$$

We search for a function

$$u^m(t)=\sum_{j=1}^m a^j(t)\omega^j$$

where $a^j(t)$ is a unknown function such that for any $\omega\in W^m$, and it satisfies the approximating equation

$$\begin{aligned}&\int_0^l u^m_{tt}\omega\mathrm{d}x+\int_0^l u^m_{xx}\omega_{xx}\mathrm{d}x+\mu\int_0^l u^m_{xxt}\omega_{xx}\mathrm{d}x+\eta\int_0^l u^m_t\omega\mathrm{d}x\\&+\left[M\left(\|u^m_x\|^2\right)+N\left(\int_0^l u^m_xu^m_{xt}\mathrm{d}x\right)\right]\int_0^l u^m_x\omega_x\mathrm{d}x\\&+\left[f\left(u^m(l,t)\right)+g\left(u^m_t(l,t)\right)\right]\omega(l)\\&=\int_0^l q(x,t)\omega\mathrm{d}x\end{aligned} \tag{11}$$

with the initial conditions

$$u^m(0)=u^0 \text{ and } u^m_t(0)=u^1 \tag{12}$$

Thus (11) and (12) are equivalent to the Cauchy problem of ODES in the variable *t*, which is known to have a local solution $u^m(t)$ in an interval $[0,t_m)$ $(t_m<T)$ for any given $T>0$.

Estimate 1. By integration of (11) over $[0,t]$ $(t<t_m)$ with $\omega=u^m_t(t)$, we see that

$$\begin{aligned}&\|u^m_t\|^2+\|u^m_{xx}\|^2+2\mu\int_0^t\|u^m_{xxt}\|^2\mathrm{d}s+\hat{M}\left(z(t)\right)\\&+2\int_0^t N\left(z_t(s)\right)z_t(s)\mathrm{d}s+2\int_0^t g\left(u^m_t(l,s)\right)u^m_t(l,s)\mathrm{d}s\\&+2\hat{f}\left(u^m(l,t)\right)\\&=\|u^m_t(0)\|^2+\|u^m_{xx}(0)\|^2+\hat{M}\left(z(0)\right)\\&+2\hat{f}\left(u^m(l,0)\right)+2\int_0^t\int_0^l q(x,s)u^m_t\mathrm{d}s\end{aligned}$$

where $z(t)=\|u^m_x\|^2$ and $z_t(t)=\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|u^m_x\|^2$.

Considering that

$$g\left(u^m_t(l,s)\right)u^m_t(l,s)\ge\rho\left|u^m_t(l,s)\right|^2>0,$$

$N\left(z_t(s)\right)z_t(s)>0$ and the initial conditions, we get

$$\|u^m_t\|^2-\int_0^t\|u^m_t(s)\|^2\mathrm{d}s\le C+\int_0^t\|q(x,s)\|^2\mathrm{d}s.$$

Using Gronwall inequality, we have $\|u^m_t\|^2\le Ce^T$. Then there exists a constant M_1 depending only on *T* such that

$$\|u^m_t\|^2+\|u^m_{xx}\|^2\le M_1. \tag{13}$$

for any $t\in[0,T]$ and for all $m\in N$.

In this paper, *C* is a constant independent of *m*, *t* and denotes different value in different mathematical expression.

Estimate 2. Integrating by parts (11) with $\omega=u^m_{tt}(0)$ and $t=0$, and considering the compatibility condition (3) we get

$$\begin{aligned}&\| u_{tt}^m(0) \|^2\\
&=\left[M\left(z(0)\right)+N\left(z_t(0)\right)\right]\int_0^l u_{xx}^m(0)u_{tt}^m(0)\mathrm{d}x\\
&\quad-\mu\int_0^l u_{xxxxt}^m(0)u_{tt}^m(0)\mathrm{d}x-\int_0^l u_{xxxx}^m(0)u_{tt}^m(0)\mathrm{d}x\\
&\quad-\eta\int_0^l u_t^m(0)u_{tt}^m(0)\mathrm{d}x+\int_0^l q(x,0)u_{tt}^m(0)\mathrm{d}x\\
&\le\left\|u_{tt}^m(0)\right\|\Big\{\left[M\left(z(0)\right)+N\left(z_t(0)\right)\right]\left\|u_{xx}^m(0)\right\|+\left\|u_{xxxx}^m(0)\right\|\\
&\qquad+\mu\left\|u_{xxxxt}^m(0)\right\|+\eta\left\|u_t^m(0)\right\|+\left\|q(0)\right\|\Big\}\end{aligned}$$

Thus there exists a positive constant M_2 such that

$$\left\|u_{tt}^m(0)\right\|\le M_2,\forall m\in N\ . \tag{14}$$

Estimate 3. Let us fix t, $\xi > 0$ such that $\xi < T - t$. Taking the difference of (11) with $t = t + \xi$ and $t = t$, and replacing ω by $u_t^m(t+\xi)-u_t^m(t)$, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|u_t^m(t+\xi)-u_t^m(t)\right\|^2+\left\|u_{xx}^m(t+\xi)-u_{xx}^m(t)\right\|^2\right]\\
&+\mu\left\|u_{xxt}^m(t+\xi)-u_{xxt}^m(t)\right\|^2+\eta\left\|u_t^m(t+\xi)-u_t^m(t)\right\|^2\\
&+\left[f\left(u^m(l,t+\xi)\right)-f\left(u^m(l,t)\right)\right]\left(u_t^m(l,t+\xi)-u_t^m(l,t)\right)\\
&+\left[g\left(u_t^m(l,t+\xi)\right)-g(\left(u_t^m(l,t)\right)\right]\left(u_t^m(l,t+\xi)-u_t^m(l,t)\right)\\
&+I_1\\
=&\int_0^l\left(q(t+\xi)-q(t)\right)\left(u_t^m(t+\xi)-u_t^m(t)\right)\mathrm{d}x\end{aligned}$$

where

$$I_1=\int_0^l\Big\{\left[M\left(z(t+\xi)\right)+N\left(z_t(t+\xi)\right)\right]u_x^m(t+\xi)-\left[M\left(z(t)\right)+N\left(z_t(t)\right)\right]u_x^m(t)\Big\}\left(u_{xt}^m(t+\xi)-u_{xt}^m(t)\right)\mathrm{d}x\,.$$

Let us estimate $\left|I_1\right|$. Since

$$u^m(0,t)=u_x^m(l,t)=u_{xx}^m(0,t)=0\,,$$

we have

$$\left\|u^m\right\|_\infty\le\sqrt{l}\left\|u_x^m\right\|,\left\|u_x^m\right\|_\infty\le\sqrt{l}\left\|u_{xx}^m\right\|,\left\|u_x^m\right\|\le l\left\|u_{xx}^m\right\| \tag{16}$$

Noting that $\Delta M_1 = M(z(t + \xi) - M(z(t))$ and $\Delta M_2 = N(z_t(t + \xi)) - N(z_t(t))$, then integrating by parts we have

$$\begin{aligned}\left|I_1\right|=&\Big|\left[M\left(z(t+\xi)\right)+N\left(z_t(t+\xi)\right)\right]\\
&\times\int_0^l\left(u_{xx}^m(t+\xi)-u_{xx}^m(t)\right)\left(u_t^m(t+\xi)-u_t^m(t)\right)\mathrm{d}x\\
&+\left(\Delta M_1+\Delta M_2\right)\int_0^l u_{xx}^m(t)\left(u_t^m(t+\xi)-u_t^m(t)\right)\mathrm{d}x\Big|\end{aligned}$$

Since $M(\cdot)\in C^1[0,\infty)$, by the Mean value theorem, from estimates 1 and (16) we have

$$\begin{aligned}\left|\Delta M_1\right|&=\left|M'(\eta_1)\left(\left\|u_x^m(t+\xi)\right\|^2-\left\|u_x^m(t)\right\|^2\right)\right|\\
&\le C\left\|u_{xx}^m(t+\xi)-u_{xx}^m(t)\right\|\end{aligned}$$

where η_1 is between $\left\|u_x^m(t+\xi)\right\|^2$ and $\left\|u_x^m(t)\right\|^2$.

By the Mean value theorem, we also have

$$\left|\Delta M_2\right|\le C\left\|u_{xx}^m(t+\xi)-u_{xx}^m(t)\right\|+C\left\|u_t^m(t+\xi)-u_t^m(t)\right\|$$

Considering that $M(z(t + \xi) \le C$ and $N(z(t + \xi) \le C$, we conclude that there exists constants $k_1 > 0$ and $k_2 > 0$ such that

$$\left|I_1\right|\le k_1\left\|u_{xx}^m(t+\xi)-u_{xx}^m(t)\right\|^2+k_2\left\|u_t^m(t+\xi)-u_t^m(t)\right\|^2 \tag{17}$$

A argument for f yields

$$\begin{aligned}&\left|f\left(u_t^m(l,t+\xi)\right)-f\left(u_t^m(l,t)\right)\right|\left|u_t^m(l,t+\xi)-u_t^m(l,t)\right|\\
&\le k_3\left\|u_{xx}^m(t+\xi)-u_{xx}^m(t)\right\|^2+\rho\left|u_t^m(l,t+\xi)-u_t^m(l,t)\right|^2\end{aligned} \tag{18}$$

where $k_3 > 0$ is a constant. Putting

$$\phi_m(t,\xi)=\left\|u_{xx}^m(t+\xi)-u_{xx}^m(t)\right\|^2+\left\|u_t^m(t+\xi)-u_t^m(t)\right\|^2$$

and taking into account of (17)-(18) and the assumptions of g, we deduce from (15) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_m(t,\xi)\le k_4\varphi_m(t,\xi),\forall t\in(0,T) \tag{19}$$

where $k_4 = \max\{k_1 + k_3, k_2\}$. Therefore

$$\phi_m(t,\xi)\le\phi_m(0,\xi)e^{k_4T} \tag{20}$$

Dividing the above inequality by ξ^2 and letting $\xi \to 0$ gives

$$\left\|u_{xxt}^m(t)\right\|^2+\left\|u_{tt}^m\right\|^2\le\left(\left\|u_{xxt}^m(0)\right\|^2+\left\|u_{tt}^m(0)\right\|^2\right)e^{k_4T}\,.$$

From estimate 2 we find a constant $M_3 > 0$ such that

$$\left\|u_{xxt}^m(t)\right\|^2+\left\|u_{tt}^m\right\|^2\le M_3,\forall m\in N,\forall t\in[0,T]\,.$$

With the estimates 1 - 3 we can use Lions-Aubin Lemma to get the necessary compactness in order to pass (11) to the limit. Then it is a matter of routine to conclude the existence of the global solution in [0, T].

Theorem 2. The solution $u(t)$ of theorem 1 is unique.

Proof. Let u, v be two solutions of (1)-(4) with the same initial data. Then writing $p = u - v$, putting $\omega = p_t$ in (10) and using mean value theorem, chauchy-schwarz inequality and Gronwall inequality, we may get $p = 0$. Thus $u = v$.

4. The Exponential DECAY of the Energy of System

In order to establish our decay result, we define the energy of the system by

$$E(t)=\frac{1}{2}\|u_t(t)\|^2+\frac{1}{2}\|u_{xx}(t)\|^2 +\frac{1}{2}\hat{M}\left(\|u_x(t)\|^2\right)+\hat{f}\left(u(l,t)\right)$$

where $\hat{M}(s)=\int_0^s M(z)\mathrm{d}z$. We have

Theorem 3. Let $u(t)$ be the solution given by theorem 1 as $q(x, t) = 0$ and $g = 0$. And assume that $N(s)s \ge 0$ and $f(s)s \ge 0$. Then there exist constants λ_2, $\lambda_4 > 0$ and $\lambda_3 < 0$ such that $E(t)\le\lambda_2 E(0)e^{\lambda_3 t}+\lambda_4 t e^{\lambda_3 t}$.

To prove Theorem 3, we firstly introduce two lemmas.

Let us define $\psi(t)=\int_0^l uu_t\mathrm{d}x+\frac{\eta}{2}\int_0^l u^2\mathrm{d}x$.

Then we have the following lemmas.

Lemma 1. Let $E_\varepsilon(t) = \mu E(t) + \varepsilon\psi(t)$. Then there exists a constant $k_5 > 0$ such that

$$\left|E_\varepsilon(t)-\mu E(t)\right|\le\varepsilon k_5 E(t),\forall\varepsilon>0.$$

Proof. By $\|u\|_\infty\le\sqrt{l}\|u_x\|$, $\|u_x\|_\infty\le\sqrt{l}\|u_{xx}\|$ and $\|u\|\le l\|u_{xx}\|$ there exists $k_5 > 0$ such that

$$\begin{aligned}\left|E_\varepsilon(t)-\mu E(t)\right| &=\varepsilon\int_0^l uu_t\mathrm{d}x+\frac{\eta\varepsilon}{2}\int_0^l u^2\mathrm{d}x\\ &\le\frac{\varepsilon l^2}{2}\|u_{xx}\|^2+\frac{\varepsilon}{2}\|u_t\|^2+\frac{\eta\varepsilon l^2}{2}\|u_{xx}\|^2\\ &\le\varepsilon k_5 E(t)\end{aligned}$$

where $k_5=(1+\eta)l^2$.

Lemma 2. There exist constants $\lambda_0 > 0$ and λ_1 such that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon(t)\le-2\lambda_0 E(t)+\lambda_1.$$

Proof. Taking the inner product of (1) with u_t and considering that $N(s)s \ge 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)\le-\mu\|u_{xxt}\|^2-\eta\|u_t\|^2.$$

Taking the inner product of (1) with u, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\psi(t)&=\int_0^l u_t^2\mathrm{d}x-\int_0^l u_{xx}^2\mathrm{d}x-\mu\int_0^1 u_{xxt}u_{xx}\mathrm{d}x\\ &-\left(M\left(\|u_x\|^2\right)\right)+N\left(\int_0^l u_x u_{xt}\mathrm{d}x\right)\int_0^l u_x^2 dx\\ &-f\left(u(l,t)\right)u(l,t)\end{aligned}$$

Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon(t)&\le-\mu^2\|u_{xxt}\|^2-\varepsilon^2\int_0^l u_{xx}^2\mathrm{d}x-\varepsilon\mu\int_0^l u_{xxt}u_{xx}dx\\ &-\left(\varepsilon-\varepsilon^{2`}\right)\int_0^l u_{xx}^2\mathrm{d}x-(\eta\mu-\varepsilon)\|u_t\|^2\\ &-\varepsilon M\left(\|u_x\|^2\right)\int_0^l u_x^2\mathrm{d}x-\varepsilon N(\int_0^l u_x u_{xt}\mathrm{d}x)\int_0^l u_x^2\mathrm{d}x\\ &-\varepsilon f\left(u(l,t)\right)u(l,t)\end{aligned}.$$

Set $h(u_{xxt})=-\mu^2\|u_{xxt}\|^2-\varepsilon^2\int_0^l u_{xx}^2\mathrm{d}x-\varepsilon\mu\int_0^l u_{xxt}u_{xx}\mathrm{d}x$.

Since $\Delta=(\varepsilon\mu)^2(u_{xx})^2-4\mu^2\varepsilon^2u_{xx}^2=-3(\varepsilon\mu)^2u_{xx}^2\le0$, we have $h(u_{xxt})\le0$. Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon(t)&\le-\left(\varepsilon-\varepsilon^{2`}\right)\int_0^l u_{xx}^2\mathrm{d}x-(\eta\mu-\varepsilon)\|u_t\|^2\\ &-\varepsilon M\left(\|u_x\|^2\right)\times\int_0^l u_x^2\mathrm{d}x-\varepsilon N\left(\int_0^l u_x u_{xt}\mathrm{d}x\right)\int_0^l u_x^2\mathrm{d}x.\\ &-\varepsilon f\left(u(l,t)\right)u(l,t)\end{aligned}$$

From the Mean value theorem, there exists a constant λ_1 such that

$$-\varepsilon N\left(\int_0^l u_x u_{xt}\mathrm{d}x\right)\int_0^l u_x^2\mathrm{d}x\le\lambda_1.$$

On writing $\lambda_0=\min\left\{\eta\mu-\varepsilon,\varepsilon-\varepsilon^2,2\varepsilon\right\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon(t)\le-2\lambda_0E(t)+\lambda_1$$

The proof of theorem 3. From lemma 1, we have

$$(\mu-\varepsilon k_5)E(t)\le E_\varepsilon(t)\le(\mu+\varepsilon k_5)E(t). \tag{21}$$

From Lemma 2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon(t)\le-2\lambda_0E(t)+\lambda_1. \tag{22}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon(t)\le-2\lambda_0\left(\frac{1}{\mu+\varepsilon k_5}\right)E_\varepsilon(t)+\lambda_1.$$

By Gronwall inequality and combing (21), we have

$$\begin{aligned}(\mu-\varepsilon k_5)E(t)&\le\left[E_\varepsilon(0)+\lambda_1 t\right]e^{\left(-2\lambda_0\frac{1}{\mu+\varepsilon k_5}t\right)}\\ &\le\left[(\mu+\varepsilon k_5)E(0)+\lambda_1 t\right]e^{\left(-2\lambda_0\frac{1}{\mu+\varepsilon k_5}t\right)}\end{aligned}.$$

Hence, for sufficiently small $\varepsilon > 0$

$$E(t)\le\frac{\mu+\varepsilon k_5}{\mu-\varepsilon k_5}E(0)e^{\left(-2\lambda_0\frac{1}{\mu+\varepsilon k_5}t\right)}+\frac{\lambda_1}{\mu-\varepsilon k_5}e^{\left(-2\lambda_0\frac{1}{\mu+\varepsilon k_5}t\right)}.$$

On writing $\lambda_2=\frac{\mu+\varepsilon k_5}{\mu-\varepsilon k_5},\lambda_3=\frac{-2\lambda_0}{\mu+\varepsilon k_5}<0$

and $\lambda_4=\frac{\lambda_1}{\mu-\varepsilon k_5}$,

we have $E(t)\le\lambda_2E(0)e^{\lambda_3t}+\lambda_4te^{\lambda_3t}$.

The proof of theorem 3 is now completed.

REFERENCES

[1] S. Woinowsky-Krieger, “The Effect of Axial Force on the

Vibration of Hinged Bars," *Journal of Applied Mechanics*, Vol. 17, 1950, pp. 35-36.

[2] J. M. Ball, "Initial-Boundary Value Problems for an Extensible Beam," *Journal of Mathematical Analysis and Applications*, Vol. 42, No. 1, 1973, pp. 61-90. doi:10.1016/0022-247X(73)90121-2

[3] L. A. Mederios, "On a New Class of Nonlinear Wave Equations," *Journal of Mathematical Analysis and Applications*, Vol. 69, No. 1, 1979, pp. 252-262.

[4] M. Tucsnak, "Semi-Internal Stabilization for a Nonlinear Euler-Bernoulli Equation," *Mathematical Methods in the Applied Sciences*, Vol. 19, No. 11, 1996, pp. 897-907. doi:10.1002/(SICI)1099-1476(19960725)19:11<897::AID-MMA801>3.0.CO;2-#

[5] S. Kouemou Patcheu, "On a Global Solution and Asymptotic Behavior for the Generalized Damped Extensible Beam Equation," *Journal of Differential Equations*, Vol. 135, No. 2, 1997, pp. 299-314. doi:10.1006/jdeq.1996.3231

[6] F. M. To, "Boundary Stabilization for a Non-Linear Beam on Elastic Bearings," *Mathematical Methods in the Applied Sciences*, Vol. 24, No. 8, 2001, pp. 583-594. doi:10.1002/mma.230

[7] J. M. Ball, "Stability Theory for an Extensible Beam," *Journal of Differential Equations*, Vol. 14, No. 3, 1973, pp. 61-90. doi:10.1016/0022-0396(73)90056-9