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Comparative Study of Analytical Solutions for Time-Dependent Solute Transport along Unsteady Groundwater Flow in Semi-infinite Aquifer

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Abstract

A comparative study is made among Laplace Transform Technique (LTT) and Fourier Transform Technique (FTT) to obtain one-dimensional analytical solution for conservative solute transport along unsteady groundwater flow in semi-infinite aquifer. The time-dependent source of contaminant concentration is considered at the origin and at the other end of the aquifer is supposed to be zero. Initially, aquifer is not solute free which means that the solute concentration exits in groundwater system and it is assumed as a uniform concentration. The aquifer is considered homogeneous and semi-infinite. The time-dependent velocity expressions are considered. The result may be used as preliminary predictive tools in groundwater management and benchmark the numerical code and solutions.

Keywords: Solute Transport, Unsteady, Aquifer, Analytical Solutions

1. Introduction

As we know, groundwater constituents are an important component of many natural water resource systems which supply water for domestic, industrial and agricultural purposes. It is generally a good source of drinking water. It is believed that groundwater is more risk free in compare to the surface water. But these days pollution of groundwater is growing continuously in the various developing countries particularly India due to the indiscriminate discharge of waste water from the various industries, especially coal based industries, which do not have sufficient treatment facilities. These industries discharge their waste water into the neighboring ponds, streams; rivers etc. The chemical constituents of the waste material often infiltrate from these ponds and mixed with the groundwater system causes groundwater contamination [1-4]. Groundwater modeling is specially used in the hydrological sciences for the assessment of the resource potential and prediction of future impact under different conditions. Many experimental and theoretical studies were undertaken to improve the understanding, management, and prediction of the movement of contaminant behavior in groundwater system. These investigations are primarily motivated by concerns about possible contamination of the subsurface environment. Hydrologist, Civil engineers, Scientists etc. are doing their best to solve this type of serious problem by various means. The subsurface solute transport is generally described with the advection-diffusion (AD) equation. In the deterministic approach, explicit closed-form solutions for transport problem can often be derived if the model parameters are constant with respect to time and position [5]. Mathematical modelling is one of the powerful tools to project the existing problems and its appropriate solutions. Although many transport problems must be solved numerically, analytical solutions are still pursued by many scientists because they can provide better physical insight into problems. Groundwater transport and its mathematical models were presented significantly [6-11]. Analytical approach of solute transport problems in groundwater reservoirs is explored [12-14]. In the present work, our objective is to find the analytical solutions using Fourier Transform Technique (FTT) and to compare the result with the solution obtained by Laplace Transform Technique (LTT). To predict the nature of the contaminant concentration along unsteady groundwater flow in semi-infinite aquifer, a comparative study is made by the proposed methods. Time-dependent velocity expressions are considered to illustrate the obtained result.

2. Mathematical Formation of the Problem

Consider a homogeneous semi-infinite aquifer. The time-dependent source of contaminant concentration is considered at the origin, *i.e.*, at $x=0$ and at the other end of the aquifer is supposed to be zero. The groundwater flow in the aquifer is unsteady where the velocity follows either a sinusoidal form or an exponential decreasing form. The sinusoidal form of velocity represents the seasonal variation in a year often observed in tropical regions like Indian sub-continent. In order to mathematically formulate the problem, let $c(x,t)$ be the concentration of contaminants in the aquifer [ML^{-3}], u the groundwater velocity [LT^{-1}], and D the dispersion coefficient [L^2T^{-1}] at time t [T]. Initially the groundwater is not supposed to be solute free *i.e.*, at time $t=0$, the aquifer is not clean which means that some initial background concentration exists in aquifer system. It is represented by uniform concentration c_i. The problem can be formulated as follows:

$$D\frac{\partial^2 c}{\partial x^2}-u\frac{\partial c}{\partial x}=\frac{\partial c}{\partial t} \quad (1)$$

$$u=u_0V(t) \quad (2)$$

where u_0 is the initial groundwater velocity [LT^{-1}] at distance x [L]. Here, two forms of $V(t)$ are considered such as

$$V(t)=1-\sin mt$$

and $$V(t)=\exp(-mt), mt<1$$

where m is the flow resistance coefficient [T^{-1}]. In aquifers in tropical regions, groundwater velocity and water level may exhibit seasonally sinusoidal behavior. In tropical regions like in Indian sub-continent, groundwater velocity and water level are minimum during the peak of the summer season (the period of greatest pumping), which falls in the month of June, just before rainy season. Maximum values are observed during the peak of winter season around December, after the rainy season (the period of lowest pumping). In these regions, groundwater infiltration is from rainfall and rivers. However, exponentially decreasing velocity expression is taken into consideration, keeping the views of literature [15]. The initial and boundary conditions can be expressed as:

$$c(x,t)=c_i;\ x\geq 0, t=0 \quad (3)$$

$$c(x,t)=c_0\left[1+\exp(-qt)\right];\ t>0, x=0 \quad (4a)$$

$$=0;\ t>0, x\to\infty \quad (4b)$$

$$\frac{\partial c}{\partial x}=0;\ x\to\infty \quad (5)$$

where c_i is the initial concentration [MT^{-3}] describing distribution of the contaminant concentration at all point *i.e.*, at $x=0$, c_0 is the solute concentration [MT^{-3}] and q is the decay rate coefficients [T^{-1}]. The physical system of the problem is shown in the **Figure 1**.

The dispersion coefficient, vary approximately directly to seepage velocity for various types of porous media [16]. Also it was found that such relationship established for steady flow was also valid for unsteady flow with sinusoidally varying seepage velocity [17]. Let $D=au$ where the coefficient of dimension length is a and depends upon pore system geometry and average pore size diameter of porous medium. However, molecular diffusion is not included in the present discussion only because the value of molecular diffusion does not vary significantly for different soil and contaminant combinations and they range from 1×10^{-9} to 2×10^{-9} m^2/sec [18].

Using Equation (2), we get $D=D_0V(t)$ Here $D_0=au_0$ is an initial dispersion coefficient. Equation (1) can now be written as follows:

$$D_0\frac{\partial^2 c}{\partial x^2}-u_0\frac{\partial c}{\partial x}=\frac{1}{V(t)}\frac{\partial c}{\partial t} \quad (6)$$

A new time variable is introduced by the transformation [19]

$$T^*=\int_0^t V(t)\,\mathrm{dt} \quad (7)$$

and Equation (6) becomes

$$D_0\frac{\partial^2 c}{\partial x^2}-u_0\frac{\partial c}{\partial x}=\frac{\partial c}{\partial T^*} \quad (8)$$

Now the set of dimensionless parameters are defined as follows

$$C=\frac{c}{c_0},\ X=\frac{x\,u_0}{D_0},\ T=\frac{u_0^2\,T^*}{D_0},\ Q=\frac{qD_0}{u_0^2} \quad (9)$$

The PDE (8) in the form of non-dimensional variable may be written as

$$\frac{\partial^2 C}{\partial X^2}-\frac{\partial C}{\partial X}=\frac{\partial C}{\partial T} \quad (10)$$

$$C(X,T)=\frac{c_i}{c_0};X\geq 0, T=0 \quad (11)$$

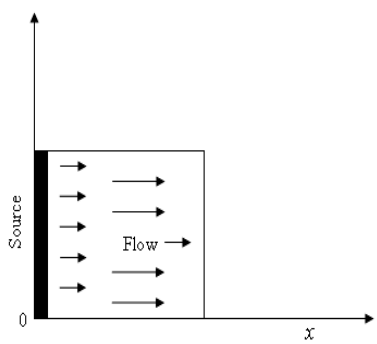


Figure 1. Physical system depicting the problem.

$$C(X,T) = 2 - QT;\ T > 0, X = 0 \quad (12a)$$

$$= 0; T > 0,\ X \to \infty \quad (12b)$$

$$\frac{\partial C(X,T)}{\partial X} = 0; X \to \infty \quad (13)$$

3. Analytical Solutions

As we all know analytical solution of the problem provide closed form solution which gives more realistic result rather than numerical solution which provide approximate solution confining the percentage of error. These days, numerical solution of the complicated problem for which analytical solution is not available, is being obtained frequently by the various scientists and researchers in India and abroad. For example, the following contributions must be cited: A solution of the differential equation of longitudinal dispersion in porous medium was presented [20]. Analytical solutions of one-dimensional convective-dispersion solute transport equations were very well presented [21-22]. Dispersion of pollutants in semi-infinite porous media with unsteady velocity distribution was discussed [23]. Analytical solutions for convective dispersive transport in confined aquifers with different initial and boundary conditions were obtained [24]. Analytical solution of a convection-dispersion model with time-dependent transport coefficients was presented [25]. Analytical solution of one dimensional time-dependent transport equation was presented [26]. Analytical solutions of the solute transport equation with rate-limited desorption and decay was explored [27]. One-dimensional virus transport homogeneous porous media with time dependent distribution coefficient was presented [28]. A Solute transport in porous media with scale-dependent dispersion and periodic boundary conditions was also presented [29]. Analytical solution for solute transport with depth dependent transformation or sorption coefficient was presented [30]. Solute Dispersion along unsteady groundwater flow in a semi-infinite aquifer was reported [31]. Analytical solutions for solute transport in saturated porous media with semi-infinite or finite thickness were presented [32]. A parametric study of one dimensional solute transport in deformable porous medium was explored [33]. In recent works, one-dimensional analytical approach of solute transport models in homogeneous as well as inhomogeneous aquifer are also explored [34,35]. Investigation of consolidation-induced solute transport: effects of consolidation on solute transport parameters were discussed and it was further extended in which experimental and numerical results were explored [36,37]. Analytical solutions for contaminant diffusion in double-layered porous media were presented [38]. All these analytical solutions are having some limitations though significant contribution for the scientific community is very well reported.

3.1. Solution Using Laplace Transform

Using the transformation

$$C(X,T) = K(X,T)\exp\left(\frac{X}{2} - \frac{T}{4}\right) \quad (14)$$

The solution (15) of above problem was obtained with same initial and boundary conditions [35].

3.2. Solution Using Fourier Transform

Using transformation given in (14) in Equations (10)-(13) and applying Fourier Transform, we can get the solution of given boundary value problem as follows:

$$K_s(p,T) = \sqrt{\frac{2}{\pi}}\left[(2-QT)\frac{p\exp\left(\frac{T}{4}\right)}{\left(p^2+\frac{1}{4}\right)} + Q\frac{p\exp\left(\frac{T}{4}\right)}{\left(p^2+\frac{1}{4}\right)^2}\right] - \sqrt{\frac{2}{\pi}}\left[\left(2-\frac{c_i}{c_0}\right)\frac{p\exp\left(-p^2T\right)}{\left(p^2+\frac{1}{4}\right)} - Q\frac{p\exp\left(-p^2T\right)}{\left(p^2+\frac{1}{4}\right)^2}\right] \quad (16)$$

where $K_s(p,T) = \sqrt{\frac{2}{\pi}}\int_0^\infty K(X,T)\sin pX\,dX,$ (17)

Taking inverse Fourier Transform for (16) and substituting the value of $K(X,T)$ in (14), we may obtained the desired solution is (See in Appendix)

$$C(X,T) = (2-QT) + QX - \frac{1}{2}\left(2 - \frac{c_i}{c_0}\right)F_1(X,T) + \frac{Q}{2}\left[TF_1(X,T) - F_2(X,T) + \exp(X)F_3(X,T)\right] \quad (18)$$

$$C(X,T) = \frac{1}{2}\left(2 - \frac{c_i}{c_0}\right)\left[erfc\left(\frac{X}{2\sqrt{T}} - \frac{\sqrt{T}}{2}\right) + \exp(X)\,erfc\left(\frac{X}{2\sqrt{T}} + \frac{\sqrt{T}}{2}\right)\right] + \frac{c_i}{c_0} - \frac{Q}{2}\left[(T - X)\,erfc\left(\frac{X}{2\sqrt{T}} - \frac{\sqrt{T}}{2}\right) + (T + X)\,erfc\left(\frac{X}{2\sqrt{T}} + \frac{\sqrt{T}}{2}\right)\right] \quad (15)$$

where

$$F_1(X,T) = erfc\left(\frac{\sqrt{T}}{2} - \frac{X}{2\sqrt{T}}\right) - \exp(X)\, erfc\left(\frac{\sqrt{T}}{2} + \frac{X}{2\sqrt{T}}\right) \tag{18a}$$

$$F_2(X,T) = X erfc\left(\frac{\sqrt{T}}{2} - \frac{X}{2\sqrt{T}}\right) + \frac{2\sqrt{T}}{\sqrt{\pi}} \exp\left[-\left(\frac{\sqrt{T}}{2} - \frac{X}{2\sqrt{T}}\right)^2\right] \tag{18b}$$

$$F_3(X,T) = X erfc\left(\frac{\sqrt{T}}{2} + \frac{X}{2\sqrt{T}}\right) - \frac{2\sqrt{T}}{\sqrt{\pi}} \exp\left[-\left(\frac{\sqrt{T}}{2} + \frac{X}{2\sqrt{T}}\right)^2\right] \tag{18c}$$

4. Illustration and Discussion

We consider the sinusoidally varying and exponentially decreasing forms of velocities which are valid fortransient groundwater flow too [15, 23]. Now from Equation (2) the velocity expressions are as follows:

$$u(t) = u_0(1 - \sin mt) \tag{19a}$$

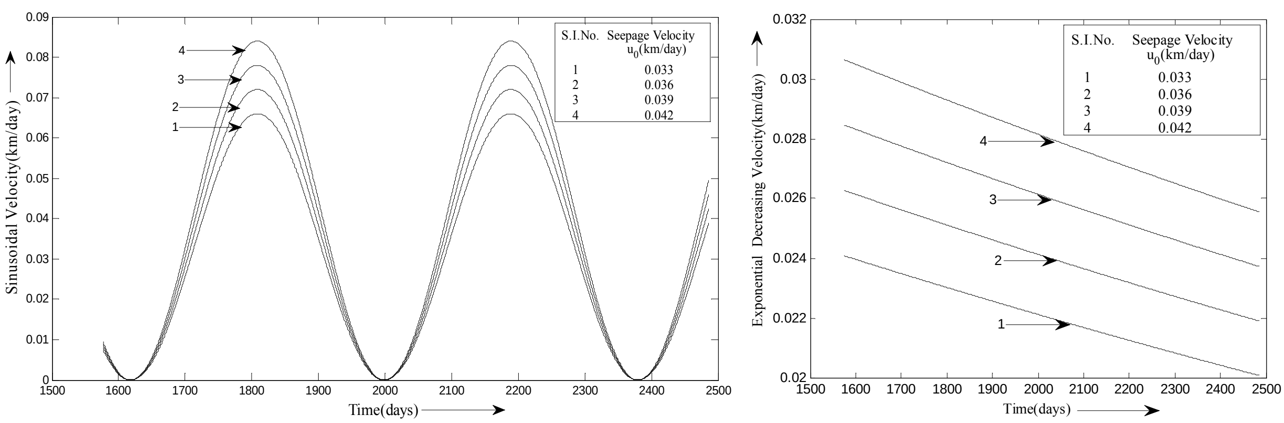
$$u(t) = u_0 \exp(-mt), mt < 1 \tag{19b}$$

where m(/day) is the flow resistance coefficient. For both the expressions, the non-dimensional time variable T may be written as

$$T = \frac{u_0^{\,2}}{mD_0}\left[mt - (1 - \cos mt)\right] \tag{20a}$$

$$T = \frac{u_0^{\,2}}{mD_0}\left[1 - \exp(-mt)\right] \tag{20b}$$

where $mt = 3k + 2$, k is a whole number are chosen. For $m = 0.0165$ (/day), (19a) yields, t (days) $=182k + 121$, approximately. For these values of mt, the velocity u, is alternatively minimum and maximum. Hence it represents the groundwater level and velocity minimum during the month of June and maximum during December just after six months (Approximately 182 days) in one year. The next data of t represents minimum and maximum records during June and December respectively in the subsequent years. The sinusoidally varying and exponentially decreasing form of velocity representations are made graphically with respect to time at different values of seepage velocity and dispersion parameters and shown in the **Figures 2(a)** and **(b)**. As we increase the seepage velocity parameter, the peak of sinusoidal form of velocity increases which reveals in **Figure 2(a)**. This representation can often be observed in tropical region of India. An analytical solutions (15) and (18) are computed for the values $c_i = 0.1$, $c_0 = 1.0$, $u_0 = 0.033$ - 0.045 km/day, $D_0 = 0.33$ - 0.45 km^2/day, $q = 0.0009$(/day), and $x = 10$ km. The time-dependent concentration values are depicted from the table 1(a-d) for sinusoidal form of velocity expression 19(a) at the seepage velocity u_0 ranging from 0.033 km/day to 0.042 km/day and dispersions parameter D_0 ranging from 0.33 km^2/day to 0.42 km^2/day. The concentration values at different positions are obtained for both the methods LTT and FTT in row (i) and row (ii) respectively shown in **Tables 1**and **2**. It is observed that concentration values decreases rapidly in row (i) in comparison to row (ii). However, in **Tables 3** and **4** the concentration values also decreases rapidly in row (i) and slowly and gradually converges at a common point



(a) (b)

Figure 2. Time-dependent (a) sinusoidally varying velocity and (b) exponentially decreasing velocity representations subject to seepage velocity u_0 = 0.033 - 0.045 km/day. Curves No. 1 - 5 represent the contaminant concentrations in 5th year, 6th year, and 7th year December and June, respectively.

Table 1. Contaminant Concentration values in sinusoidal form of velocity with u_0 = 0.033, D_0 = 0.33 using (i) Laplace Transform Technique and (ii) Fourier Transform Technique.

X (km)	0	1	2	3	4	5	6	7	8	9	10
						mt = 26, *t* = 1576 days					
(i)	1.8477	1.4911	1.1343	0.8585	0.5759	0.3340	0.1887	0.1259	0.1058	0.1010	0.1001
(ii)	1.8477	1.7564	1.4278	0.9765	0.5663	0.2966	0.1648	0.1165	0.1032	0.1005	0.1001
						mt = 29, *t* = 1758 days					
(i)	1.8381	1.4816	1.0805	0.7616	0.4798	0.2692	0.1572	0.1146	0.1028	0.1004	0.1000
(ii)	1.8381	1.7144	1.3475	0.8798	0.4859	0.2487	0.1441	0.1099	0.1017	0.1002	0.1000
						mt = 32, *t* = 1940 days					
(i)	1.8109	1.4586	0.9673	0.5667	0.3056	0.1688	0.1167	0.1029	0.1004	0.1000	0.1000
(ii)	1.8109	1.6049	1.1514	0.6648	0.3281	0.1681	0.1149	0.1024	0.1003	0.1000	0.1000
						mt = 35, *t* = 2122 days					
(i)	1.8034	1.4531	0.9447	0.5296	0.2756	0.1541	0.1120	0.1019	0.1002	0.1000	0.1000
(ii)	1.8034	1.5771	1.1048	0.6183	0.2981	0.1552	0.1111	0.1016	0.1002	0.1000	0.1000
						mt = 38, *t* = 2304 days					
(i)	1.7745	1.4349	0.8833	0.4331	0.2031	0.1229	0.1036	0.1004	0.1000	0.1000	0.1000
(ii)	1.7745	1.4783	0.9489	0.4762	0.2169	0.1251	0.1037	0.1004	0.1000	0.1000	0.1000
						mt = 41, *t* = 2486 days					
(i))	1.7683	1.4315	0.8745	0.4199	0.1938	0.1194	0.1028	0.1003	0.1000	0.1000	0.1000
(ii)	1.7683	1.4584	0.9194	0.4516	0.2046	0.1212	0.1029	0.1003	0.1000	0.1000	0.1000

Table 2. Contaminant Concentration values in sinusoidal form of velocity with u_0 = 0.036, D_0 = 0.36 using (i) Laplace Transform Technique and (ii) Fourier Transform Technique.

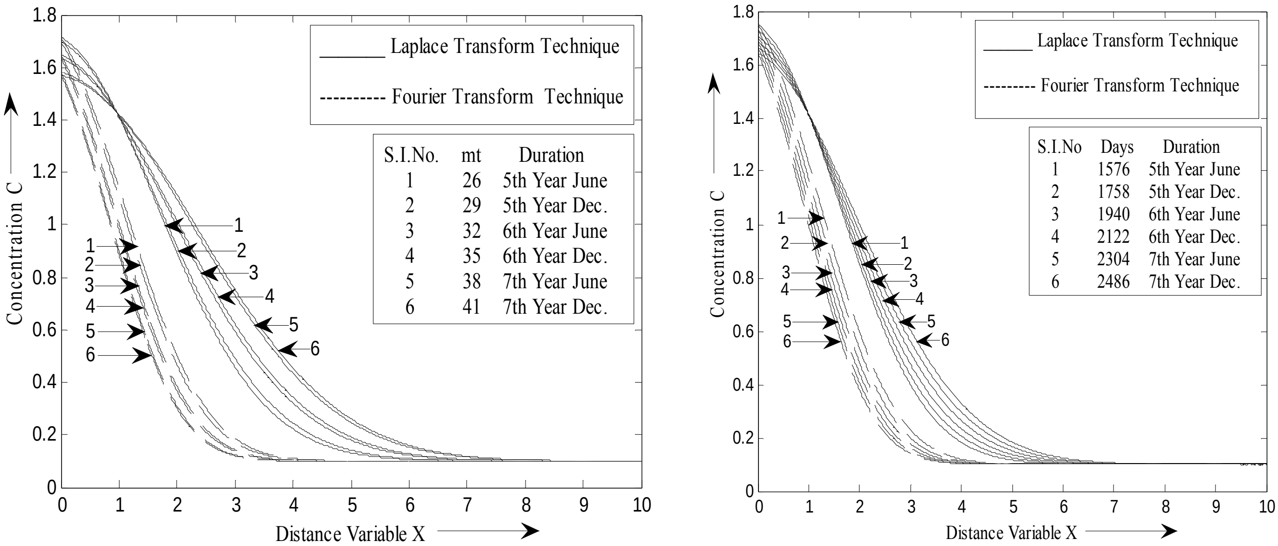
X(km)	0	1	2	3	4	5	6	7	8	9	10
						mt = 26, *t* = 1576 days					
(i)	1.8187	1.4536	0.9357	0.5190	0.2688	0.1514	0.1113	0.1018	0.1002	0.1000	0.1000
(ii)	1.8187	1.5784	1.0993	0.6122	0.2944	0.1538	0.1107	0.1015	0.1002	0.1000	0.1000
						mt = 29, *t* = 1758 days					
(i)	1.8074	1.4457	0.9057	0.4712	0.2321	0.1350	0.1066	0.1009	0.1001	0.1000	0.1000
(ii)	1.8074	1.5354	1.0288	0.5455	0.2545	0.1381	0.1066	0.1008	0.1001	0.1000	0.1000
						mt = 32, *t* =1940 days					
(i)	1.7750	1.4274	0.8548	0.3943	0.1777	0.1138	0.1017	0.1001	0.1000	0.1000	0.1000
(ii)	1.7750	1.4245	0.8615	0.4045	0.1824	0.1148	0.1018	0.1001	0.1000	0.1000	0.1000
						mt = 35, *t* = 2122 days					
(i)	1.7660	1.4234	0.8477	0.3845	0.1713	0.1117	0.1012	0.1001	0.1000	0.1000	0.1000
(ii)	1.7660	1.3967	0.8227	0.3755	0.1698	0.1115	0.1012	0.1001	0.1000	0.1000	0.1000
						mt = 38, *t* = 2304 days					
(i)	1.7317	1.4107	0.8400	0.3764	0.1665	0.1102	0.1010	0.1001	0.1000	0.1000	0.1000
(ii)	1.7317	1.2989	0.6961	0.2903	0.1378	0.1046	0.1003	0.1000	0.1000	0.1000	0.1000
						mt = 41, *t* = 2486 days					
(i)	1.7242	1.4085	0.8416	0.3793	0.1684	0.1109	0.1011	0.1001	0.1000	0.1000	0.1000
(ii)	1.7242	1.2793	0.6726	0.2762	0.1333	0.1038	0.1003	0.1000	0.1000	0.1000	0.1000

Table 3. Contaminant Concentration values in sisoidal form of velocity with $u_0 = 0.039$, $D_0 = 0.39$ using Laplace Transform Technique and (ii) Fourier Transform Technique.

X(km)	0	1	2	3	4	5	6	7	8	9	10
	mt = 26, *t* = 1576 days										
(i)	1.7872	1.4274	0.8461	0.3846	0.1723	0.1121	0.1013	0.1001	0.1000	0.1000	0.1000
(ii)	1.7872	1.4145	0.8395	0.3867	0.1744	0.1127	0.1014	0.1001	0.1000	0.1000	0.1000
	mt = 29, *t* = 1758 days										
(i)	1.7739	1.4217	0.8384	0.3744	0.1658	0.1100	0.1010	0.1001	0.1000	0.1000	0.1000
(ii)	1.7739	1.3719	0.7811	0.3446	0.1572	0.1085	0.1008	0.1000	0.1000	0.1000	0.1000
	mt = 32, *t* = 1940 days										
(i)	1.7359	1.4098	0.8423	0.3830	0.1718	0.1122	0.1014	0.1001	0.1000	0.1000	0.1000
(ii)	1.7359	1.2634	0.6459	0.2596	0.1281	0.1029	0.1002	0.1000	0.1000	0.1000	0.1000
	mt = 35, *t* = 2122 days										
(i)	1.7254	1.4075	0.8482	0.3921	0.1777	0.1143	0.1018	0.1002	0.1000	0.1000	0.1000
(ii)	1.7254	1.2365	0.6152	0.2427	0.1233	0.1022	0.1001	0.1000	0.1000	0.1000	0.1000
	mt = 38, *t* = 2304 days										
(i)	1.6851	1.4014	0.8830	0.4424	0.2110	0.1269	0.1048	0.1006	0.1001	0.1000	0.1000
(ii)	1.6851	1.1429	0.5172	0.1949	0.1118	0.1008	0.1000	0.1000	0.1000	0.1000	0.1000
	mt = 41, *t* = 2486 days										
(i)	1.6763	1.4006	0.8925	0.4555	0.2199	0.1305	0.1058	0.1008	0.1001	0.1000	0.1000
(ii)	1.6763	1.1243	0.4993	0.1872	0.1102	0.1006	0.1000	0.1000	0.1000	0.1000	0.1000

Table 4. Contaminant Concentration values in sinusoidal form of velocity with $u_0 = 0.042$, $D_0 = 0.42$ using Laplace Transform Technique and (ii) Fourier Transform Technique

X(km)	0	1	2	3	4	5	6	7	8	9	10
	mt = 26, *t* = 1576 days										
(i)	1.7532	1.4130	0.8405	0.3824	0.1721	0.1124	0.1014	0.1001	0.1000	0.1000	0.1000
(ii)	1.7532	1.2680	0.6441	0.2576	0.1274	0.1028	0.1002	0.1000	0.1000	0.1000	0.1000
	mt = 29, *t* = 1758 days										
(i)	1.7378	1.4100	0.8513	0.3985	0.1825	0.1161	0.1022	0.1002	0.1000	0.1000	0.1000
(ii)	1.7378	1.2270	0.5976	0.2326	0.1206	0.1018	0.1001	0.1000	0.1000	0.1000	0.1000
	mt = 32, *t* = 1940 days										
(i)	1.6937	1.4055	0.8978	0.4640	0.2263	0.1333	0.1065	0.1009	0.1001	0.1000	0.1000
(ii)	1.6937	1.1239	0.4923	0.1834	0.1095	0.1006	0.1000	0.1000	0.1000	0.1000	0.1000
	mt = 35, *t* = 2122 days										
(i)	1.6816	1.4050	0.9133	0.4851	0.2410	0.1397	0.1084	0.1013	0.1001	0.1000	0.1000
(ii)	1.6816	1.0986	0.4688	0.1739	0.1077	0.1004	0.1000	0.1000	0.1000	0.1000	0.1000
	mt = 38, *t* = 2304 days										
(i)	1.6348	1.4055	0.9776	0.5718	0.3046	0.1706	0.1190	0.1039	0.1006	0.1001	0.1000
(ii)	1.6348	1.0112	0.3945	0.1474	0.1036	0.1001	0.1000	0.1000	0.1000	0.1000	0.1000
	mt = 41, *t* = 2486 days										
(i)	1.6246	1.4060	0.9921	0.5912	0.3196	0.1786	0.1221	0.1048	0.1008	0.1001	0.1000
(ii)	1.6246	0.9940	0.3811	0.1431	0.1030	0.1001	0.1000	0.1000	0.1000	0.1000	0.1000



(a) (b)

Figure 3. Time-dependent contaminant source concentrations subject to (a) a sinusoidally varying velocity (b) exponentially decreasing velocity using LTT (solid line) and FTT (Dotted line) groundwater flow with longitudinal direction only.

near by the source and after that it further decreases and reached towards minimum or harmless concentration. But in row (ii), the concentration values decreases and goes on decreasing towards minimum or harmless concentration. The concentration values are depicted graphically in the presence of time-dependent source of contaminant concentration at $mt = 3k + 2$, $8 \le k \le 13$ which represents minimum and maximum records of groundwater level and velocity during June and December in 5th, 6th and 7th years respectively. The contaminant concentration distribution behaviour along transient groundwater flow of sinusoidally varying velocity is shown in the **Figure 3(a)** at the seepage velocity $u_0 = 0.045$ km/day and dispersions parameter $D_0 = 0.45$ km^2/day. It is observed that the contaminant concentration decreases at the source and emerges at a point nearby origin. After emergence tenncy of the contaminant concentration is same reaching towards the minimum or harmless concentration. But the values of the contaminant concentration decreases andineases with time just before and after the emergence respectively. For example, before emergence 5th year Dec. concentration is less than 5th year June concentra- on while after emergence the trend is just reverse. For the same set of inputs except $m = 0.0002$ (/day) as $mt < 1$, equation (15) and (18) are also computed for exponenttially decreasing form of velocity and shown in the **Figure 3(b)**. It is also observed that the trend of contaminant concentration is almost same as discussed in sinusoidally varying velocity but the de- creasing rate is little slower at the source and nearby the origin. The decreasing tendency of concentration values depicted through the **Tables 1** and **4** and the **Figures 3(a)** and **(b)** reveals that FTT is more effective in case of increasing the seepage velocity and dispersion parameters. However, LTT is preferable in the case of decreasing seepage velocity and dispersion parameters.

5. Conclusions

A comparative study is made to obtain the analytical solution of solute transport modeling in groundwater system using LTT and FTT. A solute transport model is formulated with time-dependent source concentration in one-dimensional homogeneous semi-infinite aquifer with suitable initial and boundary conditions. To predict contaminants concentration along transient groundwater flow in homogeneous, semi-infinite aquifer FTT is more preferable than LTT with respect to sensitivity of seepage velocity and dispersion parameters. The dispersion is directly proportional to seepage velocity concept is used. Analytical solution of the problems may help to model the numerical codes and solutions. It may be used as the preliminary predictive tools in groundwater management.

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7. References

[1] S. Mohan and M. Muthukumaran, "Modeling of pollutant transport in groundwater," *Journal of Institution of Engineers (India): Environmental Division*, Vol. 85, No. 1, 2004, pp. 22-32.

[2] H. D. Sharma and K. R. Reddy, "Geo-environmental Engineering," John Willy & sons Inc., Hoboken, 2004.

[3] R. Rausch, W. Schafer, R. Therrien and C. Wagner, "Solute Transport Modelling: An Introduction to Models and Solu on Strategies," Gebrüder Borntrager Verlagsbuchhandlung Science Publishers, Berlin, 2005.

[4] M. Thangarajan, "Groundwater: Resource Evaluation, Augmentation, Contamination, Restoration, Modeling and Management," Capital Publishing Company, New Delhi, 2006.

[5] F. J. Leij, N. Toride and T. V. Genuchten, "Analytical Solution for Non-Equilibrium Solute Transport in Three-Dimension Porous Media," *Journal of Hydrology*, Vol. 151, No. 2-4, 1993, pp. 193-228. doi:10.1016/0022-1694(93)90236-3

[6] J. J. Fried, "Groundwater Pollution," Elsevier Scientific Publishing Company, Amsterdam, 1975.

[7] I. C. Javendel and C. F. Tsang, "Groundwater Transport: Hand book of Mathematics Models," Water Resource Monogrser, AGU, Washington, Vol. 10, 1984.

[8] J. Bear and A. Verruijt, "Modeling Groundwater Flow and Pollution," D. Reidel Pubulising Company, Dordrecht, 1987. doi:10.1007/978-94-009-3379-8

[9] M. P. Anderson and W. W. Woessner, "Applied Groundwater Modeling: Simulation of Flow and Advective Transport," Academic Press, London, 1992.

[10] S. N. Rai, "Role of Mathematical Modeling in Groundwater Resources Management," NGRI, Hyderabad, 2004.

[11] N. C. Ghosh and K. D. Sharma, "Groundwater Modelling and Management," Capital Publishing Company, New-Delhi, 2006

[12] G. S. Kumar, M. Sekher and D. Mishra, "Time-Dependent Dispersivity Behaviour of Non Reactive Solutes in a System of Parallel Structures," *Hydrology and Earth System Sciences*, Vol. 3, No. 3, 2006, pp. 895-923. doi:10.5194/hessd-3-895-2006

[13] M. K. Singh, V. P. Singh, P. Singh and D. Shukla, "Analytical Solution for Conservative Solute Transport in One Dimensional Homogeneous Porous Formation with Time-Dependent Velocity," *Journal of Engineering Mechanics*, Vol. 135, No. 9, 2009, pp. 1015-1021. doi:10.1061/(ASCE)EM.1943-7889.0000018

[14] A. Kumar, D. K. Jaiswal and N. Kumar, "Analytical Solution to One Dimensional Advection-Diffusion Equation with Variable Coefficient in Semi-Infinite Media," *Journal of Hydrology*, Vol. 380, No. 3-4, 2010, pp. 330-337. doi:10.1016/j.jhydrol.2009.11.008

[15] R. P. Banks and S. T. Jerasate, "Dispersion in Unsteady Porous Media Flow," *Journal of Hydraulic Division*, Vol. 88, No. HY3, 1962, pp. 1-21.

[16] E. H. Ebach and R. White "Mixing of Fluid Flowing through Beds of Packed Solids," *Journal of American Institute of Chemical Engineers*, Vol. 4, No. 2, 1958, pp. 161-164. doi:10.1002/aic.690040209

[17] R. R. Rumer, "Longitudinal Dispersion in Steady and Un-Steady Flow," *Journal of Hydraulic Division*, Vol. 88, No. HY4, 1962, pp. 147-172.

[18] J. K. Mitchell, "Fundamentals of Soil Behavior," Wiley, New York, 1976.

[19] J. Crank, "The Mathematics of Diffusion," Oxford University Press, Oxford, 1975.

[20] A. Ogata, R. B. Banks, "A Solution of the Differential Equation of Longitudinal Dispersion in Porous Medium," US Geological Survey, Washington D.C., 1961.

[21] M. Th. Van Genucheten and W. J. Alves, "Analytical Solutions for Chemical Transport with Simultaneous Adsorption, Zero-Order Production and finite Order Decay," *Journal of Hydrology*, Vol. 49, No. 3-4, 1981, pp. 213-233. doi:10.1016/0022-1694(81)90214-6

[22] M. Th. Van Genucheten and W. J. Alves, "Analytical Solution of One Dimensional Convective-Dispersion Solute Transport Equation," Technical Bulletin, 1982, pp. 1-51.

[23] N. Kumar, "Dispersion of Pollutants in Semi-Infinite Porous Media with Unsteady Velocity Distribution," *Nordic Hydrology*, Vol. 14, No. 3, 1983, pp. 167-178.

[24] F. T. Lindstrom and L. Boersma, "Analytical Solutions for Convective-Dispersive Transport in Confined Aquifers with Different Initial and Boundary Conditions," *Water Resources Research*, Vol. 25, No. 2, 1989, pp. 241-256. doi:10.1029/WR025i002p00241

[25] D. A. Barry and G. Sposito, "Analytical Solution of a Convection-Dispersion Model with Time-Dependent Transport Coefficients," *Water Recourses Research*, Vol. 25, No. 12, 1989, pp. 2407-2416. doi:10.1029/WR025i012p02407

[26] H. A. Basha and F. S. El-Habel, "Analytical Solution of One Dimensional Time-Dependent Transport Equation," *Water Recourses Research*, Vol. 29, No. 9, 1993, pp. 3209-3214. doi:10.1029/93WR01038

[27] V. A. Fry, J. D. Istok and R. B. Guenther, "An Analytical Solution of the Solute Transport Equation with Rate-Limited Desorption and Decay," *Water Recourses Research*, Vol. 29, No. 9, 1993, pp. 3201-3208. doi:10.1029/93WR01394

[28] C. V. Chrysikopoulos and Y. Sim, "One-Dimensional Virus Transport Homogeneous Porous Media with Time Dependent Distribution Coefficient," *Journal of Hydrology*, Vol. 185, No. 1-4, 1996, pp. 199-219. doi:10.1016/0022-1694(95)02990-7

[29] J. D. Logan, "Solute Transport in Porous Media with Scale-Dependent Dispersion and Periodic Boundary Conditions," *Journal of Hydrology*, Vol. 184, No. 3-4, 1996, pp. 261-276. doi:10.1016/0022-1694(95)02976-1

[30] M. Flury, Q. J. Wu, L. Wu and L. Xu, "Analytical Solution for Solute Transport with Depth Dependent Transformation or Sorption Coefficient," *Water Recourses Re-*

search, Vol. 34, No. 11, 1998, pp. 2931-2937. doi:10.1029/98WR02299

[31] N. Kumar and M. Kumar, "Solute Dispersion along Unsteady Groundwater Flow in a Semi-Infinite Aquifer," *Hydrology and Earth System Sciences*, Vol. 2, No. 1, 1998, pp. 93-100. doi:10.5194/hess-2-93-1998

[32] Y. Sim and C. V. Chrysikopoulos, "Analytical Solutions for Solute Transport In Saturated Porous Media with Semi-Infinite Or Finite Thickness," *Advances in Water Recourses*, Vol. 22, No. 5, 1999, pp. 507-519. doi:10.1016/S0309-1708(98)00027-X

[33] A. Alshawabkeh and N. Rahbar, "A Parametric Study of One Dimensional Solute Transport in Deformable Porous Medium," *Journal of Geotechnical and Geoenvironmental Engineering*, Vol. 132, No. 8, 2006, pp. 100-1010. doi:10.1061/(ASCE)1090-0241(2006)132:8(1001)

[34] V. Srinivasan and T. P. Clement, "Analytical Solutions for Sequentially Coupled One Dimensional Reactive Transport Problems-Part-I: Mathematical Derivations," *Advances in Water Recourses*, Vol. 31, No. 2, 2008, pp. 203-218. doi:10.1016/j.advwatres.2007.08.002

[35] M. K. Singh, N. K. Mahato and P. Singh, "Longitudinal Dispersion with Time Dependent Source Concentration in Semi Infinite Aquifer," *Journal of Earth System Sciences*, Vol. 117, No. 6, 2008, pp. 945-949. doi:10.1007/s12040-008-0079-x

[36] J. Lee, P. J. Fox and J. J. Lenhart, "Investigation of Consolidation-Induced Solute Transport. I. Effect of Consolidation on solute Transport Parameters," *Journal of Geotechnical and Geoenvironmental Engineering*, Vol. 135, No. 9, 2009, pp. 1228-1238. doi:10.1061/(ASCE)GT.1943-5606.0000047

[37] J. Lee and P. J. Fox, "Investigation of Consolidation-Induced Solute Transport. II. Experimental and Numerical Results," *Journal of Geotechnical and Geoenvironmental Engineering*, Vol. 135, No. 9, 2009, pp. 1239-1253. doi:10.1061/(ASCE)GT.1943-5606.0000048

[38] Y. C. Li and P. J. Cleall, "Analytical solutions for Contaminant Diffusion in Double-Layered Porous Media," *Journal of Geotechnical and Geoenvironmental Engineering*, Vol. 136, No. 11, 2010, pp. 1542-1554. doi:10.1061/(ASCE)GT.1943-5606.0000365

Appendix

Analytical Solution using Fourier Transform Tech- nique (FTT):

$$D\frac{\partial^2 c}{\partial x^2} - u\frac{\partial c}{\partial x} = \frac{\partial c}{\partial t} \quad (1)$$

$$u = u_0 V(t) \quad (2)$$

The initial and boundary conditions can be expressed as:

$$c(x,t) = c_i;\ x \ge 0, t = 0 \quad (3)$$

$$c(x,t) = c_0\left[1 + \exp(-qt)\right];\quad t > 0, x = 0 \quad (4a)$$

$$= 0;\quad t > 0, x \to \infty \quad (4b)$$

$$\frac{\partial c}{\partial x} = 0\ ; x \to \infty \quad (5)$$

Let $D = au$, where the coefficient of dimension length is a and depends upon pore system geometry and average pore size diameter of porous medium. Using Equation (2), we get $D = D_0 V(t)$ .Here $D_0 = au_0$ is an initial dispersion coefficient. Equation (1) can now be written as

$$D_0\frac{\partial^2 c}{\partial x^2} - u_0\frac{\partial c}{\partial x} = \frac{1}{V(t)}\frac{\partial c}{\partial t} \quad (6)$$

A new time variable is introduced by the transformation [19]

$$T* = \int_0^t V(t)\mathrm{d}t \quad (7)$$

And Equation (6) becomes

$$D_0\frac{\partial^2 c}{\partial x^2} - u_0\frac{\partial c}{\partial x} = \frac{\partial c}{\partial T*} \quad (8)$$

and initial condition (3) and boundary conditions 4(a, b) and (5) becomes

$$c(x,T^*) = c_i; x \ge 0, T^* = 0 \quad (9)$$

$$C(x,T^*) = c_0\left(2 - QT^*\right);\ T^* > 0, x = 0 \quad (10)$$

$$= 0; T^* \ge 0,\ x \to \infty \quad (11)$$

$$\frac{\partial C(x,T^*)}{\partial x} = 0\ ;\ x \to \infty \quad (12)$$

Now the set of non dimensional variables are defined as follows

$$C = \frac{c}{c_0},\ X = \frac{x\,u_0}{D_0},\ T = \frac{u_0^2\ T*}{D_0}\ ,\ Q = \frac{qD_0}{u_0^2} \quad (13)$$

The PDE (8) in the form of non-dimensional variable may be written as

$$\frac{\partial^2 C}{\partial X^2} - \frac{\partial C}{\partial X} = \frac{\partial C}{\partial T} \quad (14)$$

$$C(X,T) = \frac{c_i}{c_0}; X \ge 0, T = 0 \quad (15)$$

$$C(X,T) = 2 - QT;\ T > 0, \mathrm{X} = 0 \quad (16)$$

$$= 0\ ;\ T > 0,\ \mathrm{X} \to \infty \quad (17)$$

$$\frac{\partial C(X,T)}{\partial X} = 0\ ;\ X \to \infty \quad (18)$$

Using the transformation

$$C(X,T)=K(X,T)\exp\left(\frac{X}{2}-\frac{T}{4}\right) \quad (19)$$

Then the Equations (14) to (18) can be written as

$$\frac{\partial^2 K}{\partial X^2}=\frac{\partial K}{\partial T} \quad (20)$$

$$K(X,T)=\frac{c_i}{c_0}\exp\left(-\frac{X}{2}\right); X\geq 0, T=0 \quad (21)$$

$$K(X,T)=(2-QT)\exp\left(\frac{T}{4}\right);\ T>0,\ \mathrm{X}=0 \quad (22)$$

$$=0; T>0,\ \mathrm{X}\to\infty \quad (23)$$

$$\frac{\partial K(X,T)}{\partial X}=0;\ T\geq 0, X\to\infty \quad (24)$$

Since $K(X,T)$ is specified at $X=0$, thus Fourier sine transform is applicable for this problem.

Taking the Fourier sine transform of the PDE (20) and using the notation

$$K_s(p,T)=\sqrt{\frac{2}{\pi}}\int_0^\infty K(X,T)\sin pX dX \quad (25)$$

and using the conditions

$$K(X,T)\to 0\ \frac{\partial K(X,T)}{\partial X}\to 0 \quad (26)$$

as $\mathrm{X}\to\infty$ then equation (20) can take the form

$$\sqrt{\frac{2}{\pi}}pK(0,T)-p^2K_s(p,T)=\frac{\mathrm{d}K_s(p,T)}{\mathrm{d}T}$$

$$\sqrt{\frac{2}{\pi}}p(2-QT)\exp\left(\frac{T}{4}\right)-p^2K_s(p,T)=\frac{\mathrm{d}K_s(p,T)}{\mathrm{d}T}$$

[Using eqn.(22)]

$$\frac{\mathrm{d}K_s(p,T)}{\mathrm{d}T}+p^2K_s(p,T)=\sqrt{\frac{2}{\pi}}p(2-QT)\exp\left(\frac{T}{4}\right) \quad (26)$$

Solving the differential equation (26) one can get the general solution as follows:

$$K_s(p,T)=\sqrt{\frac{2}{\pi}}p\left[\frac{2-QT}{\left(p^2-\frac{1}{4}\right)}-\frac{Q}{\left(p^2-\frac{1}{4}\right)^2}\right]\exp\left(\frac{T}{4}\right)+c_1\exp\left(-p^2T\right) \quad (27)$$

To remove the arbitrary constant c_1 we use the initial condition (21) then the equation (27) takes the form (28).

$$K_s(p,T)=\sqrt{\frac{2}{\pi}}p\left\{\frac{2-QT}{\left(p^2-\frac{1}{4}\right)}-\frac{Q}{\left(p^2-\frac{1}{4}\right)^2}\right\}\exp\left(\frac{T}{4}\right)-\sqrt{\frac{2}{\pi}}p\left\{\frac{\left(2-\frac{c_i}{c_0}\right)}{\left(p^2-\frac{1}{4}\right)}+\frac{Q}{\left(p^2-\frac{1}{4}\right)^2}\right\}\exp\left(-p^2T\right)$$

Taking the inverse Fourier transform on both side of equation (28) and using the transformation given in equation (19), one can get (29).

Hence, $C(X,T)$ can now be written as follows (30):

$$K_s(p,T)=\sqrt{\frac{2}{\pi}}\left[(2-QT)\frac{p\exp\left(\frac{T}{4}\right)}{\left(p^2+\frac{1}{4}\right)}+Q\frac{p\exp\left(\frac{T}{4}\right)}{\left(p^2+\frac{1}{4}\right)^2}\right]-\left[\sqrt{\frac{2}{\pi}}\left(2-\frac{c_i}{c_0}\right)\frac{p\exp\left(-p^2T\right)}{\left(p^2+\frac{1}{4}\right)}+Q\frac{p\exp\left(-p^2T\right)}{\left(p^2+\frac{1}{4}\right)^2}\right] \quad (28)$$

$$C(X,T)=(2-QT)+QX-\frac{1}{2}\left(2-\frac{c_i}{c_0}\right)\left[erfc\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right)-\exp(X)erfc\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right)\right]$$
$$+\frac{Q}{2}\left[T\left\{erfc\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right)-\exp(X)erfc\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right)\right\}\right]$$
$$+\frac{Q}{2}\left[Xerfc\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right)+2\exp(X)\sqrt{\frac{T}{\pi}}\exp\left\{-\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right)^2\right\}\right] \quad (29)$$
$$+\frac{Q}{2}X\exp(X)erfc\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right)-Q\exp(X)\sqrt{\frac{T}{\pi}}\exp\left\{-\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right)^2\right\}$$

$$C(X,T)=(2-QT)+QX-\frac{1}{2}\left(2-\frac{c_i}{c_0}\right)F_1(X,T)$$
$$+\frac{Q}{2}\left[TF_1(X,T)-F_2(X,T)+\exp(X)F_3(X,T)\right] \quad (30)$$

where

$$F_1(X,T)=erfc\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right)-\exp(X)erfc\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right)$$

$$F_2(X,T)=Xerfc\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right)$$
$$+\frac{2\sqrt{T}}{\sqrt{\pi}}\exp\left\{-\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right)^2\right\}$$

$$F_3(X,T)=Xerfc\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right)$$
$$-\frac{2\sqrt{T}}{\sqrt{\pi}}\exp\left\{-\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right)^2\right\}$$