

A Similarity Based Fuzzy System as a Function Approximator

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How to cite this paper: Pal, A., Ul Karim, I., Mondal, B. and Raha, S. (2018) A Similarity Based Fuzzy System as a Function Approximator. *International Journal of Intelligence Science*, **8**, 89-116. https://doi.org/10.4236/ijis.2018.84005

Received: July 17, 2018 Accepted: September 7, 2018 Published: September 10, 2018

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Abstract

In this paper, an attempt is made to prove that some similarity based fuzzy systems can be found to behave as function approximators. A typical similarity based fuzzy system is proposed and its behaviour is shown to have the said property. It elucidates the connection between similarity relation and similarity measure of fuzzy sets to fuzzy inference methodology. The concept of similarity relation is used in fuzzification of crisp input values. Similarity index is used in measuring approximate equality of fuzzy sets over a given universe of discourse of a linguistic variable. The similarity between the observation(s) and the antecedent of a rule is used in selecting rule(s) for possible firing and also in modifying the relation between the antecedent and consequent of the rule based on the specific observation. Inference is drawn through the usual composition and subsequently by projecting the modified fuzzy restriction acting on the variables of interest on the universe of the linguistic variable in the consequent of the rule. A specificity based defuzzification scheme is proposed for multiple-rule firing. It has been proved systematically that such a similarity based fuzzy system can uniformly approximate continuous functions to any desired degree of accuracy on a closed and bounded interval. Simulation results are presented for the well-known dc-motor problem. A comparative study is made to establish the validity and efficiency of the proposed similarity based fuzzy system.

Keywords

Fuzzification, Defuzzification, Approximate Reasoning, Similarity Relation, Similarity Index, Universal Approximator

1. Introduction

Traditional approaches to mathematical modelling require advanced mathematical skills. This is why many mathematical models are actually derived by mathematicians rather than actual observers, be it the ethologist studying animal behaviour or the sociologist studying human behaviour. However, even when the observer who studies the system cannot produce a mathematical model for it directly, (s)he might be able to describe the system and its behaviour linguistically. Thus, the question is how to transform linguistic observations into mathematical entities.

The application of fuzzy sets to the design of systems was not obvious at all, as traditionally systems were described by numerical equations. Zadeh tried several solutions to come up with a notion of fuzzy system. The idea of fuzzy system was initially viewed as a system whose state equations involve fuzzy variables or parameters, giving birth to fuzzy class of systems. A second idea was that a system is fuzzy if either its input, output or states range over a family of fuzzy sets. Later, it was suggested that a fuzzy system could be a generalization of a non-deterministic system, which moves from a state to a fuzzy set of states. So, the transition function is a fuzzy mapping. The transition equation can be captured by means of fuzzy relations along with the usual sup-min combination of fuzzy sets and fuzzy relations.

These early attempts were outlined before the emergence of the idea of fuzzy control. In 1973, it was suggested for the first time that fuzziness lies in the description of approximate rules to make the system work. This view was the result of a convergence between the idea of a system with one of the fuzzy algorithms introduced earlier and its increased focus of attention on the representation of natural language statements via linguistic variables. In this quest, systems of fuzzy if-then rules were first described, which paved the way to fuzzy controllers, built from human information, with the success of such a line of research in the early 1980s.

However, most of the applications are based not on modeling real-world phenomena, but rather on transforming the knowledge of a human expert into a fuzzy system. Today, fuzzy rule-based systems are extracted from data and serve as models of systems more than as controllers. Here, the linguistic connection is often lost, and such fuzzy systems are rather standard precise systems using membership function for interpolation, than approaches to the handling of poor knowledge in system descriptions.

For many years, fuzzy systems have been successfully applied to a wide variety of practical problems. Fuzzy rule-based systems have become popular because of their transparency and ease of tuning and modification. Rule-based fuzzy control introduced by Zadeh and Mamdani and also in the form later proposed by Takagi and Sugeno is a useful tool for systems where the exact model is either not known, or it is too complex to be tractable in real time. Notable applications of fuzzy systems include their use in control and decision making. It points out that fuzzy systems have an effective utility in the context of complex ill-defined problems, especially those which can be solved by a person without the knowledge of their underlying dynamics. However, some people were reluctant to use fuzzy systems because their effectiveness has not been mathematically proved. Some results attempted to prove that the main advantage of using fuzzy systems was the suitability for approximation with arbitrary accuracy in their universality [1] [2]. This prompted some researchers to prove that fuzzy systems are universal approximators, as a convincing proof of effectiveness. By a universal approximator we mean something that can uniformly approximate continuous functions to any degree of accuracy on a closed and bounded interval.

The first significant result in this sense was presented by Wang [3]. He proved that a certain type of fuzzy system (fuzzy rule based system with product inference, centroid defuzzification and Gaussian membership function) is a universal approximator. This result was important because it showed that the class of all fuzzy systems is a universal approximator.

Other types of fuzzy systems as universal approximators were also studied [4] [5]. Kosko [6] proved that additive fuzzy rule based systems are universal approximators, while Buckley proved that an extension of Sugeno type fuzzy logic controllers are universal approximators [4]. Castro [7] shows some types of fuzzy controllers being universal approximators.

All the types of fuzzy systems which have been showed as universal approximators are fuzzy rule based systems. There are other types of fuzzy systems which are used in applications. The aim of this paper is to study whether similarity based fuzzy systems are also universal approximators. Accordingly, we propose a similarity based fuzzy system designed on an appropriate fuzzy logical structure so that the task of the same is well-understood for a SISO model. In the sequel, it has been observed that similarity is inherent in approximate reasoning and cannot be avoided [8]. We try to compute the similarity between fuzzy sets and use it in the reasoning mechanism in such a way that change in input is always reflected in the output. Our method of inference is based on this similarity measure. We first generate a similarity relation from the fuzzy partition induced by the fuzzy sets representing the linguistic values of the variable present in the antecedent part of the rules. This similarity relation is used to fuzzify any crisp input value. Then, the similarity between the observation and a prototype of the same appearing in the conditional statement is computed to select a few rules from the rule-base. Each such rule, a conditional statement, is expressed as a fuzzy relation. This task can be performed in different ways [9]. We interpret the result as a conditional fuzzy relation. Then, the similarity value is used to modify the conditional relation [10]. Such a modification of relation may also be performed in many ways. We interpret the modified relation as a fuzzy relation induced by the specific observation. Next, we compose the observed fuzzy set with the modified fuzzy relation and project the resultant fuzzy relation on the domain of the linguistic variable defining the consequence of the rule. This is done for all the rules that match with the observation. Defuzzification, a basic operation, used in the development of fuzzy systems is extensively discussed in the light of new similarity based approximate reasoning mechanism. A new scheme for defuzzification, suitable for similarity based approximate reasoning, is defined [11]. This defuzzification method is used for a single crisp output from a number of output fuzzy sets resulting from rule-firing. The efficiency of such a similarity based fuzzy system has been proved mathematically. Simulation is performed with some real data. Results are tabulated along with their pictorial representations.

The paper consists of seven sections. After a brief introductory section we present some basic concepts necessary for the development of a similarity based fuzzy system and describe the way of connecting fuzzy set and their semantic background. A knowledge of this enables the user to estimate the possibilities of applying a typical similarity based fuzzy system in practice. Section 3 presents a discussion on the similarity measure of two fuzzy sets, similarity relation and fuzzification and its use in developing a theory of similarity based approximate reasoning methodology. Specificity measure and its incorporation into similarity based fuzzy system defuzzification is also discussed. The proposed fuzzy system can be shown to be a universal approximator to any continuous function on a compact set if normal, bounded, continuous and convex fuzzy sets with Ruspini partition on the input space are used in the *IF*-part of the fuzzy rules with the proposed fuzzifier, the relation modification procedure and specificity based defuzzifier. A case study on dc-motor is presented in Section 5. The paper is briefly concluded in section 6 followed by a list of references in the last section.

2. Similarity Based Fuzzy System (SBFS)

Definition 1. Let U be a non-empty set. F(U) is the fuzzy power set of U, *i.e.*, $F(U) = \{A | A : U \rightarrow [0,1]\}$. A is said to be normal if there exists $u \in U$ s.t. A(u) = 1. A is said to be convex if U is a linear space and

 $\forall \lambda \in [0,1] \text{ and } u, v \in U, A(\lambda u + (1-\lambda)v) \ge \min\{A(u), A(v)\}.$

The support of A is denoted by Supp(A) and is defined as

$$Supp(A) = \{ u \mid A(u) > 0 \}.$$

The height of A is denoted by Hgt(A) and is defined as

$$Hgt(A) = \sup \{A(u) \mid u \in U\}.$$

A is bounded if Supp(A) is a bounded set. $\mathcal{F}(U)$ is to denote the space of fuzzy sets which are *bounded*, *normal*, *convex* and *continuous*. Clearly, $\mathcal{F}(U) \subseteq F(U)$. Let P be an arbitrary collection of fuzzy subsets over U, *i.e.*,

$$P = \left\{ A_k \right\}_{k=1}^n \text{ each } A_k \subseteq \mathcal{F}(U).$$

P is said to form a fuzzy partition on *U* if

$$U \subseteq \bigcup_{k=1}^{n} Supp(A_k).$$

It is also called a complete partition. P is said to be consistent if $A_k(u) = 1$

then $A_i(u) = 0$ for any $j \neq k$.

Definition 2. Let [a,b] be the given range of values on which *n* linguistic terms are defined. These are considered as the primary terms which the linguistic variable can have. All other linguistic values of the variable concerned are generated by using a few linguistic hedges. Let us set $h = \frac{b-a}{n}$ and; m = n-1.

We mark A_0, A_1, \dots, A_m as the *n* fuzzy sets defined over [a, b] as a fuzzy partition of [a, b] where each A_i is normal at $u_i, i = 0, 1, \dots, m$. Clearly, $u_0 = a$; $u_{i+1} = \lfloor u_i + h \rfloor, i = 0, 1, \dots, m-2$ and $u_m = b$ is a classical partition of [a, b]; $Supp(A_0) = [u_0, u_1)$; $Supp(A_i) = (u_{i-1}, u_{i+1})$, $i = 1, 2, \dots, m-1$;

 $Supp(A_m) = (u_{m-1}, u_m]$. The membership of elements in the fuzzy sets A_0, A_1, \dots, A_m is defined as in the following:

$$A_0(u) = \frac{u_1 - u}{u_1 - u_0}; A_m(u) = \frac{u - u_{m-1}}{u_m - u_{m-1}}$$
(1)

and

$$A_{i}(u) = \begin{cases} \frac{u - u_{i-1}}{u_{i} - u_{i-1}}, & u_{i-1} \le u \le u_{i} \\ \frac{u_{i+1} - u}{u_{i+1} - u_{i}}, & u_{i} \le u \le u_{i+1}, i = 1, 2, \cdots, m-1. \end{cases}$$
(2)

Example 1. Let the domain set be $U = \{0.0, 0.5, 1.0, \dots, 10.0\}$. Let A_1, A_2, \dots, A_6 be the fuzzy sets corresponding to the points *Null*/0.0, *Zero*/2.0, *Small*/4.0, *Medium*/6.0, *Large*/8.0, *VeryLarge*/10.0, *i.e.*, the set of fuzzy sets $\{A_i \in \mathcal{F}(U) | i \in N\}$, from the Equation (1) and Equation (2), covers the domain U. A typical Ruspini fuzzy partition is as shown in Figure 1.

Many fuzzy systems are based on Zadeh's compositional rule of inference. Despite their success in various systems, researchers have indicated certain limitations [12] in the technique. This motivates the introduction of similarity based reasoning techniques as proposed in [12] [13] [14]. A brief review work on similarity based approximate reasoning, in general, can be found in [10] [15]. A detailed description of similarity related reasoning can also be found in [12].

Different approaches to similarity based reasoning are found in the literature. In [8] [16], the authors showed how a crisp set induces a fuzzy set with respect to a fuzzy equivalence relation. Thus, assuming the indistinguishability modeled by a fuzzy equivalence relation as a basic concept, fuzzy sets were viewed as induced concepts [17], *i.e.*, membership degrees of elements can be obtained from the indistinguishability. These works are mainly concerned with the connection between fuzzy sets and indistinguishability. They defined the degree to which two elements of the universe U cannot be distinguished by a collection of fuzzy sets. In the reasoning procedure, the observation (a crisp value) is first translated into a fuzzy set induced by the indistinguishability measure. The interpretation of the rules in a particular logic provides a semantic background for the composition of the specific observation with the relation and subsequent derivation of a fuzzy set as output of the fuzzy system.



Figure 1. A typical Ruspini fuzzy partition.

In this section, we would like to develop a typical SBFS as in the following and measure its performance.

Definition 3. A SBFS is defined as

$$\mathcal{R} = \langle \mathcal{R}, f, s, M, \Sigma, d \rangle$$

where

 \mathcal{R} is a fuzzy if-then rule-base defined on the fuzzy partitions $\{A_i\}$, $\{B_j\}$ of the universe of discourses U and V for two linguistic variables X and Y respectively;

 $f: U \to \mathcal{F}(U)$ is a fuzzification function which maps U to some fuzzy subset of U;

 $s: \mathcal{F}(U) \times \mathcal{F}(U) \rightarrow [0,1]$ is a similarity matching function which maps pair of fuzzy subsets of *U* to some number in [0,1];

 $M:[0,1]\times[0,1]\to[0,1]$ is a modification function which takes into consideration the similarity matching degree *s* and modifies the membership degree of a pair of element from $U\times V$ in the fuzzy relation R(u,v) obtained from the translation of the rule

if X is
$$A(\subseteq \mathcal{F}(U))$$
 then Y is $B(\subseteq \mathcal{F}(V))$

according to

$$R'(u,v) = M(s, R(u,v)); \forall u \in U \text{ and } \forall v \in V;$$

 $\Sigma: \{\mathcal{F}(U)\}^i \to \mathcal{F}(U)$ is an aggregation function which is used in case of multiple rule firing

$$B' = \sum_{j=1}^{i} B_j \text{ and;}$$

 $d: \mathcal{F}(V) \to V$ is a defuzzification function which converts a given fuzzy subset of *V* to an element of *V*.

2.1. Similarity Relation—Fuzzification

Assigning similarity modeled by a fuzzy equivalence relation as the basis, fuzzy

sets were viewed as an induced concept. Here we consider a fuzzy equivalence relation to describe the similarity between elements of a given set.

Definition 4. An equivalence relation (with respect to the operation *) on the set *U* is a mapping $E: U \times U \rightarrow [0,1]$ satisfying axioms:

(E1)	E(u,u)=1,	(reflexivity),
(E2)	E(u,v) = E(v,u),	(symmetry) and;
(E3)	$E(u,v) * E(v,w) \le E(u,w)$	$\forall u, v, w \in U(\text{transitivity}).$

In [8] the authors showed that the notion of membership is a gradual property of fuzzy sets. They have considered a fuzzy equivalence relation to describe the indistinguishability or similarity between elements of a fuzzy set. Similarity is an important concept for which, a crisp model is often found inadequate. There, they showed how a crisp set induces a fuzzy set as its extensional hull with respect to a fuzzy equivalence relation. Two elements cannot be distinguished by a fuzzy set if they are both either elements of the same set or its complement. They have shown how membership functions of fuzzy sets can be calculated from the fuzzy equivalence relation.

Let $\mathcal{F}_1(U) \subseteq \mathcal{F}(U)$ be a collection of fuzzy sets. Then

$$E(u,v) = \bigwedge_{A \in \mathcal{F}_{1}(U)} \left(A(u) \leftrightarrow A(v) \right)$$
(3)

is the coarsest fuzzy equivalence relation on U such that all fuzzy sets in $\mathcal{F}_1(U)$ are extensional with respect to E. The fuzzy equivalence relation defined by Equation (3) can be interpreted in the following way—two elements "cannot be distinguished by a (fuzzy) set" if they are both elements of the same set or its complement, but not one in the set and the other one in its complement. Thus, $A(u) \leftrightarrow A(v)$ represents the degree to which the elements u and v cannot be distinguished by the fuzzy set A. Therefore, E(u,v) is the degree to which u and v cannot be distinguished by the set $\mathcal{F}_1(U)$ of fuzzy sets.

Klawonn in [8] specify bi-implication operators with respect to a fixed implication operator, fixed conjunction and fixed negation operator. This is shown in **Table 1**.

Definition 5. A fuzzy set $A \in \mathcal{F}_1(U)$ is called extensional with respect to the fuzzy equivalence relation *E* on *U* if and only if

Table 1. Interpretation of the logical operations under different fuzzy logics.

$\alpha * \beta$	$\max\left\{\alpha+\beta-1,0\right\}$	$\min\{\alpha, \beta\}$	$lpha \cdot eta$
$\alpha \rightarrow \beta$	$\min\{1-\alpha+\beta,1\}$	$\begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha \leq \beta \\ \frac{\beta}{\alpha} & \text{otherwise} \end{cases}$
$\alpha \leftrightarrow \beta$	$1- \alpha-\beta $	$\begin{cases} 1 & \text{if } \alpha = \beta \\ \min\{\alpha, \beta\} & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha = \beta \\ \frac{\min\{\alpha, \beta\}}{\max\{\alpha, \beta\}} & \text{otherwise} \end{cases}$
$\neg \alpha$	$1-\alpha$	$\begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$

 $A(u) * E(u,v) \le A(v)$ holds $\forall u, v \in U$.

Definition 6. Let *E* be a fuzzy equivalence relation on *U* and let $A \in \mathcal{F}(U)$. The fuzzy set

 $\hat{A} = \bigcap \{ B \mid A \le B \text{ and } B \text{ is extensional with respect to } E \}$

is called the extensional hull of A with respect to E.

Theorem 1 $\hat{A}(u) = \sup_{v \in U} \{A(v) * E(u, v)\}.$

Here, we apply the indistinguishability operator in order to model fuzzy equivalence relation on a domain of universe U, which is generated by a family of fuzzy subsets of U. Accordingly, a fuzzy relation E(u, v) can be defined as in the following:

$$E(u,v) = \bigwedge_{A \in \mathcal{F}} \left(1 - \left| A(u) - A(v) \right| \right)$$
(4)

The corresponding pictorial diagram is as shown in **Figure 2**. From this equivalence relation E(u, v) we can fuzzify any point on the domain *U*by

$$A_{a}(u) = \begin{cases} E_{F}(u,a), & \text{when } a - \delta \le u \le a + \delta \\ 0, & \text{otherwise.} \end{cases}$$
(5)

Here δ is to be chosen appropriately in such a way as to cover the Supp(A'), (accordingly *h* is a reasonable value). Therefore, $A_a(u)$ is an extensional hull of a crisp point *a*.

Definition 7. A fuzzy binary relation *S* on *U* is said to be a *similarity relation* on *U* if it is reflexive, symmetric and transitive, *i.e.*,

$$S(u,u) = 1,$$

$$S(u,v) = S(v,u) \text{ and};$$

$$S(u,v) * S(v,w) \le S(u,w), \forall u, v, w \in U;$$

where * stands for the Lukasiewicz conjunction (a *t*-norm) defined by $\alpha * \beta = \max \{ \alpha + \beta - 1, 0 \}$. Then E(u, v) is a similarity relation. It is easy to see that the equivalence relation defined in Equation (4) is a similarity relation.

Algorithm: Fuzzification

Step 1. A_1, A_2, \dots, A_n are the fuzzy sets over the domain *U*. These triangular fuzzy sets are obtained by the Equation (1) and Equation (2).

Step 2. Construct the fuzzy equivalence relation E(u,v) from A_1, A_2, \dots, A_n with the help of Equation (4).

Step 3. Construct a fuzzy set with respect to a point *a* with the help of this fuzzy equivalence relation E(u, v) as given by the following Equation (6)

$$A_{a}(u) = \begin{cases} E(u,a), & \text{when } a - \delta \le u \le a + \delta \\ 0, & \text{otherwise.} \end{cases}$$
(6)

The extensional hulls of the crisp values 4.0 and 7.0 with respect to this fuzzy equivalence relation are the fuzzy sets

$$A_{4,0}(u) = E(u, 4.0)$$
 if $4.0 - 2 \le u \le 4.0 + 2$, otherwise $A_{4,0}(u) = 0$



Figure 2. An example equivalence relation induced by the fuzzy partition of Example 1.

and

$$A_{7.0}(u) = E(u, 7.0)$$
 if $7.0 - 2 \le u \le 7.0 + 2$, otherwise $A_{7.0}(u) = 0$

that is

$$A_{4.0}(u) = 0.0/2.0 + 0.25/2.5 + 0.50/3.0 + 0.75/3.5 + 1.00/4.0 + 0.75/4.5 + 0.50/5.0 + 0.25/5.5 + 0.0/6.0;$$
$$A_{7.0}(u) = 0.0/5.0 + 0.50/5.5 + 0.50/6.0 + 0.75/6.5 + 1.00/7.0 + 0.75/7.5 + 0.50/8.0 + 0.50/8.5 + 0.0/9.0.$$

Figure 3 illustrate the fuzzification of the crisp values 4.0 and 7.0.

2.2. Similarity Measure—Inference

The similarity between two objects suggest the degree to which properties of one may be inferred from those of the other. There could be several such measures, one such simple measure can be given as in the following:

Definition 8. Let

$$A = \sum_{u \in U} A(u)/u$$
 and $A' = \sum_{u \in U} A'(u)/u$

be two fuzzy sets defined over the same universe of discourse *U*. Let $s = S(A', A) \in [0,1]$ be the similarity index of the pair $\{A', A\}$ and is defined by

$$S(A', A) = \max_{u \in U} \min(A'(u), A(u)).$$
⁽⁷⁾

Clearly,

1) S(A, A) = 1; S(A', A) = 1 if A' = A;2) S(A', A) = S(A, A') and; 3) $0 \le S(A', A) \le 1.$

Let us now propose a different strategy for inference in fuzzy system based on the concept of similarity. The concept of similarity between fuzzy sets is used in selecting rules from the rule-base, to be fired for the particular input specification and then in deriving the typical action.

The knowledge base of a fuzzy system consists of a data base and a rule-base. The basic function of the data base is to provide necessary information for proper functioning of the fuzzification module, the rule-base and the defuzzification module. Whereas, function of the rule-base is to represent, in a structured way, the policy of an experienced system engineer given in the form of a set of rules as described in the following:

$$R_i$$
: If X is A_i then Y is B_i ; $i = 1, 2, \dots, n$.

Let X and Y be two linguistic variables denoting respectively the *input* and *output* of a SISO fuzzy system. Here each A_i ; $i = 1, 2, \dots, n$ is a fuzzy subset over U and each B_i ; $i = 1, 2, \dots, n$ is a fuzzy subset over V. Let U, V be respectively the Universe of discourse of the linguistic variables X and Y. Let

$$A_i = \sum_{u \in U} A_i(u)/u$$
 and $B_i = \sum_{v \in V} B_i(v)/v$.

Let us consider a pattern for approximate reasoning with imprecise knowledge, as presented in the Table 2.

For a given input X is A our similarity based reasoning scheme measure the compatibility/similarity $S(A_i, A); i = 1, 2, \dots, n$. We choose those rules for which $S(A_i, A) \ge \epsilon$, a given threshold value set for a particular system. Let there be p such rules for that system and we call them R_1, R_2, \dots, R_n (with possible rearrangement). Let

$$s_i = S(A_i, A); i = 1, 2, \cdots, n.$$

Translate each rule R_i to fuzzy relation R_i on $U \times V$ according to the relation



Figure 3. Fuzzification of the point 4.0 and 7.0.

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R_i	:	if	X is A_i	then	Y is B_i
Q	:		X is A'		
Conclusion	\leftarrow				Y is B'

$$R_i(u,v) = I(A_i(x), B_i(y)); u \in U \text{ and } v \in V$$

where *I* is an implication function.

Modification function M:

Let R: X is $A \to Y$ is B be a fuzzy rule and X is A be a specific observation. The fuzzy rule is modelled by a fuzzy conditional relation given by

$$R(u,v) = I(A(u),B(v))$$

where,

$$A = \sum_{u \in U} A(u)/u$$
; $B = \sum_{v \in V} B(v)/v$ and *I* is an implication function.

First of all, let the fuzzy conditional relation be translated to

$$R(u,v) = I(A(u), B(v))$$

= min(1,1-A(u)+B(v)); using the Lukasiewicz *t*-norm as a model for *I*.

Next, with the similarity matching index s = S(A', A) computed for the expected value and the observed value of the linguistic variable *X*, the conditional relation is modified as in the following:

$$R'(u,v) = M(s, R(u,v)) = 1 - s \cdot (1 - R(u,v)).$$
(8)

Thus, $M:[0,1]_s \times [0,1]_R \to [0,1]_{R'}$ is explicitly given by the Equation (8). Any transformation/modification of the kind given above can be understood as

$$R'(u,v) = M(s,R(u,v)) = I(s,R(u,v))$$
(9)

where *I* is an implication function (hence satisfying I(u,0)=1-u, I(0,u)=1, I(1,u)=u, is decreasing continuously in the first variable and is continuously increasing in the second variable) fulfils the postulates

- 1) If s = 1, then R'(u, v) = R(u, v);
- 2) If s = 0, then R'(u, v) = 1;

3) As *s* increase from 0 to 1, R'(u,v) decrease uniformly from 1 to R(u,v). In particular, taking *I* as the strong product implication

$$I(u,v) = 1 - u + uv$$

we can obtain our scheme for modification as given in Equation (8) on substitution in Equation (9).

Then, the composition of the observed value and the modified relation generates as a conclusion B'(v) which serves as the output of if X is A then Y is B and X is A and is explicitly given by

$$B'(v) = \sup_{u \in U} \{A'(u) * R'(u, v)\}.$$
 (10)

Next, with s_i we modify R_i to compute the modified relation R'_i on $U \times V$ as in the following:

$$R'_{i} = M(s_{i}, R_{i}) = I(s_{i}, R_{i}) = 1 - (1 - R_{i}) \cdot s_{i}; i = 1, 2, \cdots, n.$$

Now, we compute B' according to

$$B'_{i}(v) = \sup_{u \in U} \left\{ A'(u) * R'_{i}(u, v) \right\}; i = 1, 2, \cdots, n.$$
(11)

We compute the specificity of B'_i ; $i = 1, 2, \dots, n$. Choose the one with max specificity. Break tie, if there be any, according to maximum height Defuzzification.

Algorithm SAR (Similarity based approximate reasoning)

Step 1. Compute $S(A', A_i)$ for $i = 1, 2, \dots, n$ using Equation (7).

Step 2. Set ϵ and then find rules for which $S(A', A_i) \ge \epsilon$.

Step 3. Translate premise *p* and compute $R_i(u, v)$ for those rules coming out of **Step 2** using any suitable translating rule possibly, a T-norm operator.

Step 4. Modify R(u,v) with $S(A', A_i)$ to obtain the modified conditional relation R'(u,v) as on Equation (8).

Step 5. Take a composition "*" of A' and R', preferably, a T-norm and use sup-projection operation on $A' * R'_i$ to obtain B'_i as given in Equation (10).

Step 6. Compute the output using maximum specificity measure. Break tie, if there be any, according to maximum height defuzzification.

2.3. Specificity Measure—Defuzzification

The result of rule firing is a class of clipped fuzzy sets defined over the same universe of discourse. We are required to determine a single real value from those fuzzy outputs. Earlier [11], we discussed several methods of defuzzification like Center-of-gravity, Center-of-sums, height, etc. For this schema, the basic idea is as follows: If the membership grade of a particular element, in the output fuzzy set, is high then this contributes more to the defuzzified output.

Such concepts cannot be used in the present case of similarity based reasoning paradigm. Here, the lower the similarity value between the rule-antecedent and the observation, the closer the output to the least specific case (*i.e.*, unknown) with the membership grade of elements in the output fuzzy set close to 1. In such cases, a natural choice would be to use specificity information of the output fuzzy sets in defuzzification. In our scheme, the basic idea will be: the element with high membership value should come from the most specific output fuzzy set. Our first defuzzification scheme is based on this concept. The most specific among the output fuzzy sets has the maximum impact on the resultant choice. For that we compute specificity value of each output fuzzy set separately.

The result of rule firing, for a typical observation is a fuzzy set. This is interpreted at the semantic level as the desired output. Often, we need to determine a precise action as output. The purpose of defuzzification is to obtain a scalar value $u \in U$, from the said output fuzzy set, as the action. Then, if necessary, de-normalisation is performed on the output so as to obtain the corresponding action on its physical domain.

Specificity measure of fuzzy set estimates how precise is an information when represented by the fuzzy set rather than an estimate of its fuzziness which is measured by the entropy of the fuzzy set. In order to provide a definition for any specificity index, a number of observations must be considered. A fuzzy set with maximum specificity value corresponds to a precise assessment of the values of a variable. In trying to capture the form of the specificity index, a number of properties are required or desirable.

According to Dubois and Prade, a specificity measure Sp(A) [18] should satisfy the following properties. Let X be a linguistic variable defined on a universe of discourse U. A and B are normalised fuzzy subsets of U.

- **A1.** $\forall A \subseteq U$, $Sp(A) \in [0,1]$.
- **A2.** Sp(A) = 1 if and only if A is a singleton of S.
- **A3.** If $A \subseteq B$ then $Sp(A) \ge Sp(B)$.

Yager [19] introduced one such measure of specificity that satisfies the above properties. When *U* is finite, Yager proposed an expression for defining the specificity. Let us assume that *A* be a fuzzy set defined over the universal set *U* and A_{α} be the *a*-level set of *A*. The specificity associated with *A* is denoted as *Sp*(*A*) and is defined as

$$Sp(A) = \int_0^{\alpha_{\max}} \frac{1}{CardA_{\alpha}} d\alpha$$
 (12)

Let us now list some properties [19] associated with the above definition.

P1. For all A, Sp(A) assumes its maximum value 1, when $A = \{1/u\}$ for some particular $u \in U$.

P2. For all A, $Sp(A) \in [0,1]$ and it assumes its minimum value 0, when $A = \Phi$.

P3. If A(u) = k for all $u \in U$ then $Sp(A) = \frac{k}{n}$ where *n* is the cardinality of the ordinary set *U*.

Defuzzification is a procedure applied to reduce the anxiety in a decision. Specificity estimates how precise is the information on the values of a linguistic variable restricted by a fuzzy set. As suggested by the axioms and further properties of specificity a crisp set can be less specific than a fuzzy set for restricting the possible values of a variable. Accordingly, we propose a new technique for defuzzification based on measure of precision. Let there be *m*-clipped fuzzy sets $\{A^{(k)}; k = 1, 2, \dots, m\}$ and let $\{s_{(k)}, p^{(k)}; k = 1, 2, \dots, m\}$ be the specificity associated with $A^{(k)}$ as well as the peak point of the consequent fuzzy set of the k^{th} -rule. Let $p^{(k)}$ be the peak value of $A^{(k)}$ and $h^{(k)}$ be the corresponding height of the clipped version of $A^{(k)}$. Then using height method the defuzzified value will be given by

$$u^* = \frac{\sum_{k=1}^{m} p^{(k)} \cdot h^{(k)}}{\sum_{k=1}^{m} h^{(k)}}.$$
 (13)

Whereas, the specificity based defuzzified value u^* will be given by

$$u^* = \frac{\sum_{k=1}^{m} p^{(k)} \cdot s_{(k)}}{\sum_{k=1}^{m} s_{(k)}}.$$
 (14)

DOI: 10.4236/ijis.2018.84005

Algorithm SBFS (Similarity based fuzzy system)

Step 1. Let $x^t = (x_1^t, x_2^t, \dots, x_n^t)$ be the system state vector at time t.

Step 2. Fuzzify each x_i^t , the real value for the system state variable, using triangular membership function forming a Ruspini type fuzzy partition.

Step 3. Compute similarity relation $E(x_i^t, x_j^t)$ with the help of state vector x^t .

Step 4. Use the similarity relation to fuzzify the input data by **fuzzification** algorithm viz., $A_a(x) = E(a, x)$, where $a, x \in x^t$.

Step 5. For each rule R_k , compute similarity index between the input fuzzy set and the antecedent fuzzy set, for all variables. Obtain ϵ_k , the minimum of all the similarity indices. Take this as a matching grade of the rule.

Step 6. Perform similarity based approximate reasoning, taking one/more rule(s) at a time for which $\epsilon_k \ge \alpha$; $0 \le \alpha \le 1$.

Step 7. Defuzzify the fuzzy sets, as obtained in **Step 6**, by using the specificity based defuzzification scheme.

Step 8. Use the defuzzified result, as found in **Step 7**, as the system input for the next time interval. Set $t \leftarrow t+1$. Go to **Step 4**.

3. Universal Approximation

During the past several years, fuzzy logic control (FLC) has been successfully applied to a wide variety of practical problems. Notable applications of that FLC systems include the control of warm water, robot, heat exchange, traffic junction, cement kiln, automobile speed, automotive engineering, model car parking and turning, power system and nuclear reactor, etc. It points out that fuzzy control has been effectively used in the context of complex ill-defined processes, especially those which can be controlled by a skilled human operator without the knowledge of their underlying dynamics. In this sense, neural and adaptive fuzzy systems have been compared to classical control methods by B. Kosko. There, it is observed that they are model-free estimators, *i.e.*, they estimate a function without requiring a mathematical description of how the output functionally depends on the input; they learn from samples.

However, some people criticize fuzzy control because its effectiveness has not been proved viz., the very fundamental theoretical question "Why does a fuzzy rule-based system have such good performance for a wide variety of practical problems?", remains unanswered. There exist some qualitative explanations, e.g., fuzzy rules utilize linguistic information, fuzzy inference simulates human thinking procedure, fuzzy rule systems capture the approximate and inexact nature of the real world, etc., but mathematical proofs have not been obtained.

In this section, we prove that the proposed SBFS can serve as a universal approximator [20] to any continuous function on a closed and bounded interval on real line.

Theorem 2: For any continuous function $f:[a,b] \to R$ and $\epsilon > 0$, there is a SBFS such that $\sup_{x \in [a,b]} |f(x) - g(x)| < \epsilon$, where $g:[a,b] \to R$ is a function which assigns the crisp input x to the crisp output y of the fuzzy system.

Proof: Here, *f* is continuous on [a,b]. Since [a,b] is compact, *f* is uniformly continuous on [a,b], *i.e.*,

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } |x - x'| < \delta \Longrightarrow |f(x) - f(x')| < \epsilon.$$

Let us first proceed to choose the normal points. Let $(n-1) = \left\lceil \frac{b-a}{\delta} \right\rceil$ and

 $\frac{b-a}{n} = \delta_1$, then obviously $\delta_1 < \delta$. We now introduce (n-1) points in be-

tween *a* and *b* as in the following.

Set $x_0 = a, x_1 = x_0 + \delta_1, x_2 = x_1 + \delta_1, \dots, x_i = x_{i-1} + \delta_1, \dots, x_n = b$ and thereby we generate *n* number of subintervals

 $[x_i, x_{i-1}], i = 1, 2, 3, \dots, n \text{ of length } \delta_1(<\delta), \text{ i.e., } |x_i - x_{i-1}| < \delta \forall i = 1, \dots, n.$

These x_i are the normal points in the input space and $y_i = f(x_i)$ are the normal points in the output space.

We now try to construct (n+1) fuzzy subsets A_i , $i = 0, 1, 2, \dots, n$ over the input space such that each A_i is

1) centered and normal at x_i ;

2) continuous and convex;

3) $Supp(A_0) = [a, x_1)$, $Supp(A_n) = (x_n, b]$ and $Supp(A_i) = (x_{i-1}, x_{i+1})$ for $i = 1, \dots, n-1$;

4) $\{A_i\}_{i=1}^n$ forms a Ruspini partition as given in **Figure 4**.

Similarly, we construct fuzzy subsets $B_0, B_1, B_2, \dots, B_n$ of the output space, where each B_i is

1) centered and normal at y_i ;

2) continuous and convex;

3) Supp $(B_i) = (y_i - \epsilon, y_i + \epsilon)$ for $i = 0, 1, 2, \dots, n$;

4) $\{B_i\}_{i=0}^n$ forms a fuzzy partition as given in **Figure 5**.

Here, we note that for any $y \in f([a,b]) \exists x \in [a,b]$, more precisely $x \in [x_k, x_{k+1}]$ for some $k \in \{0,1,2,\dots,n\}$. As we have $|x_k - x_{k+1}| < \delta$ therefore, $|x_k - x| < \delta$ and this implies $|y - y_k| < \epsilon$. This, in turn, means that $\{B_i\}_{i=0}^n$ forms a complete fuzzy partition on the output space as we claimed it to be.

As we are unaware of the order of B_i s in the output space we therefore, take five arbitrary B_i s, viz., B_i, B_j, B_k, B_l, B_m which is shown in the fuzzy partition of the output space in **Figure 5**.

Next, we construct the rule base with n+1 rules as in the following.

If x is A_i then y is B_i for $i = 0, 1, \dots, n$. Here, B_i is chosen in such a way that for the normal point y_i of B_i , there is a pre-image x_i such that $f(x_i) = y_i$ and $A_i(x_i) = 1$.

Let us now consider a crisp value $x' \in [a,b]$ as the given input. Note that if $x' = x_i$ for $i = 0, 1, \dots, n$ then choose the input fuzzy set A_i and obtain B_i as the fuzzy output firing the *i*th rule and so the crisp output is $y_i = g(x')$. Here $g(x') = y_i = f(x_i) = f(x')$ and the theorem is proved.



Figure 4. Fuzzy partition of the input space.



Figure 5. Fuzzy partition of the output space.

Assume that $x' \neq x_i$ for any $i = 0, 1, \dots, n$ then $x' \in [x_m, x_{m+1}]$ for some $m = 0, 1, 2, \dots, n$. In this case we choose

$$\delta_2 = \min\{|x' - x_m|, |x' - x_{m+1}|\}$$

and apply Fuzzification algorithm *i.e.*,

1) construct a fuzzy equivalence relation E(x, y) from A_0, A_1, \dots, A_n using $E(x, y) = \bigwedge_{i=0}^{n} (1 - |A_i(x) - A_i(y)|)$ and; 2) set $\delta_2^{i=0} \ge 0$ and define a fuzzy set A' about x' from the fuzzy equiva-

lence relation E(x, y) by

$$A'(x) = \begin{cases} E(x,x') & \text{for } x \in [x' - \delta_2, x' + \delta_2] \\ 0 & \text{otherwise.} \end{cases}$$
(15)

Here A' is defined to look like Figure 6, so that A' intersects at most A_m

and A_{m+1} .

Now we compute the similarity measure $S(A', A_m) = s_m$ and $S(A', A_{m+1}) = s_{m+1}$ using **Definition 8**.

<u>**Case (1)**</u> If one of s_m and s_{m+1} is greater than our desired threshold τ_0 . Say, $s_m > \tau_0$ and fire the m^{th} rule and obtain the modified consequence of the rule B'_m using SAR algorithm, which, in this case, is going to be the fuzzy output B', given by the equation

$$B'(y) = \sup_{x \in [a,b]} \left\{ T\left(A'(x), R'_m(x, y)\right) \right\}$$

where *T* is a continuous t-norm and $R'_m = M(s_m, R_m)$.

Here we take Lukaswiecz conjunction as T and Lukaswiecz implication as R. We now consider the following subcases:

Subcase (a) For $\{x : A_m(x) < B_m(y)\}$,

$$R_m(x, y) = 1$$

$$\Rightarrow R'_m(x, y) = 1$$

$$\Rightarrow B'_m(y) = \sup_{x} A'(x).$$

<u>Subcase (b)</u> For $\{x: A_m(x) \ge B_m(y)\}$,

$$R_m(x, y) = 1 - A_m(x) + B_m(y)$$

$$\Rightarrow R'_m(x, y) = 1 - s_m A_m(x) + s_m B_m(y)$$

$$\Rightarrow B'_m(y) = \sup_x \left(\max\left(A'(x) - s_m A_m(x) + s_m B_m(y), 0\right) \right)$$

Note that $B'_m(y_m) = 1$ *i.e.*, $Ker(B'_m)$ is nonempty and $y_m \in Ker(B'_m)$. Further, observe that for $\{y : |y - y_m| \ge \epsilon\}$,

$$R_m(x, y) = 1 - A_m(x, y)$$

$$\Rightarrow R'_m(x, y) = 1 - s_m A_m(x, y)$$

$$\Rightarrow B'_m(y) = \sup_x (A'(x) - s_m A_m(x)) < 1.$$

As, B_m is convex, continuous and symmetric so is B'_m .

This prompts us to draw B'_m as in **Figure 7**. Now, it becomes obvious that in this case the defuzzified output is y_m , *i.e.*, $g(x') = y_m = f(x_m)$.

As $x' \in [x_m, x_{m+1}]$ and $|x_m - x_{m+1}| < \delta$ therefore, $|x' - x_m| < \delta$. This will imply

$$\left| f(x') - f(x_m) \right| < \epsilon$$

$$\Rightarrow \left| f(x') - y_m \right| < \epsilon$$

$$\Rightarrow \left| f(x') - g(x') \right| < \epsilon.$$

Again, as $x' \in [a, b]$ is arbitrary,

$$\sup_{x\in[a,b]}\left|f(x)-g(x)\right|<\epsilon.$$



Figure 6. A typical fuzzy input.



Figure 7. Modification of Bm.

<u>Case (2)</u> If both s_m and $s_{m+1} \ge \tau_0$ then fire both m^{th} and $(m+1)^{th}$ rule and obtain B'_m and B'_{m+1} using SAR algorithm.

It may be noted that in this case we may obtain at most a τ_0 similar output to the consequent as the input is τ_0 similar to the antecedent. Now, we compute $S(B'_m, B_m)$ and $S(B'_m, B_{m+1})$ and choose the maximum and we call it s'_m . Likewise, we compute $S(B'_{m+1}, B_m)$ and $S(B'_{m+1}, B_{m+1})$ and choose the maximum and call it s'_{m+1} .

Subcase (a) If one of s'_m and s'_{m+1} is greater than our desired threshold value $\tau_1 (\leq \tau_0)$, then we take the corresponding B'_i for i = m, m+1 as the final output B'. Suppose $s'_m > \tau_1$, then the final fuzzy output would be $B = B'_m$ and we proceed similarly as in **Case(1)** to prove the theorem.

<u>Subcase (b)</u> If both s'_m , $s'_{m+1} > \tau_1$, then $B' = B'_m \cap B'_{m+1}$ which looks like **Figure 8** and we defuzzify B' using specificity based defuzzification as suggested in [10]. Crisp output

$$y' = g(x') = \frac{y_m \times Sp(B'_m) + y_{m+1} \times Sp(B'_{m+1})}{Sp(B'_m) + Sp(B'_{m+1})}$$

where Sp(B') is the specificity index of B'. Thus, $y' \in [y_m, y_{m+1}]$ or $[y_{m+1}, y_m]$ depending on $y_m < y_{m+1}$ or $y_m > y_{m+1}$. Without loss of generality, let us assume $y' \in [y_m, y_{m+1}]$.

We have $f(x_m) = y_m$ and $f(x_{m+1}) = y_{m+1}$ then by the intermediate value property of $f:[x_m, x_{m+1}] \rightarrow R \exists u \in [x_m, x_{m+1}]$ such that f(u) = y'. Again, $u, x' \in [x_m, x_{m+1}]$ and $|x_m - x_{m+1}| < \delta \Rightarrow |u - x'| < \delta$.

This will imply

$$|f(x') - f(u)| < \epsilon$$

$$\Rightarrow |f(x') - y'| < \epsilon$$

$$\Rightarrow |f(x') - g(x')| < \epsilon.$$
(16)

Since, $x' \in [a,b]$ is arbitrary, $\sup_{x \in [a,b]} |f(x) - g(x)| < \epsilon$. Hence, in either case, $\sup_{x \in [a,b]} |f(x) - g(x)| < \epsilon$.

4. A Case Study on dc-Motor

In this section, let us consider the dc-motor as in [9]. The human expert observed the behaviour of the dc-motor and described the relation between current and speed in the form of fuzzy conditional statements as in the following. We present a diagram of the dc-motor in **Figure 9** [21].

Let the domain of the antecedent part be $U = \{0.0, 0.5, 1.0, \dots, 9.5, 10.0\}$. Let

 $current(I) = \{Null, Zero, Small, Medium, Large and Huge\}$

be a typical fuzzy cover on *U*, corresponding to the points 0.0, 2.0, 4.0, 6.0, 8.0, 10.0, *i.e.*, the set of the antecedent fuzzy sets defined through Equation (1) and Equation (2). Moreover, let the domain of the consequent part be

 $V = \{400, 500, \dots, 1800, 2000\}$ and

speed $(N) = \{$ Zero, Small, Medium, Large and Huge $\}$

be a fuzzy cover of V, corresponding to the points 400, 800, 1200, 1600 and 2000. Now, the authors in [22] define the behaviour of the dc-motor (the specific relation between *current* and *speed*) through fuzzy rules and data as in **Table 3** and **Table 4**. The membership values of fuzzy sets for current(I) and speed(N) are as given in **Table 5** and **Table 6** respectively. Figure 10 and Figure 11 illustrated respectively the primary fuzzy sets considered in this paper for linguistic variables current(I) and speed(N) of the dc-motor. Figure 12 and Figure 13 represent the static data of the dc-motor for the two cases presented as Example 1 and Example 2.

Here, we illustrated the behaviour of dc-motor through the comparison be-

tween real input-output and that obtained by the application of the proposed methodology. The **Figure 14**, **Figure 15** and **Figure 16**, **Figure 17** illustrated respectively the output of simulation and a comparison of the same with the real values for the two cases.

Table 3. Data for Example-1 and corresponding fuzzy model.

Data						Rules				
Ι	Ν	Ι	N	Ι	Ν					
0.0	2000	3.5	600	7.0	1600					
0.5	1800	4.0	400	7.5	1800	r1:	If	<i>I</i> =Null	then N = Huge	also
1.0	1600	4.5	600	8.0	2000	r2:	If	I=Zero	then N = Medium	also
1.5	1400	5.0	800	8.5	1800	r3:	If	<i>I</i> = Small	then $N = $ Zero	also
2.0	1200	5.5	1000	9.0	1600	r4:	If	<i>I</i> = Medium	then N = Medium	also
2.5	1000	6.0	1200	9.5	1400	r5:	If	I = Large	then N = Huge	also
3.0	800	6.5	1400	10.0	1200	r6:	If	I = Huge	then N = Medium	

 Table 4. Data for Example-2 and corresponding fuzzy model.

Data						Rules				
Ι	Ν	Ι	Ν	Ι	Ν					
0.0	2000	3.5	1700	7.0	1400					
0.5	2000	4.0	1600	7.5	1300	rl:	If	I = Null	then N = Huge	also
1.0	2000	4.5	1600	8.0	12000	r2:	If	I=Zero	then N = Medium	also
1.5	2000	5.0	1600	8.5	1000	r3:	If	<i>I</i> = Small	then $N = $ Zero	also
2.0	2000	5.5	1600	9.0	800	r4:	If	<i>I</i> = Medium	then N = Medium	also
2.5	1900	6.0	1600	9.5	600	r5:	If	<i>I</i> = Large	then N = Huge	also
3.0	1800	6.5	1500	10.0	400	r6:	If	<i>I</i> = Huge	then N = Medium	

Table 5. Primary fuzzy sets for current, the linguistic variable I.

U	Null	Zero	Small	Medium	Large	Huge
0.00	1.00	0.00	0.00	0.00	0.00	0.00
0.50	0.75	0.25	0.00	0.00	0.00	0.00
1.00	0.50	0.50	0.00	0.00	0.00	0.00
1.50	0.25	0.75	0.00	0.00	0.00	0.00
2.00	0.00	1.00	0.00	0.00	0.00	0.00
2.50	0.00	0.75	0.25	0.00	0.00	0.00
3.00	0.00	0.50	0.50	0.00	0.00	0.00

Continued						
3.50	0.00	0.25	0.75	0.00	0.00	0.00
4.00	0.00	0.00	1.00	0.00	0.00	0.00
4.50	0.00	0.00	0.75	0.25	0.00	0.00
5.00	0.00	0.00	0.50	0.50	0.00	0.00
5.50	0.00	0.00	0.25	0.75	0.00	0.00
6.00	0.00	0.00	0.00	1.00	0.00	0.00
6.50	0.00	0.00	0.00	0.75	0.25	0.00
7.00	0.00	0.00	0.00	0.50	0.50	0.00
7.50	0.00	0.00	0.00	0.25	0.75	0.00
8.00	0.00	0.00	0.00	0.00	1.00	0.00
8.50	0.00	0.00	0.00	0.00	0.75	0.25
9.00	0.00	0.00	0.00	0.00	0.50	0.50
9.50	0.00	0.00	0.00	0.00	0.25	0.75
10.00	0.00	0.00	0.00	0.00	0.00	1.00

Table 6. Primary fuzzy sets for speed, the linguistic variable N.

V	Zero	Small	Medium	Large	Huge
400	1.00	0.00	0.00	0.00	0.00
500	0.75	0.25	0.00	0.00	0.00
600	0.50	0.50	0.00	0.00	0.00
700	0.25	0.75	0.00	0.00	0.00
800	0.00	1.00	0.00	0.00	0.00
900	0.00	0.75	0.25	0.00	0.00
1000	0.00	0.50	0.50	0.00	0.00
1100	0.00	0.25	0.75	0.00	0.00
1200	0.00	0.00	1.00	0.00	0.00
1300	0.00	0.00	0.75	0.25	0.00
1400	0.00	0.00	0.50	0.50	0.00
1500	0.00	0.00	0.25	0.75	0.00
1600	0.00	0.00	0.00	1.00	0.0
1700	0.00	0.00	0.00	0.75	0.25
1800	0.00	0.00	0.00	0.50	0.50
1900	0.00	0.00	0.00	0.25	0.75
2000	0.00	0.00	0.00	0.00	1.00







Figure 9. Block diagram of a dc-motor.







Figure 11. Fuzzy sets for N.



Figure 12. The real static behaviour of the dc-motor for Example-1.



Figure 13. The real static behaviour of the dc-motor for Example-2.



Figure 14. The behaviour of the proposed SBFS for Example-1.



Figure 15. A comparative study with real behaviour of dc-motor for Example-1.



Figure 16. The behaviour of the proposed SBFS for Example-2.



Figure 17. A comparative study with real behaviour for Example-2.

5. Conclusions

The significance of function approximation becomes obvious in the wide spectrum of computational activities for modelling and analysis ranging from applied mathematics to soft computing. Function approximation is a method of generalization in such a way that it specifies a function in an aim to develop a representative function being approximate to the target function. We have seen that conventional fuzzy systems characterized by a fixed fuzzy inference, a typical fuzzy relation along with a defuzzification procedure do act as universal function approximators with a large number of fuzzy rules, thereby increasing the complexity of the system further. In this research, we attempted to show that any real continuous function on a closed and bounded real interval can also be approximated by an appropriate SBFS to any degree of accuracy with fewer rules if similarity is considered in reasoning. From the simulation of dc-motor system, it is demonstrated that the proposed scheme approximates with good accuracy a non-linear system with manipulation of fewer fuzzy rules than the ordinary fuzzy system and its control performance is comparable to that of any model non-linear controller. Here, we have considered the design of the system with only six rules. It has been observed that an induction of a few more rules in the rule base for the system will make the system performance highly satisfactory.

In doing so, we have discussed similarity based approximate reasoning methodology which provides solutions to difficult problems in the construction of intelligent systems in which, the available information is supplied by human experts which, at times are found to be incomplete, imprecise or even uncertain in nature and therefore, inherently ambiguous. Human beings perform qualitative reasoning relying heavily on analogy and similarity in situations where there is no direct knowledge to come up with a plausible conclusion from the available information. We have seen that similarity relations provide an interesting framework to understand the concepts in the design of a fuzzy system. A fuzzification technique has been employed based on similarity relation which is found to be inherent in approximate reasoning. In the process, we have developed a mechanism to compute the matching degree of two fuzzy sets—representation of imprecise concepts (e.g., **low** speed and **very low** speed of a dc-motor). This concept of similarity measure is used at different stages of reasoning—in rule selection and in relation modification.

Proposed similarity based approximate reasoning technique is a combination of Zadeh's Compositional Rule of Inference and Turksen's similarity based reasoning. It is shown that this method is a more general characterization of similarity based approximate reasoning and Turksen's method is a special case of the proposed method. Interesting results have been presented from a typical case study.

We have suggested relevant issues involved in the design of fuzzy systems—introduced similarity in reasoning, similarity relation in fuzzification and the concept of specificity measure in defuzzification. It is hoped that the introduction of the specificity based defuzzification technique will prove to be a powerful one in qualitative control of fuzzy systems. This actually broadens the universal acceptability of fuzzy models.

As it is based on fuzzy logic, the flexibility provided by fuzzy logics allow choices in the selection of a fuzzy partition, the interpretation of different operations—conjunction, disjunction, implication, the fuzzification, the modification and the defuzzification techniques for a particular fuzzy system.

Acknowledgements

This research has been primarily supported by the UGC SAP-DRS Phase-III programme under the Department of Mathematics at Visva-Bharati. It is also supported by the UGC fellowship (Serial no. 2121540996 and Roll no. 423697 under the UGC scheme). UGC's financial support is highly appreciated.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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