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Existence of Equilibrium Points in the R3BP with Variable Mass When the Smaller Primary is an Oblate Spheroid

M. R. Hassan[1], Sweta Kumari[2], Md. Aminul Hassan[3]

[1]Department of Mathematics, S. M. College, T. M. Bhagalpur University, Bhagalpur, India
[2]Research Scholar, T. M. Bhagalpur University, Bhagalpur, India
[3]GTE, Bangalore, India
Email: hassansmc@gmail.com, mahassan012@gmail.com

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Abstract

The paper deals with the existence of equilibrium points in the restricted three-body problem when the smaller primary is an oblate spheroid and the infinitesimal body is of variable mass. Following the method of small parameters; the co-ordinates of collinear equilibrium points have been calculated, whereas the co-ordinates of triangular equilibrium points are established by classical method. On studying the surface of zero-velocity curves, it is found that the mass reduction factor has very minor effect on the location of the equilibrium points; whereas the oblateness parameter of the smaller primary has a significant role on the existence of equilibrium points.

Keywords

Restricted Three-Body Problem, Jean's Law, Space-Time Transformation, Oblateness, Equilibrium Points, Surface of Zero-Velocity

1. Introduction

Restricted problem of three bodies with variable mass is of great importance in celestial mechanics. The two-body problem with variable mass was first studied by Jeans [1] regarding the evaluation of binary system. Meshcherskii [2] assumed that the mass was ejected isotropically from the two-body system at very high velocities and was lost to the system. He examined the change in orbits, the variation in angular momentum and the energy of the system. Omarov [3] has discussed the restricted problem of perturbed motion of two bodies with variable mass. Following Jeans [1], Verhulst [4] discussed the two body problem with slowly decreasing mass, by a non-linear, non-autonomous system of differential equations. Shrivastava and Ishwar [5] derived the equations of motion in the circular restricted problem of three bodies with variable mass with the assumption that the mass of the infinitesimal body varies with respect to time.

Singh and Ishwar [6] showed the effect of perturbation on the location and stability of the triangular equilibrium points in the restricted three-body problem. Das *et al.* [5] developed the equations of motion in elliptic restricted problem of three bodies with variable mass. Lukyanov [7] discussed the stability of equilibrium points in the restricted problem of three bodies with variable mass. He found that for any set of parameters, all the equilibriums points in the problem (Collinear, Triangular and Coplanar) are stable with respect to the conditions considered in the Meshcherskii space-time transformation. El Shaboury [8] discussed the equation of motion of Elliptic Restricted Three-body Problem (ER3BP) with variable mass and two triaxial rigid bodies. He applied the Jeans law, Nechvili's transformation and space-time transformation given by Meshcherskii in a special case.

Plastino *et al.* [9] presented techniques for the problems of Celestial Mechanics, involving bodies with varying masses. They have emphasized that Newton's second law is valid only for the body of fixed masses and the motion of a body losing mass is isotropically unaffected by this law. Bekov [10] [11] has discussed the equilibrium points and Hill's surface in the restricted problem of three bodies with variable mass. He has also discussed the existence and stability of equilibrium points in the same problem. Singh *et al.* [12] has discussed the non-linear stability of equilibrium points in the restricted problem of three bodies with variable mass. They have also found that in non-linear sense, collinear points are unstable for all mass ratios and the triangular points are stable in the range of linear stability except for three mass ratios which depend upon β, the constant due to the variation in mass governed by Jean's law.

At present, we have proposed to extend the work of Singh [12] by considering smaller primary as an oblate spheroid in the restricted problem of three bodies as shown in **Figure 1** and to find the co-ordinates of equilibrium points $L_i \ (i = 1,2,3,4,5)$ by the method of small parameters.

2. Equations of Motion

Let m be the mass of the infinitesimal body varying with time. The primaries of masses μ and $1-\mu$ are moving on the circular orbits about their centre of mass as shown in **Figure 1**. We consider a bary-centric rotating co-ordinate system (O, xyz), rotating relative to inertial frame with angular velocity ω. The line joining the centers of μ and $1-\mu$ is considered as the x-axis and a line lying on the plane of motion and perpendicular to the x-axis and through the centre of mass as the y-axis and a line through the centre of mass and perpendicular to the plane of motion as the z-axis. Let $(\mu, 0, 0)$ and $(\mu-1, 0, 0)$ respectively be the co-ordinates of the primaries P_1 and P_2 and (x, y, z) be the co-ordinates of the infinitesimal mass P. The equation of motion of the infinitesimal body of variable mass m can be written as

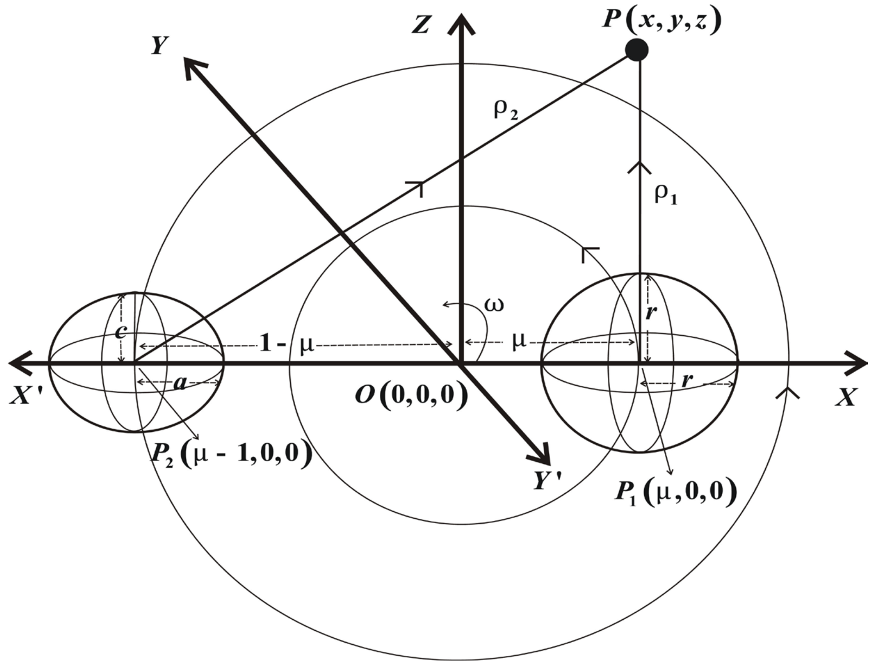

Figure 1. Rotating frame of reference in the R3BP in 3-Dimension about Z-axis.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)=-Gm\left[\frac{1-\mu}{\rho_1^3}\boldsymbol{\rho}_1+\frac{\mu}{\rho_2^3}\boldsymbol{\rho}_2+\frac{9A\mu}{2\rho_2^5}\boldsymbol{\rho}_2\right], \tag{1}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\boldsymbol{\omega}\times$ is the defined operator under consideration,

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}=\frac{\partial\boldsymbol{r}}{\partial t}+\boldsymbol{\omega}\times\boldsymbol{r}\,. \tag{2}$$

The oblateness parameter of the smaller primary is given by

$$A=\frac{a^2-c^2}{5R^2},$$

where a and c are the equatorial and polar radii of the oblate primary, R is the dimensional distance between the primaries,

$$\rho_1^2=(x-\mu)^2+y^2+z^2,$$

and $\rho_2^2=(x-\mu+1)^2+y^2+z^2$.

Now from Equation (1),

$$\left(\frac{\partial}{\partial t}+\boldsymbol{\omega}\times\right)\left(m\frac{\partial\boldsymbol{r}}{\partial t}+m\boldsymbol{\omega}\times\boldsymbol{r}\right)=-Gm\left[\frac{1-\mu}{\rho_1^3}\boldsymbol{\rho}_1+\frac{\mu}{\rho_2^3}\boldsymbol{\rho}_2+\frac{9A\mu}{2\rho_2^5}\boldsymbol{\rho}_2\right],$$

$$\left.\begin{aligned}&\Rightarrow m\ddot{\boldsymbol{r}}+\dot{m}(\dot{\boldsymbol{r}}+\boldsymbol{\omega}\times\boldsymbol{r})+2m\boldsymbol{\omega}\times\boldsymbol{r}=\omega^2m(x\hat{i}+y\hat{j})-Gm\left[\frac{1-\mu}{\rho_1^3}\boldsymbol{\rho}_1+\frac{\mu}{\rho_2^3}\boldsymbol{\rho}_2+\frac{9A\mu}{2\rho_2^5}\boldsymbol{\rho}_2\right]\\&=m\left[\omega^2x\hat{i}+\omega^2y\hat{j}-\frac{(1-\mu)(x-\mu)\hat{i}+y\hat{j}+z\hat{k}}{\rho_1^3}-\frac{\mu}{\rho_2^3}\left\{(x-\mu+1)\hat{i}+y\hat{j}+z\hat{k}\right\}\right.\\&\left.-9\frac{A\mu}{2\rho_2^5}\left\{(x-\mu+1)\hat{i}+y\hat{j}+z\hat{k}\right\}\right],\end{aligned}\right\} \tag{3}$$

where units are so chosen that the sum of the masses of the primaries and the gravitational constant G both are unity.

The equations of motion in the Cartesian form are

$$\left.\begin{aligned} \ddot{x}+\frac{\dot{m}}{m}\left(\dot{x}-\omega y\right)+2\omega\dot{x} &= -\frac{1}{m}\frac{\partial U}{\partial x}, \\ \ddot{y}+\frac{\dot{m}}{m}\left(\dot{y}+\omega x\right)-2\omega\dot{y} &= -\frac{1}{m}\frac{\partial U}{\partial y}, \\ \ddot{z}+\frac{\dot{m}}{m}\dot{z} &= -\frac{1}{m}\frac{\partial U}{\partial z}, \end{aligned}\right\} \tag{4}$$

where

$$U=-m\left[\frac{\omega^2}{2}\left(x^2+y^2\right)+\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}+\frac{3A\mu}{2\rho_2^3}\right]. \tag{5}$$

i.e.,

$$\left.\begin{aligned} &-\frac{1}{m}\frac{\partial U}{\partial x}=\omega^2 x-\frac{(1-\mu)(x-\mu)}{\rho_1^3}-\frac{\mu(x-\mu+1)}{\rho_2^3}-\frac{9A\mu(x-\mu+1)}{2\rho_2^5}, \\ &-\frac{1}{m}\frac{\partial U}{\partial y}=\omega^2 y-\frac{(1-\mu)y}{\rho_1^3}-\frac{\mu y}{\rho_2^3}-\frac{9A\mu y}{2\rho_2^5}, \\ &\text{and} \\ &-\frac{1}{m}\frac{\partial U}{\partial z}=-\frac{(1-\mu)z}{\rho_1^3}-\frac{\mu^2}{\rho_2^3}-\frac{9A\mu z}{2\rho_2^5}. \end{aligned}\right\} \tag{6}$$

By Jeans law, the variation of mass of the infinitesimal body is given by

$$\frac{\mathrm{d}m}{\mathrm{d}t}=-\alpha m^n \quad \text{i.e.,} \quad \frac{\dot{m}}{m}=-\alpha m^{n-1}, \tag{7}$$

where α is a constant coefficient and the value of exponent $n\in[0.4,\ 4.4]$ for the stars of the main sequence.

Let us introduce space time transformations as

$$\left.\begin{aligned} &x=\xi\gamma^{-q},\quad y=\eta\gamma^{-q},\quad z=\zeta\gamma^{-q},\quad \mathrm{d}t=\gamma^{-k}\mathrm{d}\tau, \\ &\rho_1=r_1\gamma^{-q},\quad \rho_2=r_2\gamma^{-q},\quad \gamma=\frac{m}{m_0}<1, \end{aligned}\right\} \tag{8}$$

where m_0 is the mass of the satellite at $t=0$.

From Equations ((7) and (8)), we get

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}=-\beta\gamma^{n-1}, \tag{9}$$

where

$$\beta=\alpha m_0^{n-1}=\text{constant}.$$

Differentiating x, y and z with respect to t twice, we get

$$\begin{aligned} \dot{x}&=\xi'\gamma^{k-q}+\beta q\xi\gamma^{n-q-1}, \\ \dot{y}&=\eta'\gamma^{k-q}+\beta q\eta\gamma^{n-q-1}, \\ \dot{z}&=\zeta'\gamma^{k-q}+\beta q\zeta\gamma^{n-q-1}, \end{aligned}$$

$$\ddot{x}=\xi''\gamma^{2k-q}+\beta\xi'(2q-k)\gamma^{n+k-q-1}-\beta^2 q\xi(n-q-1)\gamma^{2n-q-2},$$
$$\ddot{y}=\eta''\gamma^{2k-q}+\beta\eta'(2q-k)\gamma^{n+k-q-1}-\beta^2 q\eta(n-q-1)\gamma^{2n-q-2},$$
$$\ddot{z}=\zeta''\gamma^{2k-q}+\beta\zeta'(2q-k)\gamma^{n+k-q-1}-\beta^2 q\zeta(n-q-1)\gamma^{2n-q-2}.$$

Also,

$$\frac{\dot{m}}{m}=-\alpha m^{n-1}=-\alpha(m_0\gamma)^{n-1}=-\alpha m_0^{n-1}\gamma^{n-1},$$
$$\text{i.e.,}\quad \frac{\dot{m}}{m}=-\beta\gamma^{n-1}.$$

Now,

$$-\frac{1}{m}\frac{\partial U}{\partial x}=-\frac{1}{m_0\gamma}\frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x}=-\frac{\gamma^q}{m_0\gamma}\frac{\partial U}{\partial \xi}=-\frac{\gamma^{q-1}}{m_0}\left(\frac{\partial U}{\partial \xi}\right),$$
$$-\frac{1}{m}\frac{\partial U}{\partial y}=-\frac{\gamma^{q-1}}{m_0}\frac{\partial U}{\partial \eta},$$
$$-\frac{1}{m}\frac{\partial U}{\partial z}=-\frac{\gamma^{q-1}}{m_0}\frac{\partial U}{\partial \varsigma}.$$

Putting the values of $\dot{x},\ \dot{y},\ \dot{z},\ \ddot{x},\ \ddot{y},\ \ddot{z},\ \frac{\partial U}{\partial x},\ \frac{\partial U}{\partial y},\ \frac{\partial U}{\partial z}$ and $\frac{\dot{m}}{m}$ in Equation (4), we get

$$\left.\begin{aligned}
&\Big[\xi''+(2q-k-1)\beta\xi'\gamma^{n-k-1}-2\omega\eta'\gamma^{-k}-(n-q)\beta^2 q\xi\gamma^{2(n-k-1)}\\
&\qquad -(2q-1)\beta\omega\eta\gamma^{n-q-1}\Big]=-\frac{\gamma^{2q-2k-1}}{m_0}\frac{\partial U}{\partial \xi},\\
&\Big[\eta''+(2q-k-1)\beta\eta'\gamma^{n-k-1}+2\omega\xi'\gamma^{-k}-(n-q)\beta^2 q\eta\gamma^{2(n-k-1)}\\
&\qquad +(2q-1)\beta\omega\xi\gamma^{n-q-1}\Big]=-\frac{\gamma^{2q-2k-1}}{m_0}\frac{\partial U}{\partial \eta},\\
&\Big[\zeta''+(2q-k-1)\beta\zeta'\gamma^{n-k-1}-(n-q)\beta^2 q\zeta\gamma^{2(n-k-1)}\Big]=-\frac{\gamma^{2q-2k-1}}{m_0}\frac{\partial U}{\partial \zeta},
\end{aligned}\right\}\tag{10}$$

where

$$U=-m_0\left[\frac{1}{2}(\xi^2+\eta^2)\omega^2\gamma^{1-2q}+\frac{1-\mu}{r_1}\gamma^{q+1}+\frac{\mu}{r_2}\gamma^{q+1}+\frac{3A\mu}{2r_2^3}\gamma^{3q+1}\right].\tag{11}$$

In order to make the Equation (10) free from the non-variational factor, it is sufficient to put

$$\left.\begin{aligned}
&n-k-1=0,\quad 2q-k-1=0,\quad n=1\in[0.4,\ 4.4],\\
&\text{i.e., } k=0, q=\frac{1}{2}, n=1 \text{ and } \beta=\alpha
\end{aligned}\right\}\tag{12}$$

Thus the System (10) reduces to

$$\left.\begin{aligned}
\text{i.e. }\xi''-2\omega\eta'&=-\frac{1}{m_0}\frac{\partial U}{\partial \xi}+\frac{\alpha^2}{4}\xi,\\
\eta''-2\omega\xi'&=-\frac{1}{m_0}\frac{\partial U}{\partial \eta}+\frac{\alpha^2}{4}\eta,\\
\zeta''&=-\frac{1}{m_0}\frac{\partial U}{\partial \zeta}+\frac{\alpha^2}{4}\zeta,
\end{aligned}\right\}\tag{13}$$

where

$$U=-m_0\left[\frac{1}{2}\left(\xi^2+\eta^2\right)\omega^2+\frac{(1-\mu)}{r_1}\gamma^{\frac{3}{2}}+\frac{\mu}{r_2}\gamma^{\frac{3}{2}}+\frac{3A\mu}{2r_2^3}\gamma^{\frac{5}{2}}\right]. \tag{14}$$

From System (13),

$$\left.\begin{aligned}
-\frac{1}{m_0}\frac{\partial U}{\partial \xi}&=\omega^2\xi-\frac{(1-\mu)\left(\xi-\mu\sqrt{\gamma}\right)}{r_1^3}\gamma^{\frac{3}{2}}\\
&\quad-\frac{\mu\left(\xi-\mu\sqrt{\gamma}+\sqrt{\gamma}\right)}{r_2^3}\gamma^{\frac{3}{2}}-\frac{9A\mu\left(\xi-\mu\sqrt{\gamma}+\sqrt{\gamma}\right)}{2r_2^5}\gamma^{\frac{5}{2}},\\
-\frac{1}{m_0}\frac{\partial U}{\partial \eta}&=\omega^2\eta-\frac{(1-\mu)\eta}{r_1^3}\gamma^{\frac{3}{2}}-\frac{\mu\eta}{r_2^3}\gamma^{\frac{3}{2}}-\frac{9A\mu\eta}{2r_2^5}\gamma^{\frac{5}{2}},\\
-\frac{1}{m_0}\frac{\partial U}{\partial \zeta}&=-\frac{(1-\mu)\zeta}{r_1^3}\gamma^{\frac{3}{2}}-\frac{\mu\zeta}{r_2^3}\gamma^{\frac{3}{2}}-\frac{9A\mu\zeta}{2r_2^5}\gamma^{\frac{5}{2}}.
\end{aligned}\right\} \tag{15}$$

The Jacobi's Integral is

$$\begin{aligned}
\xi'^2+\eta'^2+\zeta'^2&=2\left[\frac{\omega^2}{2}\left(\xi^2+\eta^2\right)+\frac{1-\mu}{r_1}\gamma^{\frac{3}{2}}+\frac{\mu}{r_2}\gamma^{\frac{3}{2}}+\frac{3A\mu}{2r_2^3}\gamma^{\frac{5}{2}}\right]\\
&\quad+\frac{\alpha^2}{4}\left(\xi^2+\eta^2+\zeta^2\right)+C.
\end{aligned} \tag{16}$$

3. Existence of Equilibrium Points

For the existence of equilibrium points $\xi'=\eta'=\zeta'=\xi''=\eta''=\zeta''=0,$ then from Systems (13) and (15)

$$\begin{aligned}
&\left(\frac{\alpha^2}{4}+\omega^2\right)\xi-\frac{(1-\mu)\left(\xi-\mu\sqrt{\gamma}\right)}{r_1^3}\gamma^{\frac{3}{2}}\\
&-\frac{\mu\left(\xi-\mu\sqrt{\gamma}+\sqrt{\gamma}\right)}{r_2^3}\gamma^{\frac{3}{2}}-\frac{9A\mu\left(\xi-\mu\sqrt{\gamma}+\sqrt{\gamma}\right)}{2r_2^5}\gamma^{\frac{5}{2}}=0,\\
&\left(\frac{\alpha^2}{4}+\omega^2\right)\eta-\frac{(1-\mu)\eta}{r_1^3}\gamma^{\frac{3}{2}}-\frac{\mu\eta}{r_2^3}\gamma^{\frac{3}{2}}-\frac{9A\mu}{2r_2^5}\eta\gamma^{\frac{5}{2}}=0,\\
\text{and}\quad&\left(\frac{\alpha^2}{4}\right)\zeta-\frac{(1-\mu)\zeta}{r_1^3}\gamma^{\frac{3}{2}}-\frac{\mu\zeta}{r_2^3}\gamma^{\frac{3}{2}}-\frac{9A\mu}{2r_2^5}\zeta\gamma^{\frac{5}{2}}=0.
\end{aligned}$$

For solving the above equations, let us change these equations in Cartesian form as

$$\left.\begin{aligned}
&\left(\frac{\alpha^2}{4}+\omega^2\right)x-\frac{(1-\mu)(x-\mu)}{\rho_1^3}-\frac{\mu(x-\mu+1)}{\rho_2^3}-\frac{9A\mu(x-\mu+1)}{2\rho_2^5}=0,\\
&\left(\frac{\alpha^2}{4}+\omega^2\right)y-\frac{(1-\mu)y}{\rho_1^3}-\frac{\mu y}{\rho_2^3}-\frac{9A\mu y}{2\rho_2^5}=0,\\
&\text{and}\\
&\left(\frac{\alpha^2}{4}\right)z-\frac{(1-\mu)z}{\rho_1^3}-\frac{\mu z}{\rho_2^3}-\frac{9A\mu z}{2\rho_2^5}=0.
\end{aligned}\right\} \tag{17}$$

4. Existence of Collinear Equilibrium Points

For the Collinear equilibrium points, $y = z = 0$, then $\rho_1 = |x-\mu|$, $\rho_2 = |x-\mu+1|$.

From Equation (17), we get

$$\left(\frac{\alpha^2}{4}+\omega^2\right)x - \frac{(1-\mu)(x-\mu)}{|x-\mu|^3} - \frac{\mu(x-\mu+1)}{|x-\mu+1|^3} - \frac{9A\mu(x-\mu+1)}{2|x-\mu+1|^5} = 0. \tag{18}$$

Let $L_1(\xi_1,0,0)$ be the first collinear equilibrium point lying to the left of the second primary $M_2(\mu-1,0,0)$ *i.e.*, $\xi_1 < \mu-1$ as shown in Figure 2 then $\xi_1 < \mu$,

$$\Rightarrow \xi_1 - \mu + 1 < 0 \text{ and } \xi_1 - \mu < 0,$$
$$\Rightarrow |\xi_1 - \mu + 1| = -(\xi_1 - \mu + 1) \text{ and } |\xi_1 - \mu| = -(\xi_1 - \mu).$$

Thus from Equation (18),

$$\left(\frac{\alpha^2}{4}+\omega^2\right)\xi_1 + \frac{(1-\mu)}{(\xi_1-\mu)^2} + \frac{\mu}{(\xi_1-\mu+1)^2} + \frac{9A\mu}{2(\xi_1-\mu+1)^4} = 0. \tag{19}$$

as $\xi_1 < \mu-1$, so let $\xi_1 = \mu-1-\rho$ where ρ is a small quantity.

For the first equilibrium point $L_1(\xi_1,0,0)$, we have

$$\Rightarrow \left(\frac{\alpha^2}{4}+\omega^2\right)(\mu-1-\rho) + \frac{(1-\mu)}{(-1-\rho)^2} + \frac{\mu}{\rho^2} + \frac{9A\mu}{2\rho^4} = 0,$$
$$\Rightarrow \left(\frac{\alpha^2}{4}+\omega^2\right)(\mu-1-\rho) + \frac{(1-\mu)}{(1+\rho)^2} + \frac{\mu}{\rho^2} + \frac{9A\mu}{2\rho^4} = 0,$$

$$\begin{aligned} &2\left(\frac{\alpha^2}{4}+\omega^2\right)(\mu-1-\rho)(1+\rho)^2\rho^4 + 2(1-\mu)\rho^4 \\ &+2\mu\rho^2(1+\rho)^2 + 9A\mu(1+\rho)^2 = 0. \end{aligned} \tag{20}$$

Here, Equation (20) is seven degree polynomial equation in ρ, so there are seven values of ρ. If we put $\mu = 0$ then from Equation (20), we get

$$-2\left(\frac{\alpha^2}{4}+\omega^2\right)(1+\rho)^3\rho^4 + 2\rho^4 = 0. \tag{21}$$

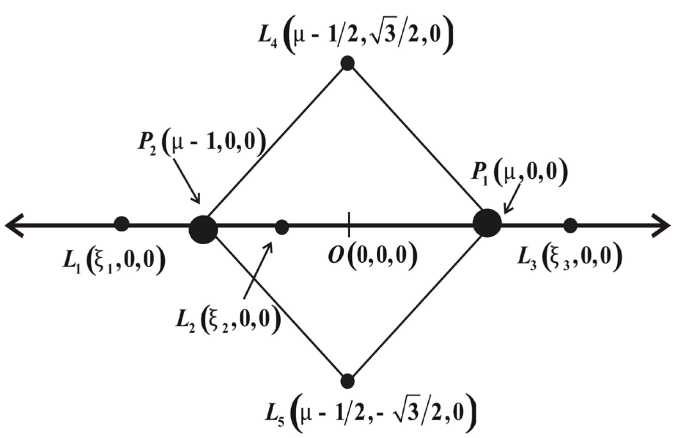

Figure 2. Locations of collinear and triangular equilibrium points.

Thus $\rho = 0,0,0,0$, are four roots of Equation (21) when $\mu = 0$, so $\rho^4 = o(\mu)$ i.e., $\rho = o\left(\mu^{\frac{1}{4}}\right) = o(\upsilon)$ *i.e.*, $\upsilon = \mu^{\frac{1}{4}} \Rightarrow \mu = \upsilon^4$.

Thus the Equation (20) reduces to

$$\begin{aligned} &2\left(\frac{\alpha^2}{4}+\omega^2\right)\left(\upsilon^4-1-\rho\right)(1+\rho)^2\rho^4+2\left(1-\upsilon^4\right)\rho^4 \\ &+2\upsilon^4\rho^2(1+\rho)^2+9A\upsilon^4(1+\rho)^2=0. \end{aligned} \tag{22}$$

Let $\rho = a_1\upsilon + a_2\upsilon^2 + a_3\upsilon^3 + a_4\upsilon^4 + a_5\upsilon^5 + a_6\upsilon^6 + a_7\upsilon^7 + \cdots$ where $a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7\cdots$ are small parameters, then

$$\left.\begin{aligned} \rho^2 &= a_1^2\upsilon^2 + 2a_1a_2\upsilon^3 + \left(a_2^2+2a_1a_3\right)\upsilon^4 + 2\left(a_1a_4+a_2a_3\right)\upsilon^5 \\ &\quad + \left(a_3^2+2a_1a_5+2a_2a_4\right)\upsilon^6 + 2\left(a_2a_5+a_1a_6+a_3a_4\right)\upsilon^7+\cdots, \\ \rho^3 &= a_1^3\upsilon^3 + 3a_1^2a_2\upsilon^4 + 3\left(a_1a_2^2+a_1^2a_3\right)\upsilon^5 + \left(a_2^3+3a_1^2a_4+6a_1a_2a_3\right)\upsilon^6 \\ &\quad + 3\left(a_1a_3^2+a_2^2a_3+a_1^2a_5+2a_1a_2a_4\right)\upsilon^7+\cdots, \\ \rho^4 &= a_1^4\upsilon^4 + 4a_1^3a_2\upsilon^5 + 2\left(3a_1^2a_2^2+2a_1^3a_3\right)\upsilon^6 \\ &\quad + 4\left(a_1a_2^3+a_1^3a_4+3a_1^2a_2a_3\right)\upsilon^7+\cdots, \\ \rho^5 &= a_1^5\upsilon^5 + 5a_1^4a_2\upsilon^6 + 5\left(2a_1^3a_2^2+a_1^4a_3\right)\upsilon^7+\cdots, \\ \rho^6 &= a_1^6\upsilon^6 + 6a_1^5a_2\upsilon^7+\cdots, \\ \rho^7 &= a_1^7\upsilon^7+\cdots. \end{aligned}\right\} \tag{23}$$

Putting the value of $\rho,\ \rho^2,\ \rho^3,\ \rho^4,\ \rho^5,\ \rho^6,\ \rho^7,\cdots$ in Equation (22) and equating the co-efficient of different powers of υ to zero, we get the values of the parameters as

$$\left.\begin{aligned} a_1 &= \left[\frac{18A}{\alpha^2-6A}\right]^{\frac{1}{4}}, \\ a_2 &= Q\left[18Aa_1 - 3\left(\alpha^2+6A+4\right)a_1^5\right], \\ a_3 &= Q\left[(9A+2)a_1^2 + \left(\alpha^2+4\omega^2\right)\left(\frac{3}{2}a_1^6 - 6a_1^2a_2^2 - 15a_1^4a_2\right) + 12a_1^2a_2^2\right], \\ a_4 &= Q\Big[4a_1^3 + 2(9A+2)a_1a_2 + 18Aa_2 \\ &\quad -\left(\alpha^2+6A+4\right)\left(\frac{1}{2}a_1^7 - 9a_1^5a_2 + 15a_1^4a_3 + 30a_1^3a_2^2 + 6a_1^2a_2a_3 - 2a_1a_2^3\right)\Big], \\ &a_5, a_6, a_7, \cdots, \end{aligned}\right\} \tag{24}$$

where $Q = -\dfrac{1}{2\left(\alpha^2+6A\right)a_1^3}$.

Therefore, the co-ordinate of the first equilibrium point $L_1(\xi_1,0,0)$ is given by

$$\begin{aligned} \xi_1 &= \mu - 1 - \rho \\ &= \mu - 1 - \left(a_1\upsilon + a_2\upsilon^2 + a_3\upsilon^3 + a_4\upsilon^4 + a_5\upsilon^5 + a_6\upsilon^6 + \cdots\right) \\ &= \mu - 1 - \left\{a_1\mu^{\frac{1}{4}} + a_2\mu^{\frac{2}{4}} + a_3\mu^{\frac{3}{4}} + a_4\mu + a_5\mu^{\frac{5}{4}} + a_6\mu^{\frac{6}{4}} + \cdots\right\} \end{aligned}$$

$$\xi_1 = \mu - 1 - \sum_{n=1}^{\infty} a_n\mu^{\frac{n}{4}}.$$

Let $L_2(\xi_2,0,0)$ be the second collinear equilibrium point between the two primaries P_1 and P_2 then $\mu-1<\xi_2<\mu$

$$\Rightarrow \xi_2-\mu+1>0 \text{ and } \xi_2-\mu<0.$$
$$\Rightarrow |\xi_2-\mu+1|=\xi_2-\mu+1 \text{ and } |\xi_2-\mu|=-(\xi_2-\mu).$$

Thus from Equation (18),

$$\left(\frac{\alpha^2}{4}+\omega^2\right)\xi_2+\frac{1-\mu}{(\xi_2-\mu)^2}-\frac{\mu}{(\xi_2-\mu+1)^2}-\frac{9A\mu}{2(\xi_2-\mu+1)^4}=0. \tag{25}$$

Since $\xi_2>\mu-1$ hence let $\xi_2=\mu-1+\rho$, thus

$$\xi_2-\mu+1=\rho \quad \Rightarrow \xi_2-\mu=-1+\rho,$$

where ρ is a small quantity.

In terms of ρ, the Equation (25) can be written as

$$\left(\frac{\alpha^2}{4}+\omega^2\right)(\mu-1+\rho)+\frac{1-\mu}{(\rho-1)^2}-\frac{\mu}{\rho^2}-\frac{9A\mu}{\rho^4}=0,$$

i.e.,

$$\begin{aligned}&2\left(\frac{\alpha^2}{4}+\omega^2\right)(\mu-1+\rho)\rho^4(\rho-1)^2+2(1-\mu)\rho^4\\&-2\mu\rho^2(\rho-1)^2-9A\mu(\rho-1)^2=0.\end{aligned} \tag{26}$$

The Equation (26) is a seven degree polynomial equation in ρ, so there are seven values of ρ in Equation (26).

If we put $\mu=0$ in Equation (26), we get

$$2\left(\frac{\alpha^2}{4}+\omega^2\right)\rho^4(\rho-1)^3+2\rho^4=0. \tag{27}$$

Here $\rho^4=0$ i.e., $\rho=0,0,0,0$ are the four roots of Equation (27) when $\mu=0$, so ρ we can choose as some order of μ *i.e.*,

$$\rho^4=o(\mu) \Rightarrow \rho=o\left(\mu^{\frac{1}{4}}\right)=o(\upsilon),$$

where $\mu^{\frac{1}{4}}=\upsilon$ i.e., $\mu=\upsilon^4$.

Let $\rho=b_1\upsilon+b_2\upsilon^2+b_3\upsilon^3+b_4\upsilon^4+b_5\upsilon^5+b_6\upsilon^6+b_7\upsilon^7+\cdots$. where $b_1, b_2, b_3, b_4, b_5, b_6, b_7\cdots$ are small parameters. Putting the values of $\rho^1,\rho^2,\rho^3,\rho^4$ and $\mu=\upsilon^4$ in Equation (26) and equating the coefficients of different powers of υ, we get

$$\left.\begin{aligned}&b_1=\left(-\frac{9A}{2+\alpha^2+6A}\right)^{\frac{1}{4}},\\&b_2=R\left[3\left(\alpha^2+4+6A\right)b_1^4+18A\right],\\&b_3=R\left[\left(\alpha^2+4+6A\right)\left(-3b_1^6+15b_4^4b_2-6b_1^2b_2^2\right)+12b_1^2b_2^2-(9A+2)b_1^2+18Ab_2\right],\\&b_4=R\begin{bmatrix}\left(\alpha^2+4+6A\right)\left(b_1^7-18b_1^5b_2+15b_1^4b_3+30b_1^2b_2^2-4b_1b_2^3-12b_1^2b_2b_3\right)\\+8\left(b_1b_2^3+3b_1^2b_2b_3+4b_1^3\right)-2(9A+2)b_1b_2+18Ab_3\end{bmatrix},\end{aligned}\right\} \tag{28}$$

where $R = \dfrac{1}{4\left(\alpha^2 + 2 + 6A\right)b_1^3}$.

Thus the co-ordinate of the second equilibrium point is given by

$$\begin{aligned}\xi_2 &= \mu - 1 + b_1\upsilon + b_2\upsilon^2 + b_3\upsilon^3 + b_4\upsilon^4 + b_5\upsilon^5 + b_6\upsilon^6 + b_7\upsilon^7 + \cdots \\ &= \mu - 1 + b_1\mu^{\frac{1}{4}} + b_2\mu^{\frac{2}{4}} + b_3\mu^{\frac{3}{4}} + b_4\mu^1 + b_5\mu^{\frac{5}{4}} + b_6\mu^{\frac{6}{4}} + \cdots \\ \xi_2 &= \mu - 1 + \sum_{n=1}^{\infty} b_n\mu^{\frac{n}{4}}.\end{aligned}$$

Let $L_3\left(\xi_3, 0, 0\right)$ be the third equilibrium point right to the first primary, then

$$\xi_3 > \xi_2 = \mu - 1 - \rho, \quad \Rightarrow \xi_3 > \mu - 1 + \rho, \quad \Rightarrow \xi_3 = \mu - 1 + 2\rho$$

Thus from Equation (18), we have

$$\left(\frac{\alpha^2}{4} + \omega^2\right)(\mu - 1 + 2\rho) - \frac{1-\mu}{(2\rho - 1)^2} - \frac{\mu}{(2\rho)^2} - \frac{9A\mu}{2(2\rho)^4} = 0,$$

$$\left(\frac{\alpha^2}{4} + \omega^2\right)(\mu - 1 + 2\rho) - \frac{1-\mu}{(2\rho - 1)^2} - \frac{\mu}{4\rho^2} - \frac{9A\mu}{32\rho^4} = 0,$$

$$\left.\begin{aligned}&32\left(\frac{\alpha^2}{4} + \omega^2\right)(2\rho - 1 + \mu)(2\rho - 1)^2\rho^4 - 32(1-\mu)\rho^4 \\ &-8\mu\rho^2(2\rho - 1)^2 - 9A\mu(2\rho - 1)^2 = 0.\end{aligned}\right\} \tag{29}$$

When $\mu = 0$, then Equation (29) reduced to

$$32\left(\frac{\alpha^2}{4} + \omega^2\right)(2\rho - 1)^3\rho^4 - 32\rho^4 = 0. \tag{30}$$

$\rho^4 = 0$ i.e., $\rho = 0, 0, 0, 0$ are the four roots of the Equation (29) when $\mu = 0$, so $\rho^4 = O(\mu) \Rightarrow \rho = O\left(\mu^{\frac{1}{4}}\right) = O(\upsilon)$ say when $\mu = \upsilon^4$.

Let $\rho = c_1\upsilon + c_2\upsilon^2 + c_3\upsilon^3 + c_4\upsilon^4 + \cdots$

where $c_1, c_2, c_3, c_4, \cdots$ are small parameters.

Thus Equation (29) reduced to

$$\begin{aligned}&8\left(\alpha^2 + 4\omega^2\right)\left(2\rho - 1 + \upsilon^4\right)(2\rho - 1)^2\rho^4 - 32\left(1 - \upsilon^4\right)\rho^4 \\ &-8\upsilon^4\rho^2(2\rho - 1)^2 - 9A\upsilon^4(2\rho - 1)^2 = 0\end{aligned}$$

By putting values of $\rho, \rho^2, \rho^3, \rho^4, \rho^5, \rho^6, \rho^7$ in Equation (29) and equating the co-efficient of different powers of υ, we get

$$\left.\begin{aligned}&c_1 = \left[-\frac{9A}{8\left(\alpha^2 + 4\omega^2 + 4\right)}\right]^{\frac{1}{4}}, \\ &c_2 = S\left[48\left(\alpha^2 + 4\omega^2\right)c_1^5\right], \\ &c_3 = S\left[16\left(\alpha^2 + 4\omega^2\right)c_1^2\left(15c_1^2c_2 - 6c_1^4 - 3c_2^2\right) - 192c_1^2c_2^2 - 3Ac_2\right], \\ &c_4 = S\left[\begin{aligned}&32c_1^3 - 3Ac_3 - 80Ac_1c_2 - 384\left(c_1c_2^3 + 3c_1^2c_2c_3\right) \\ &-16\left(\alpha^2 + 4\omega^2\right)\left(2c_1c_2^3 + 6c_1^2c_2c_3 - 15c_1^4c_3 - 30c_1^2c_2^2 + 36c_1^5c_2 - 4c_1^7\right)\end{aligned}\right], \\ &\text{and so on}\end{aligned}\right\} \tag{31}$$

where $S=\dfrac{1}{32\left(\alpha^2+4\omega^2+4\right)c_1^3}$.

Thus the co-ordinates of the third equilibrium point is given by

$$\xi_3=\mu-1-2\rho=\mu-1+2\sum_{n=1}^{\infty}c_n\upsilon^n=\mu-1+2\sum_{n=1}^{\infty}c_n\mu^{\frac{n}{4}}$$

5. Existence of Triangular Equilibrium Points

For triangular equilibrium point $x\neq 0,\ y\neq 0$ and $z=0$ then from the System (17), we have

$$a_0x-\frac{(1-\mu)(x-\mu)}{\rho_1^3}-\frac{\mu(x-\mu+1)}{\rho_2^3}-\frac{9A\mu(x-\mu+1)}{2\rho_2^5}=0. \tag{32}$$

$$a_0-\frac{1-\mu}{\rho_1^3}-\frac{\mu}{\rho_2^3}-\frac{9A\mu}{2\rho_2^5}=0. \tag{33}$$

Now Equation $(32)-(33)\times(x-\mu+1)$ gives

$$a_0-\frac{1}{\rho_2^3}=0 \text{ i.e., } \frac{1}{\rho_2^3}=a_0. \tag{34}$$

Again Equation $(32)-(33)\times(x-\mu)$ gives

$$\frac{1}{\rho_1^3}+\frac{9A}{2\rho_2^5}-a_0=0 \quad \text{i.e., } \frac{1}{\rho_2^3}+\frac{9A}{2\rho_2^5}=a_0. \tag{35}$$

Since $0<A\ll 1$, hence for the first approximation, if we put $A=0$, then from Equations ((34) and (35)), we get

$$\frac{1}{\rho_1^3}=\frac{1}{\rho_2^3}=a_0=1+\frac{\alpha^2}{4},$$

$$\Rightarrow \rho_1=\rho_2=\left(1+\frac{\alpha^2}{4}\right)^{-\frac{1}{3}}=1-\frac{1}{12}\alpha^2<1.$$

For better approximation $A\neq 0$, then the above solutions can be written as $\rho_1=1-\dfrac{\alpha^2}{12}+\alpha_1$ and $\rho_2=1-\dfrac{\alpha^2}{12}+\alpha_2$ where $0<\alpha_1,\ \alpha_2\ll 1$.

For triangular equilibrium points $z=0$, then

$$\rho_1^2=(x-\mu)^2+y^2 \text{ and } \rho_2^2=(x-\mu+1)^2+y^2.$$

Now,

$$\rho_2^2=(x-\mu)^2+y^2+2(x-\mu)+1,$$
$$\rho_2^2-\rho_1^2=2(x-\mu)+1.$$

$$\left.\begin{aligned}&\Rightarrow (x-\mu)=\alpha_2-\alpha_1-\frac{1}{2},\\ &\Rightarrow x=\mu-\frac{1}{2}+\alpha_2-\alpha_1\end{aligned}\right\} \tag{36}$$

Again,

$$
\begin{aligned}
&\rho_1^2 = (x-\mu)^2 + y^2, \\
&y^2 = \left(1+\alpha_1 - \frac{\alpha^2}{12}\right)^2 - \left(\alpha_2 - \alpha_1 - \frac{1}{2}\right)^2, \\
&y^2 = \frac{3}{4} + \alpha_1 + \alpha_2 - \frac{\alpha^2}{6}, \\
&y = \pm\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}(\alpha_1 + \alpha_2) - \frac{2\alpha^2}{9}\right].
\end{aligned} \tag{37}
$$

Putting the value of ρ_2 and a_0 in Equation (34), we get $\alpha_2 = -\frac{A}{2}$. Putting the value of ρ_2 and a_0 in Equation (35), we get

$$\alpha_1 = 1 + \frac{15}{2}A - \frac{\alpha^2}{3}.$$

Thus,

$$\alpha_2 + \alpha_1 = 1 + 7A - \frac{\alpha^2}{3} \quad \text{and} \quad \alpha_2 - \alpha_1 = -1 - 8A + \frac{\alpha^2}{3}. \tag{38}$$

Therefore,

$$x = \mu - \frac{3}{2} - 8A + \frac{\alpha^2}{3}, \tag{39}$$

$$
\begin{aligned}
&y = \pm\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}\left(1 + 7A - \frac{\alpha^2}{3}\right) - \frac{2\alpha^2}{9}\right], \\
&y = \pm\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}(1+7A) - \frac{4\alpha^2}{9}\right].
\end{aligned} \tag{40}
$$

$$L_{4,5} = \left(\mu - \frac{3}{2} - 8A + \frac{\alpha^2}{3},\ \pm\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}(1+7A) - \frac{4\alpha^2}{9}\right]\right)$$

6. Surface of Zero–Velocity

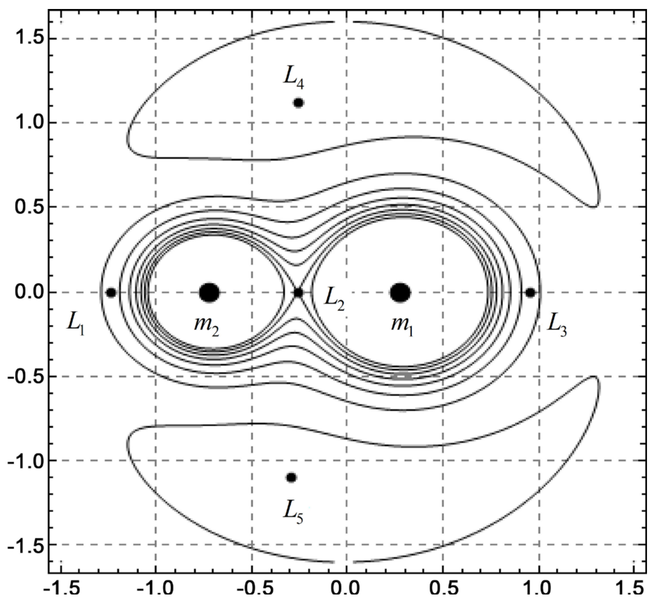

Figure 3. Zero velocity curve (ZVC) for $C = 1$ (classical case).

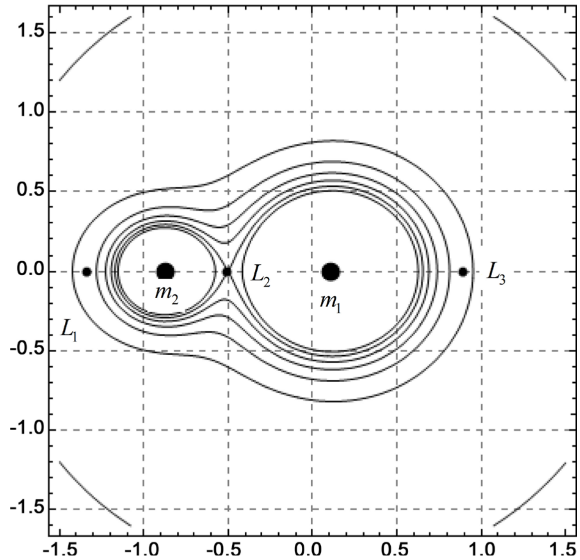

Figure 4. Zero velocity curve for $C = 2$ (classical case).

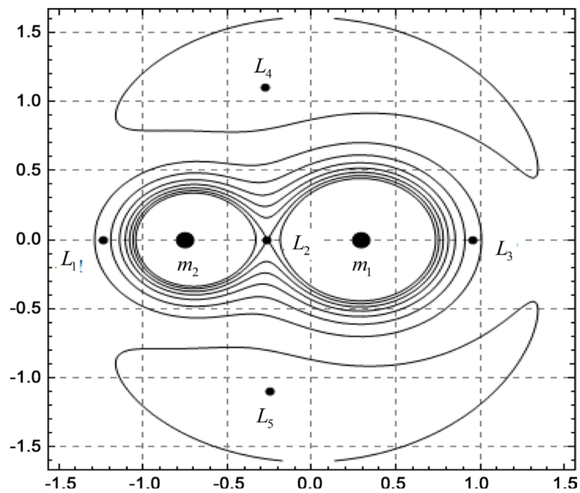

Figure 5. Zero velocity curve for $C = 3$ (classical case).

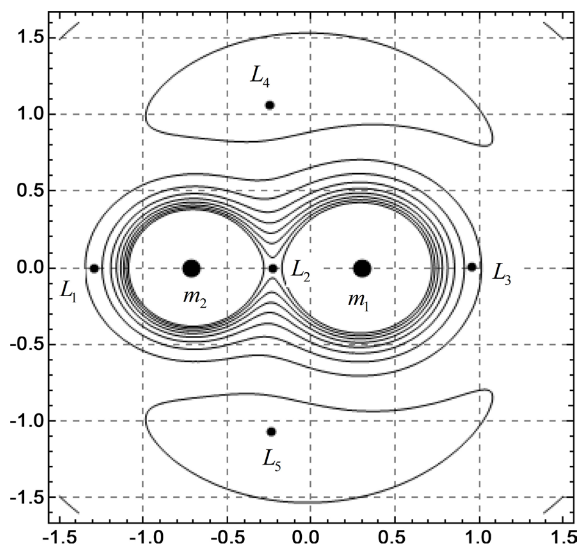

Figure 6. Zero velocity curve for $C = 1$ (perturbed case).

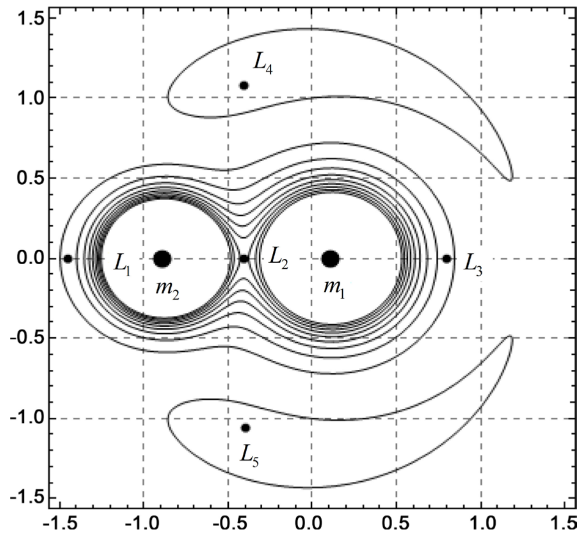

Figure 7. Zero velocity curve for $C = 2$ (perturbed case).

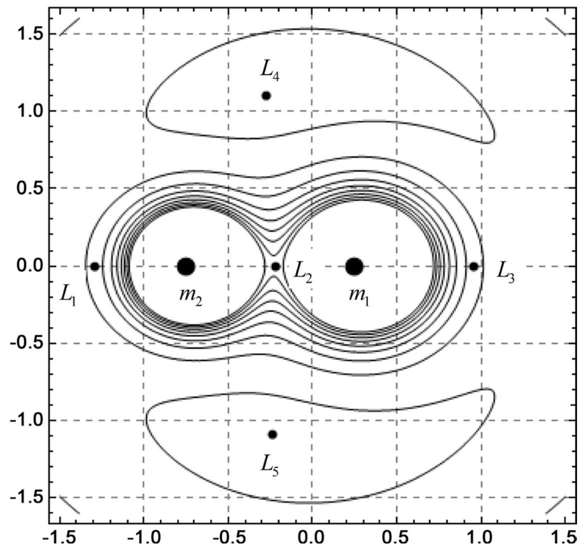

Figure 8. Zero velocity curve (ZVC) for $C = 3$ (perturbed case).

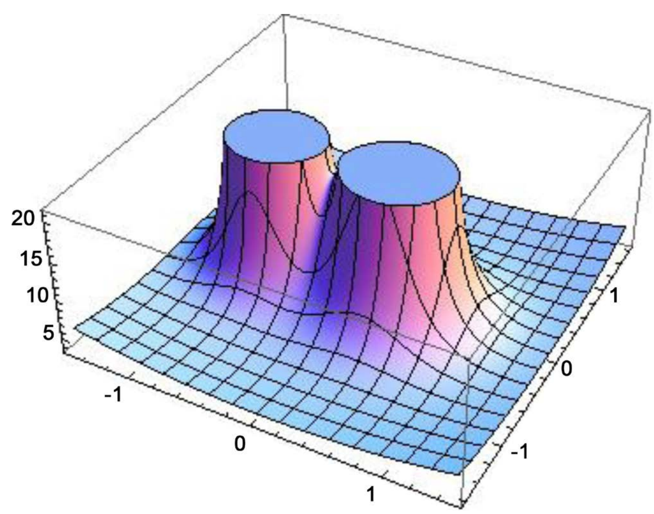

Figure 9. 3 Dimensional view of ZVC of **Figure 6**.

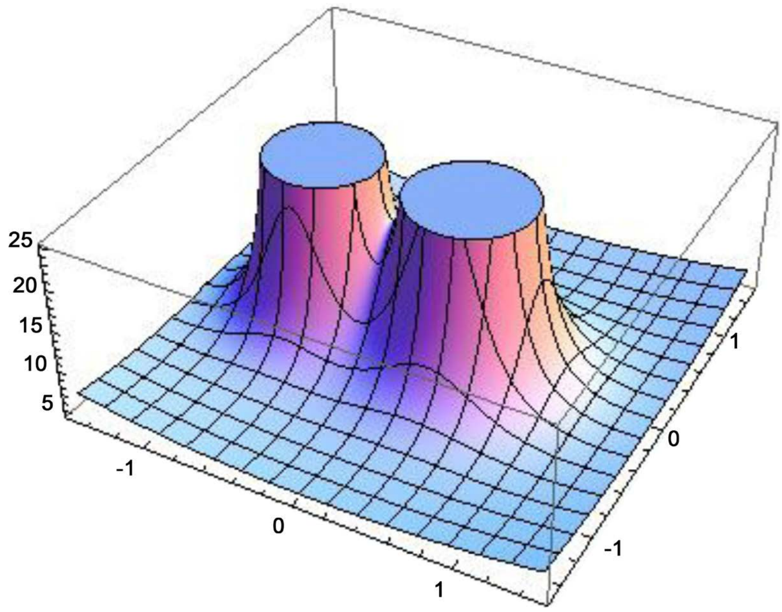

Figure 10. 3 Dimensional view of ZVC of Figure 7.

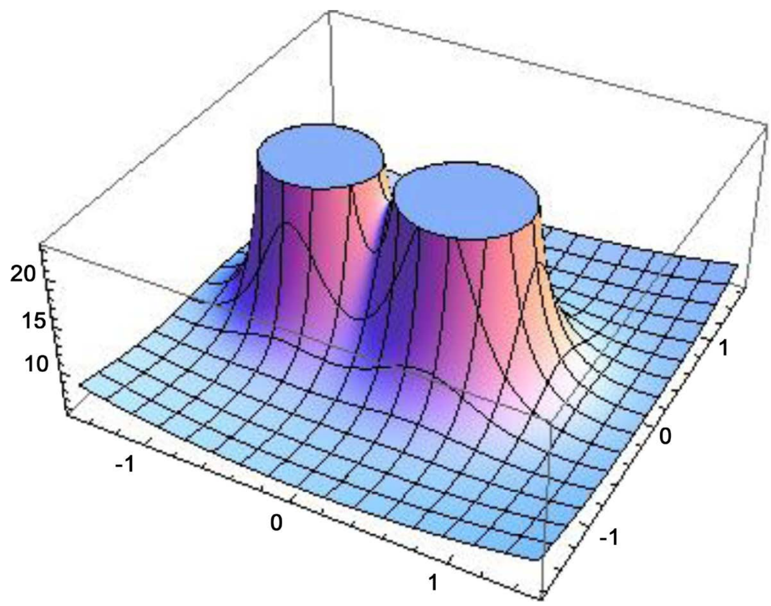

Figure 11. 3 Dimensional view of ZVC of Figure 8.

7. Discussions and Conclusions

In section 2, the equations of motion of the infinitesimal body with variable mass have been derived under the gravitational field of one oblate primary and other spherical. By Jean's law, the time rate mass variation is defined as $\frac{\mathrm{d}m}{\mathrm{d}t} = -\alpha m^n$, $n \in [0.4, 4.4]$, where α is a constant and the interval $[0.4, 4.4]$ in which exponent of the mass of the stars of the main sequence lies. The System (4) is transformed to space-time co-ordinates by the space-time transformations given in Equations (8) and (9). The Jacobi's integral has been derived in Equation (16).

In section 3, the equations for solving equilibrium points, have been derived in Equation (17) by putting $\xi' = \xi'' = 0,\ \eta' = \eta'' = 0,\ \zeta' = \zeta'' = 0$ in Equation (13). Again the equations for equilibrium points, have been transformed to original frame $(0, xyz)$ which are given in Equation (18). In section 4, for collinear equilibrium points we put $y = z = 0$, then from Equation (17), we get only one Equation (19). Applying small parameter method, we established the co-ordinates of L_1, L_2, L_3 as ξ_1, ξ_2, ξ_3 in terms of order of $\mu^{\frac{1}{4}}$. In section 5, the co-ordinates of equilateral triangular equilibrium points have been calculated by the classical method. In section 6, zero-velocity curves in **Figures 3-8** and its 3-dimensional surface in **Figures 9-11** have been drawn for $C = 1,\ C = 2$ and $C = 3$ in classical case and perturbed case.

From the above facts we concluded that in the perturbed case, first equilibrium point $L_1(\xi_1, 0, 0)$ shifted away from the second primary $P_2(\mu - 1, 0, 0)$ whereas $L_2(\xi_2, 0, 0)$ shifted towards the first primary $P_1(\mu, 0, 0)$ but $L_3(\xi_3, 0, 0)$ is not influenced by the perturbation which can be seen in **Figures 6-8**. So far, the matter is concerned with the influence of perturbation on the co-ordinates of L_4 and L_5, we can say that for $C = 1$ and $C = 3$, the triangular equilibrium configuration is maintained but for $C = 2$ in both classical and perturbed cases, the equilateral triangular configuration is not maintained. Whatever be the analytical changes in the co-ordinates of L_1, L_2, L_3, L_4 and L_5 that is due oblateness not due to the mass reduction factor α of the infinitesimal body.

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