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Chaotic and Hyperchaotic Complex Jerk Equations

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ABSTRACT

The aim of this paper is to introduce and investigate chaotic and hyperchaotic complex jerk equations. The jerk equations describe various phenomena in engineering and physics, for example, electrical circuits, laser physics, mechanical oscillators, damped harmonic oscillators, and biological systems. Properties of these systems are studied and their Lyapunov exponents are calculated. The dynamics of these systems is rich in wide range of systems parameters. The control of chaotic attractors of the complex jerk equation is investigated. The Lyapunov exponents are calculated to show that the chaotic behavior is converted to regular behavior.

Keywords: Hyperchaotic; Chaotic; Attractors; Lyapunov Exponents; Jerk Function; Control; Complex

1. Introduction

Chaos and hyperchaos can occur in systems of autonomous ordinary differential equations (ODE's) with at least three variables and two quadratic nonlinearities [1-4]. The Poincaré-Bendixson theorem shows that chaos does not exist in a two-dimensional autonomous system. In 1994, Sprott [5] found numerically fourteen chaotic systems with six terms and one quadratic nonlinearities. In 1996, Gottlieb [6] showed that some of these systems could be written as a single third-order ODE. By "jerk function" he means a function j such that the third-order ODE can be written in the form $\dddot{\theta} = j\left(\theta, \dot{\theta}, \ddot{\theta}\right)$, where j is the time derivative of the acceleration $\ddot{\theta}$ and the equation is called a jerk equation. The jerk equations describe various phenomena in engineering and physics, for example, electrical circuits, laser physics, mechanical oscillators, damped harmonic oscillators, and biological systems. In the literature some jerk equations are introduced and studied [7-13]. Kocić *et al.* [14] considered and studied two modifications of a 3-dimensional dynamic flow known as jerk dynamical systems of Sprott [13].

In this paper, we define a complex jerk equation as an autonomous third-order complex differential equation of the form:

$$\dddot{z} = j\left(z, \overline{z}, \dot{z}, \dot{\overline{z}}, \ddot{z}, \ddot{\overline{z}}\right), \tag{1}$$

where z is a complex variable, the overdot represent the time derivative and j is the complex jerk function (time derivative of acceleration $\ddot{z}$) and an overbar denotes complex conjugate variables. In recent years we introduced and studied several complex nonlinear systems which appear in many important applications [15-21].

As an example of Equation (1), we propose a chaotic complex jerk equation as:

$$\dddot{z} + \alpha\ddot{z} + \beta\dot{z} + \nu z + \eta z^2\overline{z} = 0 \tag{2}$$

where α, β and η, are positive parameters, ν is a negative parameter, z is complex variable, dots represent derivatives with respect to time and an overbar denotes complex conjugate variables.

We suggest the equation:

$$\dddot{z} + \alpha\ddot{z} - z\dot{z}^2 + z = 0, \alpha > 0, \tag{3}$$

which is an example of a hyperchaotic complex jerk equation. The corresponding real form of (3) (*i.e. z* is a real variable) was introduced in [9], and has only chaotic behavior.

The organization of rest of the paper is as follows: In Section 2, symmetry, invariance, fixed points of (2) and stability analysis of the trivial fixed point are discussed. Lyapunov exponents are calculated in Section 3, and used to classify the attractors of (2). It is clear that our Equation (2) has chaotic, periodic, quasi-periodic attractors and solutions that approach fixed points. In Section 4, we study the basic properties of the hyperchaotic complex jerk Equation (3). Numerically the range of parameters values of the system at which hyperchaotic attractors exist is calculated in Section 5. Section 6 contains the control of chaotic attractors of Equation (2) by adding a complex periodic forcing. The last section contains our concluding remarks.

2. Basic Properties of Chaotic Jerk Equation (2)

In this section we study the basic properties of our new Equation (2). Equation (2) can equivalently be written as three, first-order, ordinary differential equations as:

$$\dot{z} = x, \dot{x} = y, \dot{y} = -\alpha y - \beta x - v z - \eta z^2 \bar{z}, \qquad (4)$$

where $z = u_1 + i u_2$, $x = u_3 + i u_4$ and $y = u_5 + i u_6$ are complex variables, $i = \sqrt{-1}$, $u_j, j = 1, 2, \cdots, 6$ are real variables, dots represent derivatives with respect to time and an overbar denotes complex conjugate variables.

The real version of (4) is:

$$\dot{u}_1 = u_3, \dot{u}_2 = u_4, \dot{u}_3 = u_5, \dot{u}_4 = u_6,$$
$$\dot{u}_5 = -\alpha u_5 - \beta u_3 - v u_1 - \eta u_1 \left(u_1^2 + u_2^2 \right), \qquad (5)$$
$$\dot{u}_6 = -\alpha u_6 - \beta u_4 - v u_2 - \eta u_2 \left(u_1^2 + u_2^2 \right).$$

The basic dynamical properties of system (5) are:

2.1. Symmetry and Invariance

From (5), we note that this system is invariant under the transformation
$$\left(u_1, u_2, u_3, u_4, u_5, u_6 \right) \Leftrightarrow \left(-u_1, -u_2, -u_3, -u_4, -u_5, -u_6 \right).$$
Therefore, if $\left(u_1, u_2, u_3, u_4, u_5, u_6 \right)$ is a solution of (5), then $\left(-u_1, -u_2, -u_3, -u_4, -u_5, -u_6 \right)$ is also a solution of the same system.

2.2. Dissipation

The divergence of (5) is: $\sum_{j=1}^{6} \dfrac{\partial \dot{u}_j}{\partial u_j} = -2\alpha$.

Therefore the system (5) is dissipative for the case: $\alpha > 0$.

2.3. Equilibria and Their Stability

The fixed points of system (5) can be found by solving the following equations:

$$u_3 = 0, u_4 = 0, u_5 = 0, u_6 = 0,$$
$$-\alpha u_5 - \beta u_3 - v u_1 - \eta u_1 \left(u_1^2 + u_2^2 \right) = 0, \qquad (6)$$
$$-\alpha u_6 - \beta u_4 - v u_2 - \eta u_2 \left(u_1^2 + u_2^2 \right) = 0.$$

Therefor system (4) has trivial fixed point $E_0 = (0, 0, 0, 0, 0, 0)$. The projection in the $\left(u_1, u_2 \right)$ plane of the non trivial fixed points is a *circle*:

$$u_1^2 + u_2^2 = r^2, \qquad (7)$$

whose center is at the origin and radius is $r = \sqrt{-v/\eta}$.

The non trivial fixed point can be written in the form

$$E_\theta = \left(u_1, u_2, u_3, u_4, u_5, u_6 \right),$$

where $u_1 = r \cos\theta$, $u_2 = r \sin\theta$, $u_3 = u_4 = u_5 = u_6 = 0$, for $\theta = [0, 2\pi]$.

To study the stability of E_0 the Jacobian matrix of Equation (2) at E_0 is:

$$J_{E_0} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -v & 0 & -\beta & 0 & -\alpha & 0 \\ 0 & -v & 0 & -\beta & 0 & -\alpha \end{pmatrix}.$$

The characteristic polynomial is:

$$\left(\lambda^3 + \alpha \lambda^2 + \beta \lambda + v \right)^2 = 0. \qquad (8)$$

According to the Routh-Hurwitz condition, the real parts of the roots λ of (8) are negative if and only if

$$\alpha > 0, \beta > 0, v > 0, \alpha\beta - v > 0.$$

Since, $v < 0$ then E_0 is unstable.

The stability analysis of E_θ can be similarly studied as we did for the trivial fixed point E_0.

3. Lyapunov Exponents and Attractors of Equation (2)

This section is devoted to calculate Lyapunov exponents and used their signs to classify attractors of Equation (2). Based on these exponents, we compute parameters values of our Equation (2) at which chaotic, periodic, and quasi-periodic attractors and attractors that approach fixed point exist.

3.1. Lyapunov Exponents

System (5) in vector notation can be written as:

$$\dot{U}(t) = h\left(U(t); \eta \right), \qquad (9)$$

where $U(t) = \left[u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t) \right]^t$ is the state space vector, $h = \left[h_1, h_2, h_3, h_4, h_5, h_6 \right]^t$, η is a set of parameters and $\left[\cdots \right]^t$ denoting transpose. The equations for small deviations δU from the trajectory $U(t)$ are:

$$\delta \dot{U}(t) = L_{lj}\left(U(t); \eta \right) \delta U, \quad l, j = 1, 2, 3, 4, 5, 6 \qquad (10)$$

where $L_{l,j} = \dfrac{\partial h_l}{\partial u_j}$ is the Jacobian matrix of the form:

$$L_{l,j} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -v - \eta\left(3u_1^2 + u_2^2 \right) & -2\eta u_1 u_2 & -\beta & 0 & -\alpha & 0 \\ -2\eta u_1 u_2 & -v - \eta\left(3u_2^2 + u_1^2 \right) & 0 & -\beta & 0 & -\alpha \end{pmatrix}.$$

The Lyapunov exponents λ_l of the system is defined by:

$$\lambda_l = \lim_{t \to \infty} \frac{1}{t} \log \frac{\|\delta u_l(t)\|}{\|\delta u_l(0)\|}. \qquad (11)$$

To find λ_l, Equations (9) and (10) must be numerically solved simultaneously. Runge–Kutta method of order 4 is used to calculate λ_l.

For the choice $\alpha = 1$, $\beta = 4$, $\nu = -5$ and $\eta = 1$ and the initial conditions $t_0 = 0$, $u_1(0) = 4$, $u_2(0) = 1$, $u_3(0) = -2$, $u_4(0) = 2$, $u_5(0) = -1$ and $u_6(0) = 1$. We calculate the Lyapunov exponents which are: $\lambda_1 = 0.244$, $\lambda_2 = 0$, $\lambda_3 = 0$, $\lambda_4 = -0.488$, $\lambda_5 = -0.954$, $\lambda_6 = -1.691$.

This means that our system (4) for this choice of α, β, μ and η is a chaotic system since one of its Lyapunov exponents λ_1 is positive and dissipative system because their sum is negative.

3.2. Attractors of Equation (2)

3.2.1. Fix $\beta = 4$, $\nu = -5$, $\eta = 1$ and Vary α
In **Figures 1(a)** and **(b)** we plotted the corresponding Lyapunov exponents λ_l, $l = 1, 2, \cdots, 6$ of system (5) using the initial conditions $t_0 = 0$, $u_1(0) = 4$, $u_2(0) = 1$, $u_3(0) = -2$, $u_4(0) = 2$, $u_5(0) = -1$ and $u_6(0) = 1$. It is clear that from **Figure 1(a)**, when $\alpha \in [0.65, 0.795]$, $(0.840, 1.025]$ and $(1.26, 1.34]$ the new system (5) has chaotic attractors. It has also periodic attractors for $\alpha \in (1.34, 2.54]$, while it has quasi-periodic attractors when α lies in the interval $(0.795, 0.840)$. Solutions of system (5) that approach nontrivial fixed points are exist for $\alpha \in (2.54, 4.7)$. As is shown in **Figure 1(b)** the values of λ_4, λ_5 and λ_6 are negative.

3.2.2. Fix $\alpha = 1$, $\nu = -5$, $\eta = 1$ and Vary β
From **Figure 1(c)** one can conclude that (5) has chaotic attractors for $\beta \in [3.4, 3.590]$, $[3.610, 3.925]$, $[3.955, 4.135]$ and $(4.155, 4.880]$, and periodic attractors for β lies in $(3.590, 3.610]$, $[4.135, 4.155]$ and $(4.880, 9.8]$. Solutions of system (5) that approach nontrivial fixed points are exist for $\beta \in (10.1, 12.4)$.

3.2.3. Fix $\alpha = 1$, $\beta = 4$, $\eta = 1$ and Vary ν
As we did before we plot only λ_1, λ_2 and λ_3 in **Figure 1(d)** and we see that (5) has chaotic attractors for $\nu \in [-6.7, -6.315]$, $(-6.240, -5.955]$, $(-5.865, -5.580]$, $(-5.525, -5.105]$, $(-5.07, -4.805]$ and $(-4.775, -3.995]$. In between above values of ν our system has periodic attractors as one sees from **Figure 1(d)**.

3.2.4. Fix $\alpha = 1$, $\beta = 4$, $\nu = -5$ and Vary η
As is shown in **Figure 1(e)**, the chaotic attractors exist for $\eta \in [0.001, 1.14]$.

Using the same choice of the parameters and initial

conditions as in **Figure 1**, the chaotic attractors of (5) are plotted in **Figure 2** in (u_2, u_3, u_4), (u_2, u_4, u_6), (u_2, u_4, u_5) and (u_1, u_4, u_5) spaces respectively.

4. Some Properties of Equation (3)

This section deals with the basic dynamical properties of a hyperchaotic jerk Equation (3). As we did in Section 2, Equation (3) can be written as a system of three, first-order, ordinary differential equations such as:

$$\dot{z} = x, \dot{x} = y, \dot{y} = -\alpha y - zx^2 - z, \qquad (12)$$

where $z = u_1 + iu_2$, $x = u_3 + iu_4$ and $y = u_5 + iu_6$ are complex variables, $i = \sqrt{-1}$, u_j, $j = 1, 2, \cdots, 6$ are real variables.

The real version of (3) is:

$$\dot{u}_1 = u_3, \dot{u}_2 = u_4, \dot{u}_3 = u_5, \dot{u}_4 = u_6,$$
$$\dot{u}_5 = -\alpha u_5 + u_1(u_3^2 - u_4^2) - 2u_2 u_3 u_4 - u_1, \qquad (13)$$
$$\dot{u}_6 = -\alpha u_6 + u_2(u_3^2 - u_4^2) + 2u_1 u_3 u_4 - u_2.$$

If $(u_1, u_2, u_3, u_4, u_5, u_6)$ is a solution of (13), then $(-u_1, -u_2, -u_3, -u_4, -u_5, -u_6)$ is also a solution.

From (13) if $\alpha > 0$, then (13) is dissipative.

System (13) has only trivial fixed point $E_0 = (0, 0, 0, 0, 0, 0)$.

To study the stability of E_0 we calculate the Jacobian matrix of system (13) at E_0 to get:

$$J_{E_0} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & -\alpha & 0 \\ 0 & -1 & 0 & 0 & 0 & -\alpha \end{pmatrix}.$$

Its characteristic polynomial is:

$$P(\lambda) = (\lambda^3 + \alpha\lambda^2 + 1)^2 = 0. \qquad (14)$$

An elementary study proves that this polynomial has only one real root $\lambda_r < \dfrac{-2\alpha}{3}$ which is therefore negative. Since the characteristic equation is a cubic equation with real coefficients, we will have without loss of generality $P(\lambda) = (\lambda - \lambda_r)(\lambda - \lambda_c)(\lambda - \bar{\lambda}_c)$. where λ_c is a complex number. After expanding the above equation and comparing the coefficients with those of the original characteristic equation we come up with the relation $\text{Re}(\lambda_c) = 2^{-1}\lambda_r^{-2}$. We conclude that the real part of λ_c is positive and that the fixed point is unstable, see Ref. [9].

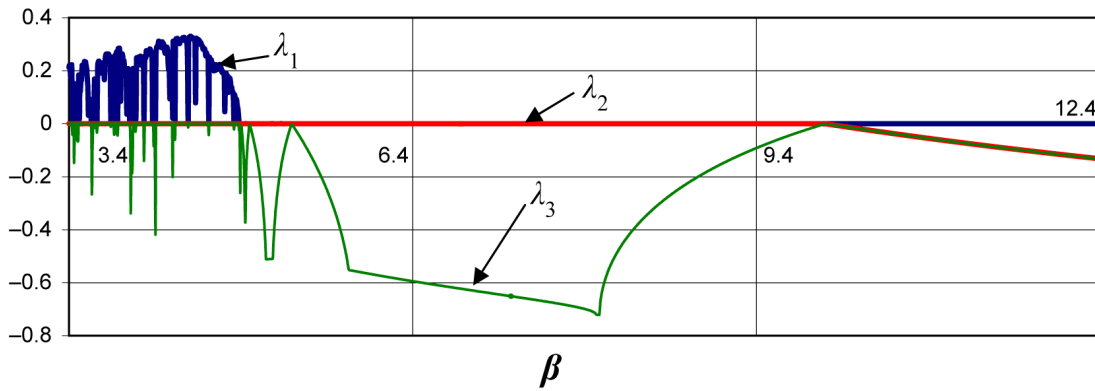## 5. Lyapunov Exponents and Attractors of Hyperchaotic Equation (3)

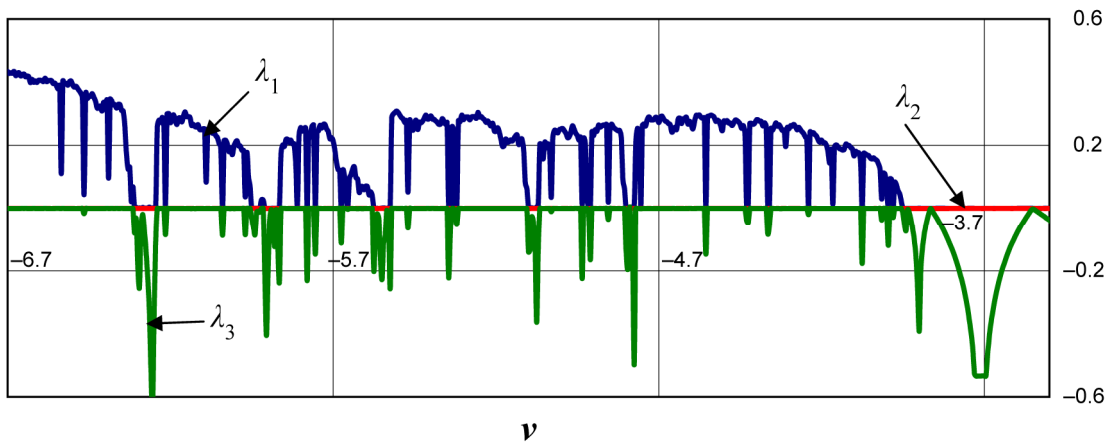In this section we calculate the Lyapunov exponents and attractors of Equation (3).
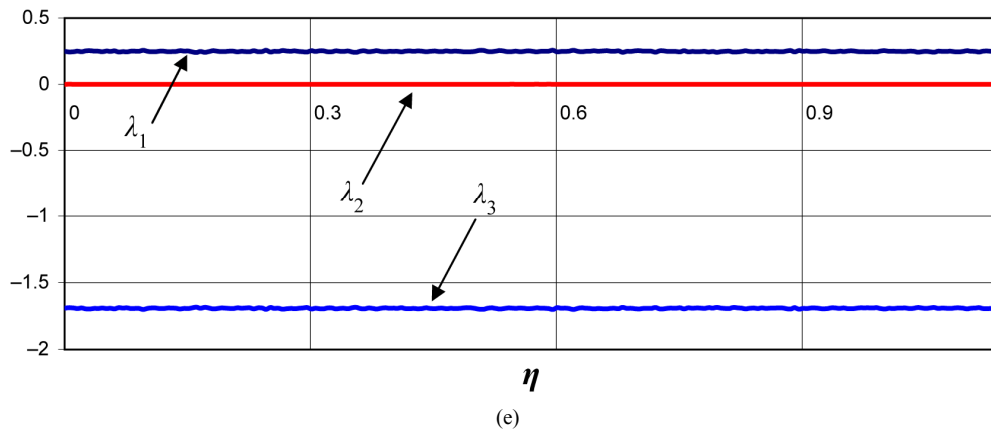
(a)

(b)

(c)

(d)

(e)
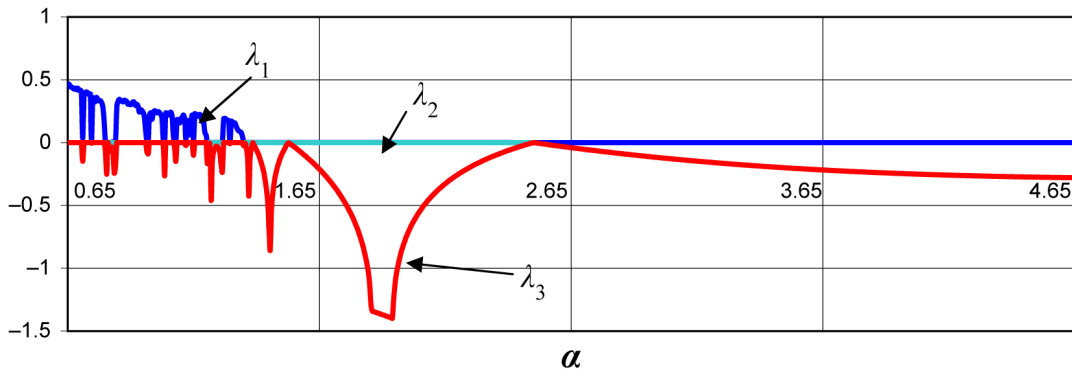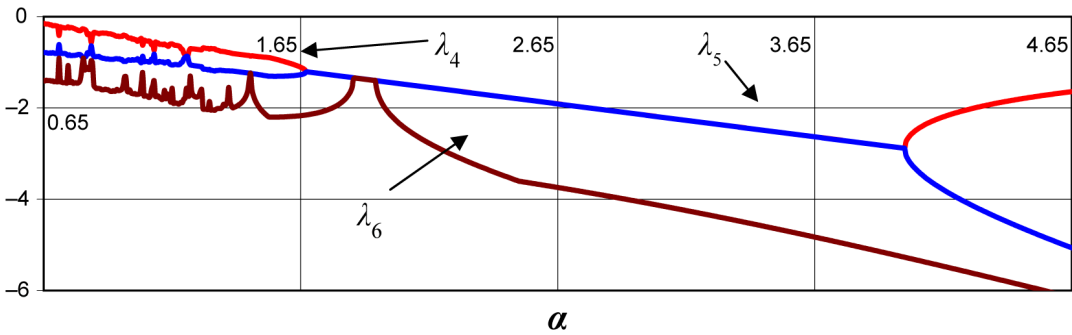
Figure 1. Lyapunov exponents of (2) with the initial conditions $t_0 = 0$, $u_1(0) = 4$, $u_2(0) = 1$, $u_3(0) = -2$, $u_4(0) = 2$, $u_5(0) = -1$ and $u_6(0) = 1$. (a) λ_1, λ_2 and λ_3 versus α; (b) λ_4, λ_5 and λ_6 versus α; (c) λ_1, λ_2 and λ_3 versus β; (d) λ_1, λ_2 and λ_3 versus ν; (e) λ_1, λ_2 and λ_3 versus η.
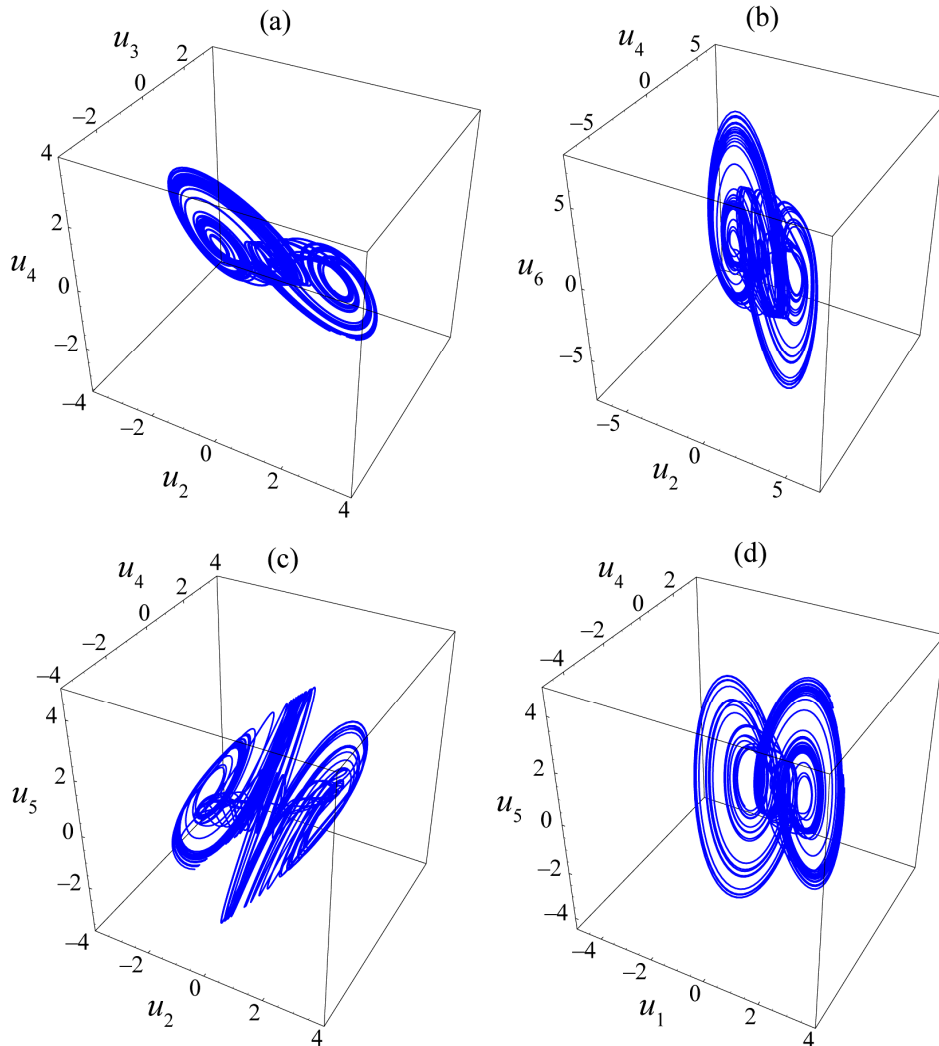

Figure 2. A chaotic attractor of (2) for $\alpha = 1$, $\beta = 4$, $\nu = -5$ and $\eta = 1$ at the same initial conditions as in Figure 1. (a) in (u_2, u_3, u_4) space; (b) in (u_2, u_4, u_6) space; (c) in (u_2, u_4, u_5) space; (d) in (u_1, u_4, u_5) space.

For the choice $\alpha = 2.03$ and the initial conditions $t_0 = 0$, $u_1(0) = 4$, $u_2(0) = 0$, $u_3(0) = 0$, $u_4(0) = 0$, $u_5(0) = -1$ and $u_6(0) = 0$, we calculate the Lyapunov exponents which are: $\lambda_1 = 0.111018$, $\lambda_2 = 0.111018$, $\lambda_3 = 0$, $\lambda_4 = 0$, $\lambda_5 = -3.03968$ and $\lambda_6 = -3.03968$. This means that our Equation (3) for this choice of α is hyper chaotic equation since two of the Lyapunov exponents λ_1 and λ_2 are positive and dissipative equation since the sum of its Lyapunov exponents is negative.

In **Figure 3(a)** we plotted the corresponding Lyapunov exponents λ_l, $l = 1, 2, \cdots, 6$ of Equation (3) using the initial conditions $t_0 = 0$, $u_1(0) = 4$, $u_2(0) = 0$, $u_3(0) = 0$, $u_4(0) = 0$, $u_5(0) = -1$ and $u_6(0) = 0$. It is clear that from **Figure 3(a)**, when $\alpha \in (2.0278, 2.0413]$, $(2.0430, 2.0539]$, $(2.0544, 2.0732]$ and $(2.0743, 2.0840]$ Equation (3) has

hyperchaotic attractors, while it has quasi-periodic attractors when α lies in the intervals, $(2.0413, 2.0430]$, $(2.0732, 2.0743]$ and $(2.0840, 2.1]$.

The hyperchaotic attractors of (3) using the same initial conditions as in **Figure 3(a)** and for $\alpha = 2.03$, are plotted in **Figures 3(b)** and **(c)** in (u_2, u_3, u_5) and (u_1, u_2, u_3) spaces respectively.

6. Control of Chaotic Attractors of System (4)

This section is devoted to study the control of chaotic attractors of system (4), based on the addition of complex periodic forcing $k \exp(i\omega t)$ to its first and second equations, so system (4) becomes:

$$\dot{z} = x + k\exp(i\omega t), \dot{x} = y, \dot{y} = -\alpha y - \beta x - vz - \eta z^2\bar{z}. \text{ (15)}$$
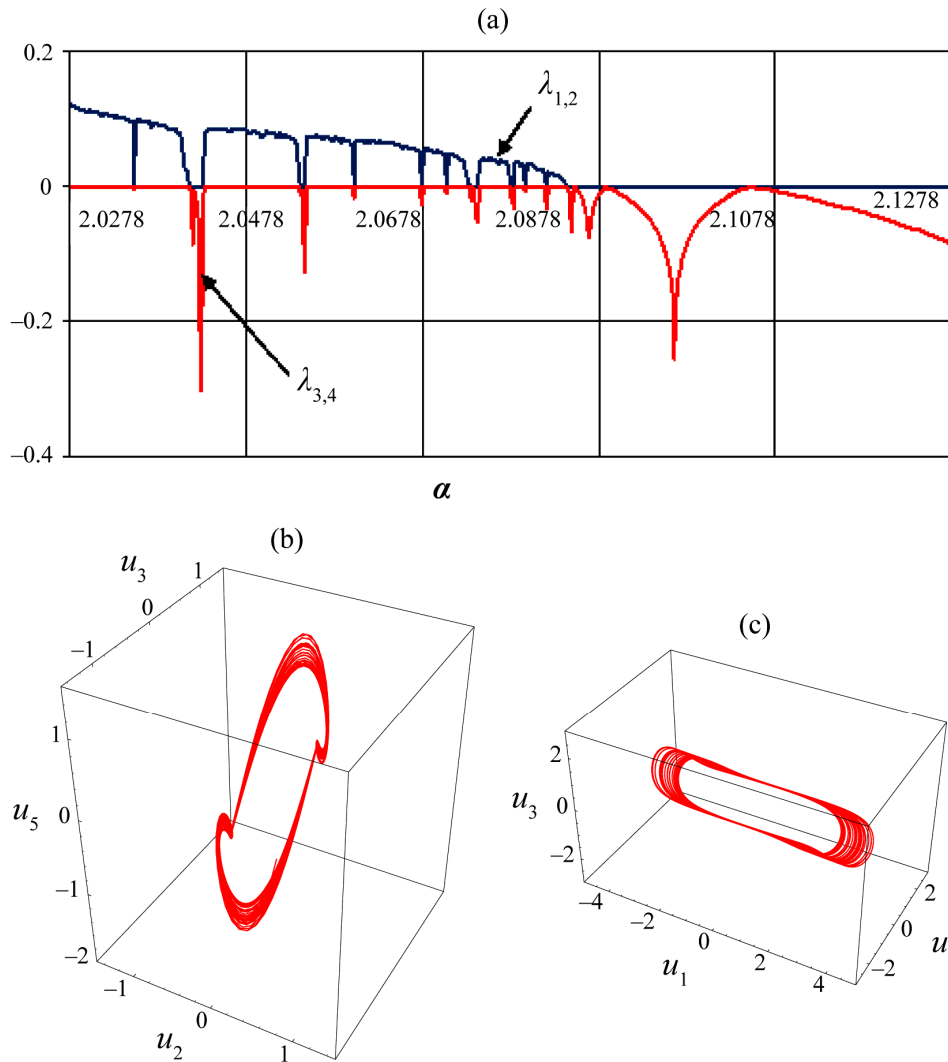
Figure 3. Lyapunov exponents and numerical calculations of the hyperchaotic attractors of (3) using the initial conditions $t_0 = 0$, $u_1(0) = 4$, $u_2(0) = 0$, $u_3(0) = 0$, $u_4(0) = 0$, $u_5(0) = -1$ and $u_6(0) = 0$. (a) λ_1, λ_2, λ_3 and λ_4 versus α; (b) A hyperchaotic attractors of (3) in (u_2, u_3, u_5) space; (c) A hyperchaotic attractors of (3) in (u_1, u_2, u_3) space.

The real version of (15) using $u_7 = \omega t$ reads:

$$\dot{u}_1 = u_3 + k\cos(u_6),$$
$$\dot{u}_2 = u_4 + k\sin(u_6),$$
$$\dot{u}_3 = u_5, \dot{u}_4 = u_6,$$
$$\dot{u}_5 = -\alpha u_5 - \beta u_3 - \nu u_1 - \eta u_1\left(u_1^2 + u_2^2\right), \quad (16)$$
$$\dot{u}_6 = -\alpha u_6 - \beta u_4 - \nu u_2 - \eta u_2\left(u_1^2 + u_2^2\right),$$
$$\dot{u}_7 = \omega.$$

What we would like to see is whether, by a suitable selection of values of k and ω, one can control the chaotic solutions of system (4) by converting them from chaotic to periodic with frequency ω.

For the choice of $\alpha = 1$, $\beta = 4$, $\nu = -4.3$, $\eta = 1$, $\omega = 10$ and $k = 10$, and the initial conditions $t_0 = 0$, $u_1(0) = 4$, $u_2(0) = 1$, $u_3(0) = -2$, $u_4(0) = 2$, $u_5(0) = -1$, $u_6(0) = 1$, $u_7(0) = 1$, we have the Lyapunov exponents $\lambda_1 = 0$, $\lambda_2 = -0.02861$, $\lambda_3 = -0.02943$, $\lambda_4 = -0.72071$, $\lambda_5 = -0.72155$, $\lambda_6 = -1.38533$, $\lambda_7 = 1$ (for more details about the calculations of Lyapunov exponents, see Ref. [16]. This means that the hyperchaotic attractor of (2) is converted to periodic behavior (see **Figures 4(a)** and **(c)** before control and **Figures 4(b)** and **(d)** after control).

7. Conclusions

In this paper we proposed both chaotic and hyperchaotic complex jerk equations and investigated their dynamics. The stability analysis of the trivial fixed points of these complex equations are studied. These equations appeared in several important applications of physics, engineering, and biology.
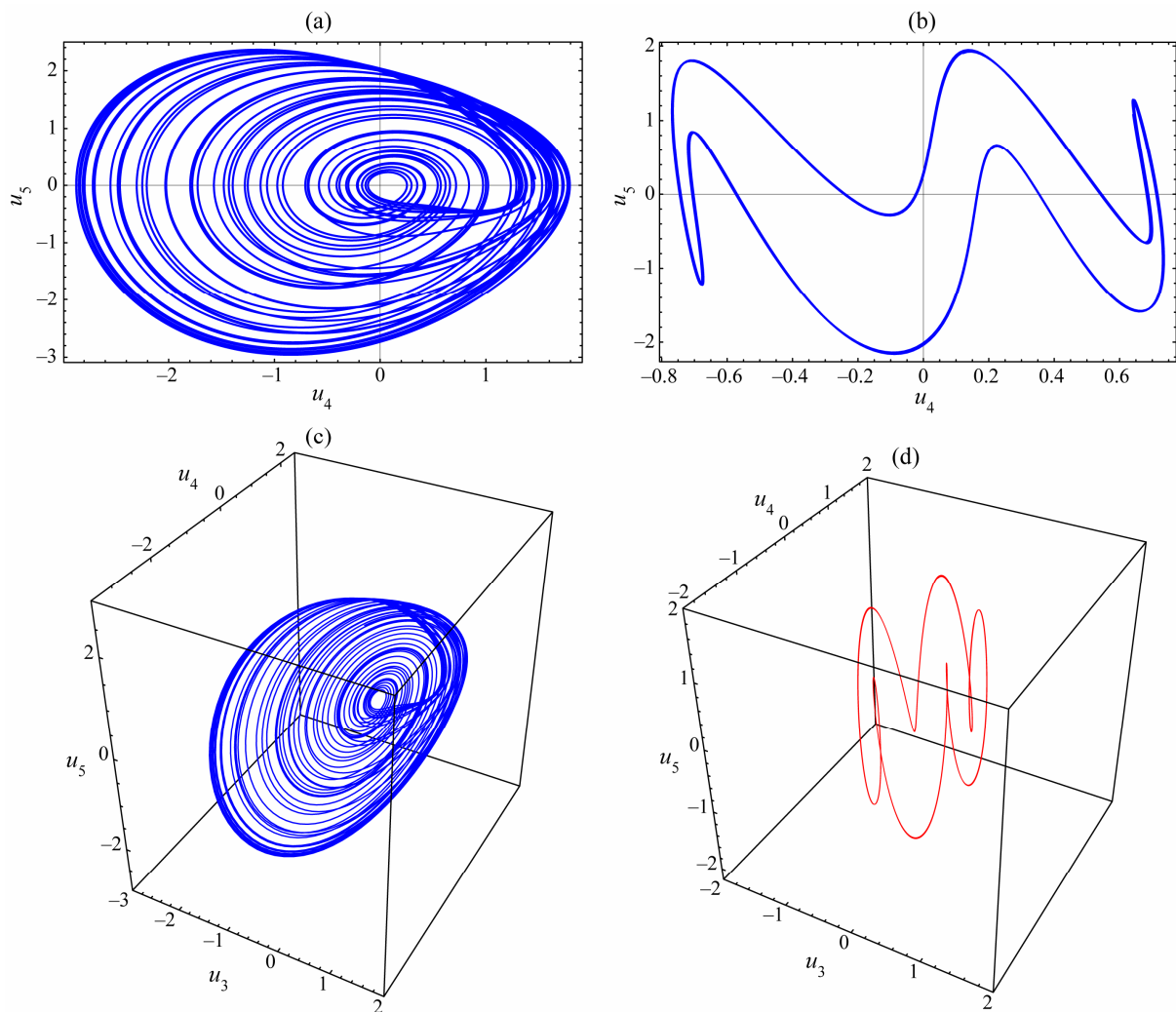
Figure 4. A chaotic attractor of system (15) for $\alpha = 1, \beta = 4$, $\nu = -4.3$, $\eta = 1$, $k = 10$, $w = 10$ **and** $u_7(0) = 1$, **with the same initial conditions as in Figure 1 and (a)** (u_4, u_5) **plane before control; (b)** (u_4, u_5) **plane after control; (c)** (u_3, u_4, u_5) **space before control; (d)** (u_3, u_4, u_5) **space after control.**

Both of our examples (2) and (3) are symmetric and dissipative under the condition $\alpha > 0$. The chaotic Equation (2) has both isolated and non-isolated fixed points. The projection of non-isolated fixed point in (u_1, u_2) space is a circle with center $(0,0)$. The dynamic of system (2) is very complicated as shown in **Figure 1**, since it has, solutions approach to fixed points, periodic solutions, quasi-periodic solutions and chaotic behavior. The hyperchaotic Equation (3) has only one fixed point. The values of the parameter α at which (3) has hyperchaotic attractors is calculated. The control of chaotic attractors of Equation (2) is studied by adding a complex periodic forcing and the results are shown in **Figure 4**. Other examples of Equation (1) can be similarly studied and investigated as we did for Equations (2) and (3).

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