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Short-Term Orbit Prediction with J_2 and Mean Orbital Elements

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Abstract

An analytical theory for calculating perturbations of the orbital elements of a satellite due to J_2 to accuracy up to fourth power in eccentricity is developed. It is observed that there is significant improvement in all the orbital elements with the present theory over second-order theory. The theory is used for computing the mean orbital elements, which are found to be more accurate than the existing Bhatnagar and Taqvi's theory (up to second power in eccentricity). Mean elements have a large number of practical applications.

Keywords: Perturbation Due to J_2, Mean Orbital Elements, Short-Periodic Terms, Osculating Orbital Elements, Fourth Power in Eccentricity

1. Introduction

The fact that the Earth is not a true sphere is one of the important causes for the deviation of the orbits of the artificial satellites from undisturbed Keplerian ellipses, the largest perturbations in the motion of such satellites being due to the oblateness of the Earth. The standard osculating, i.e. instantaneously defined elements of an elliptic orbit are "a" (semi-major axis), "e" (eccentricity), "i" (inclination), "Ω" (right ascension of ascending node), "ω" (argument of perigee) and "M" (mean anomaly). The osculating element space is related to mean element space through the short-periodic variations which are combination of short-periodic terms of period less than the orbital period of the orbital elements and are functions of the perturbations present. If ζ is an osculating element and ζ_{sp} is the short-periodic variation in the corresponding element, then the mean element ζ_m is related by $\zeta = \zeta_m + \zeta_{sp}$ where ζ_{sp} is the function of mean elements.

The motion of an artificial satellite in the axisymmetric field due to Earth's low-degree harmonics has been recognized as the "main problem" in the theory of satellite orbits, and many solutions have been published. Some of the important contributions are due to Kozai [1], Brouwer [2], Chebotarev [3], Deprit and Rom [4], Aksnes [5], Liu [6], Kinoshita [7], Bhatnagar and Taqvi [8], Gooding [9]. Sharma [10] utilized the theories of Bhatnagar and Taqvi and Liu to compute the mean elements using the iterative scheme of Gooding [11] and made a comparison between them.

Chebotarev [3] generated analytical expressions for short-periodic terms with J_2 specifically for near-circular orbits, by introducing the variables $h = e \cdot \sin\omega$, $l = e \cdot \cos\omega$ in the Lagrange's planetary equations and then computing the involved partial derivatives of the perturbing function R on the right hand side of the equations and then analytically integrating and retaining only the linear terms in h and l. By retaining the quadratic terms in h and l, Bhatnagar and Taqvi [8] generated the short periodic expressions: δa, δh, δl, $\delta\Omega$, δi and $\delta\lambda_0$, where $\lambda_0 = M + \omega$, to an accuracy of second-order in eccentricity.

In this Paper, an analytical theory for calculating the perturbations due to J_2 in the orbital elements of a satellite to accuracy up to fourth power in eccentricity is derived. The mean orbital elements, particularly semi-major axis and eccentricity generated using first-order theory (only linear terms in e) and second-order theory (linear and quadratic terms in e) are compared with those computed using developed fourth-order theory. It is observed that there is significant improvement in the important orbital elements: semi-major axis, eccentricity and inclination with the present theory over the second-order theory of Bhatnagar and Taqvi.

2. Perturbations Due to J_2

If the Earth is assumed to be an oblate spheroid and only J_2 term in the oblateness is considered, then the disturbing function is

$$R = \frac{JK^2 RE^2}{3r^3}\left(1-3\sin^2(\delta)\right), \text{ with } J = -\frac{3}{2}J_2,$$

where $K^2 = 398600.8\ \mathrm{km^3/s^2}$ is the Gaussian constant, r is the radial distance of the satellite, $RE = 6378.15$ km, is the equatorial radius of the Earth, δ the declination of the satellite, and $J_2 = 1.08263\times10^{-3}$ is the second harmonics of the Earth.

To expand R in powers of eccentricity of the satellite orbit, utilizing the relation

$$\sin(\delta) = \sin(i)\sin(\vartheta + \omega),$$

where ϑ is the true anomaly, using this in R, we obtain

$$R = \frac{JK^2RE^2}{6a^3}\Bigg[\left(2-3\sin^2(i)\right)\left(\frac{a}{r}\right)^3 + 3\sin^2(i)\cos(2\omega)\left(\frac{a}{r}\right)^3\cos(2\vartheta) - 3\sin^2(i)\sin(2\omega)\left(\frac{a}{r}\right)^3\sin(2\vartheta)\Bigg].$$

The quantities $\left(\frac{a}{r}\right)^3$, $\left(\frac{a}{r}\right)^3\cos(2\vartheta)$, $\left(\frac{a}{r}\right)^3\sin(2\vartheta)$ can be expanded in series in terms of multiples of the mean anomaly M.

Considering the terms up to sixth order in eccentricity with the help of "MAXIMA" software available in public domain [12], we obtain an expression for the disturbing function in the form

$$R = \sum_{j=0}^{8}\left[A_j\cos(jM) + B_j\cos(jM+2\omega) + C_j\cos(jM-2\omega)\right].$$

The coefficients A_j, B_j, C_j depend on the variables a, e, and i.

The transformation of the disturbing function R into the new variables h, l and λ can be carried out with the aid of the following relations

$$\lambda = M + \omega$$

$$e\sin(M) = e\sin(\lambda-\omega) = l\sin(\lambda) - h\cos(\lambda)$$

$$e\cos(M) = e\cos(\lambda-\omega) = l\cos(\lambda) + h\sin(\lambda)$$

$$e\sin(M+2\omega) = e\sin(\lambda-\omega) = l\sin(\lambda) + h\cos(\lambda)$$

$$e\cos(M+2\omega) = e\cos(\lambda-\omega) = l\cos(\lambda) - h\sin(\lambda)$$

The disturbing function R will then be a function of l, h and λ.

By introducing the variables $h = e\cdot\sin\omega$, $l = e\cdot\cos\omega$, the Lagrange's planetary equations of motion can be expressed as

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\frac{\partial R}{\partial \lambda_0},$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\sqrt{1-h^2-l^2}}{na^2}\frac{\partial R}{\partial l} - \frac{\sqrt{1-h^2-l^2}}{na^2} \times \frac{h}{1+\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial \lambda_0} - \frac{l\operatorname{cotan}(i)}{na^2\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial i},$$

$$\frac{\mathrm{d}l}{\mathrm{d}t} = -\frac{\sqrt{1-h^2-l^2}}{na^2}\frac{\partial R}{\partial h} - \frac{\sqrt{1-h^2-l^2}}{na^2} \times \frac{l}{1+\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial \lambda_0} + \frac{h\operatorname{cotan}(\mathrm{i})}{na^2\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial i},$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{\operatorname{cosec}(i)}{na^2\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial i},$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{\cot(i)}{na^2\sqrt{1-h^2-l^2}}\left(l\frac{\partial R}{\partial h} - h\frac{\partial R}{\partial l} + \frac{\partial R}{\partial \lambda_0}\right) - \frac{\operatorname{cosec}(i)}{na^2\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial \Omega},$$

$$\frac{\mathrm{d}\lambda_0}{\mathrm{d}t} = -\frac{2}{na}\frac{\partial R}{\partial a} + \frac{\sqrt{1-h^2-l^2}}{1+\sqrt{1-h^2-l^2}} \times \frac{1}{2na^2}\left(h\frac{\partial R}{\partial h} + l\frac{\partial R}{\partial l}\right) - \frac{\operatorname{cotan} i}{na^2\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial i},$$

where $\lambda_0 = M + \omega$.

Computing the derivatives of the disturbing function with the help of MAXIMA and introducing the results in the above equations of motion and integration leads to the expressions given in Appendix. They define the first-order perturbations of the orbital elements of the satellite with the fourth-order accuracy in the eccentricity. The perturbations of the orbital elements are now found from the relations

$$a = a_0 + \delta a,$$

$$i = i_0 + \delta i,$$

$$h = h_0 + At + \delta h,$$

$$l = l_0 + Bt + \delta l,$$

$$\Omega = \Omega_0 + Ct + \delta\Omega,$$

$$\lambda_0 = \lambda_0(t_0) + Dt + \delta\lambda_0,$$

where A, B, C, D are four constant coefficients of the secular perturbations. $\delta a, \delta i, \delta h, \delta l, \delta\Omega$, and $\delta\lambda_0$ contain only periodic terms. $a_0, i_0, h_0, l_0, \Omega_0$, and $\lambda_0(t_0)$ are six constants of integration.

Writing

$$a_m = a_0,$$
$$i_m = i_0,$$
$$h_m = h_0 + At,$$
$$l_m = l_0 + Bt,$$
$$\Omega_m = \Omega_0 + Ct,$$
$$\lambda_m = \lambda_0(t_0) + Dt.$$

For the mean elements of the orbit at time t, we have for the perturbed elements

$$a = a_m + \delta a,$$
$$i = i_m + \delta i,$$
$$h = h_m + \delta h,$$
$$l = l_m + \delta l,$$
$$\Omega = \Omega_0 + \delta\Omega,$$
$$\lambda_0 = \lambda_0(t_0) + \delta\lambda_0.$$

The elements a, e, h, l, Ω and λ_0 are called the osculating elements of the orbit at time t. We utilize the uniformly regular KS canonical equations of motion provided in [13] for generating the osculating orbital elements with Earth's zonal harmonics terms J_2 to J_6 with the detailed procedure provided in Sharma and Raj [14]. The mean orbital elements are then computed by using the iterative scheme of Gooding [11].

3. Results

To compare the developed fourth-order theory with the first-order and the second-order theories, various test cases are considered. A study in the variations of the important orbital parameters: semi-major axis, eccentricity and inclination due to the variations in semi-major axis, eccentricity and inclination have been carried out.

Table 1(a) provides the minimum and maximum values of the osculating and mean semi-major axis for different inclinations and eccentricities as well as the difference in maximum and minimum mean semi-major axis with respect to second-order and fourth-order theories during a revolution. The values of other orbital parameters a, Ω, ω and M used in the computation are 8000 km, 60˚, 60˚ and 0˚, respectively. It can be observed that the difference in mean semi-major axis with fourth-order theory is less than that with second-order theory. The maximum and minimum values of difference in mean semi-major as calculated with second-order theory is 16.48 m and 3209.73 m, respectively, while these values as calculated using the fourth-order theory are 4.71 m and 568.04 m, respectively. For both the cases, minimum value correspond to e = 0.05 and i = 5˚ and maximum value corresponds to e = 0.2 and i = 85˚.

Similar observation can be made from the **Tables 1(b)** and **(c)**, where the difference in the mean eccentricity and the mean inclination is compared for second-order and fourth-order theory for various inclinations and eccentricities. The minimum and maximum values of difference in mean eccentricity calculated from second-order and fourth-order theory is 1.0×10^{-5} and 3.60×10^{-4}, 0.98×10^{-5} and 1.69×10^{-4}, respectively. The minimum and maximum value of difference in mean inclination calculated from second-order theory and fourth-order theory is 0.23×10^{-4} and 2.08×10^{-3}, 0.19×10^{-4} and 2.84×10^{-4}, respectively.

Figure 1(a) shows the difference in maximum and minimum mean semi-major axis during a revolution with respect to different semi-major axis ranging from 8000 km to 20,000 km. The initial values for e, i, Ω, ω and M are 0.2, 30˚, 60˚, 0˚ and 0˚, respectively. The comparisons clearly show that the mean semi-major axis computed using the fourth-order theory is better than that with first-order as well as second-order theory. Similarly, **Figure 1(b)** gives the difference in maximum and minimum mean eccentricity and **Figure 1(c)** provides the difference in maximum and minimum mean inclination. Also, it can be observed that the difference in maximum and minimum mean values of semi-major axis, eccentricity and inclination decreases with the increase in the semi-major axis.

In **Figures 2(a)-(c)**, the eccentricity is varied from 0.001 to 0.3 keeping other orbital parameters constant. The initial orbital values for a, i, Ω, ω and M for this case are 10,000 km, 60˚, 60˚, 0˚ and 0˚, respectively. **Figure 2(a)** shows the difference in maximum and minimum mean semi-major axis computed using first-order, second-order and fourth-order theories. The difference in maximum and minimum mean eccentricity calculated using first-order, second-order and fourth-order theories is shown in **Figure 2(b)**. Similarly, **Figure 2(c)** shows the difference in mean inclination. From **Figures 2(a)-(c)**, it is clear that the fourth-order theory is much better than the first-order and the second-order theory. The difference in the maximum and the minimum mean values of semi-major axis, eccentricity and inclination increases with the increase in eccentricity.

Table 1. (a) Osculating and mean Semi-major Axis (sma) with second-order and fourth-order theory; (b) Osculating and mean Eccentricity with second-order and fourth-order theory; (c) Osculating and mean Inclination with second-order and fourth-order theory.

(a)

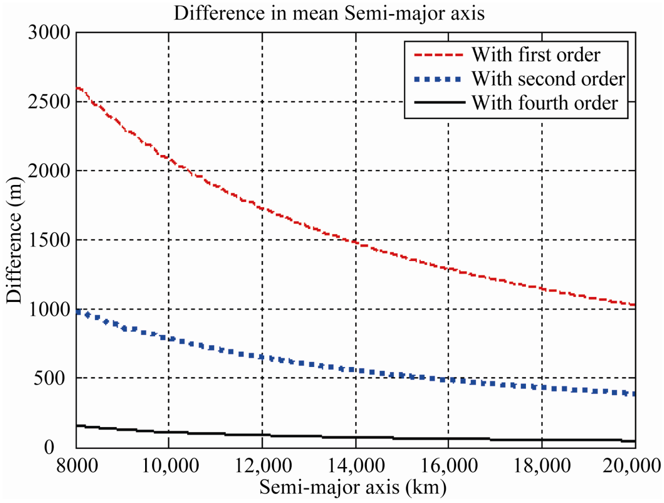
Eccentricity	Inclination (deg)	Osculating sma		Mean sma with second-order theory			Mean sma with fourth-order theory		
		Min. (km)	Max. (km)	Min. (km)	Max. (km)	Difference (m)	Min. (km)	Max. (km)	Difference (m)
0.05	5.0	7998.390134	8000.008978	7999.162941	7999.179419	16.4782	7999.168784	7999.173496	4.7118
0.10		7996.669502	8000.004093	7998.070622	7998.189655	119.0332	7998.116000	7998.130877	14.8773
0.15		7994.772549	8000.002368	7996.682878	7997.078671	395.7933	7996.825949	7996.866235	40.2867
0.20		7992.588264	8000.001516	7994.887539	7995.846327	958.7884	7995.200624	7995.312576	111.9515
0.05	30.0	7998.428413	8003.212381	8000.649839	8000.678311	28.4717	8000.648887	8000.674698	25.8111
0.10		7997.801028	8003.400363	8000.151593	8000.273466	121.8721	8000.194898	8000.226255	31.3574
0.15		7997.074459	8003.689339	7999.451909	7999.866312	414.4038	7999.645186	7999.695060	49.8735
0.20		7996.203808	8004.095670	7998.444801	7999.514272	1069.4706	7998.919588	7999.073464	153.8755
0.05	60.0	7996.557691	8010.418215	8003.692898	8003.759315	66.4171	8003.707043	8003.750959	43.9160
0.10		7996.097090	8011.827262	8004.330186	8004.654562	324.3765	8004.473164	8004.530684	57.5204
0.15		7995.515656	8013.618649	8004.927473	8005.967345	1039.8719	8005.397864	8005.519192	121.3279
0.20		7994.774613	8015.907208	8005.447230	8007.841908	2394.6783	8006.463541	8006.880513	416.9712
0.05	85.0	7995.488448	8013.911465	8005.161796	8005.284294	122.4973	8005.183165	8005.275545	92.3796
0.10		7994.919290	8015.932651	8006.343771	8006.795036	451.2649	8006.536166	8006.657413	121.2460
0.15		7994.194924	8018.482340	8007.554803	8008.939894	1385.0913	8008.177651	8008.392115	214.4640
0.20		7993.268832	8021.730186	8008.775745	8011.985476	3209.7309	8010.097622	8010.665660	568.0381

(b)

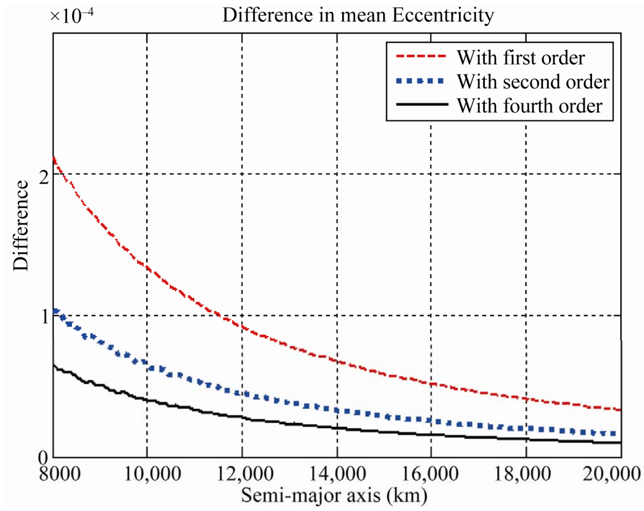
Eccentricity	Inclination (deg)	Osculating eccentricity		Mean eccentricity with second-order theory			Mean eccentricity with fourth-order theory		
		Min.	Max.	Min.	Max.	Difference	Min.	Max.	Difference
0.05	5.0	0.04794688	0.05000037	0.04889417	0.04890422	0.00001006	0.04889392	0.04890375	0.00000983
0.10		0.09791049	0.10000035	0.09879228	0.09881554	0.00002326	0.09879383	0.09881153	0.00001769
0.15		0.14784776	0.15000033	0.14866655	0.14871987	0.00005331	0.14867424	0.14870585	0.00003161
0.20		0.19775550	0.20000030	0.19851078	0.19861816	0.00010738	0.19853053	0.19858400	0.00005347
0.05	30.0	0.04878353	0.05017033	0.04942107	0.04943109	0.00001002	0.04942086	0.04943152	0.00001066
0.10		0.09877868	0.10018897	0.09936881	0.09939136	0.00002256	0.09937119	0.09939156	0.00002037
0.15		0.14875991	0.15021154	0.14929837	0.14935189	0.00005352	0.14931240	0.14934579	0.00003339
0.20		0.19872607	0.20023875	0.19920533	0.19931423	0.00010890	0.19924129	0.19929292	0.00005163
0.05	60.0	0.04970433	0.05115603	0.05049308	0.05051447	0.00002139	0.05049252	0.05051423	0.00002171
0.10		0.09966969	0.10115578	0.10053450	0.10059342	0.00005892	0.10053654	0.10058509	0.00004855
0.15		0.14962954	0.15117384	0.15056778	0.15070184	0.00013405	0.15058800	0.15067086	0.00008286
0.20		0.19958193	0.20132082	0.20058070	0.20085604	0.00027533	0.20064880	0.20077820	0.00012940
0.05	85.0	0.04963439	0.05217508	0.05101226	0.05103838	0.00002612	0.05101134	0.05103656	0.00002523
0.10		0.09959036	0.10218780	0.10109729	0.10117441	0.00007712	0.10110069	0.10116093	0.00006024
0.15		0.14954012	0.15223293	0.15117753	0.15135481	0.00017728	0.15120418	0.15131026	0.00010609
0.20		0.19948080	0.20231353	0.20124017	0.20160062	0.00036044	0.20132546	0.20149488	0.00016942

(c)

Eccentricity	Inclination (deg)	Osculating inclination		Mean inclination with second-order theory			Mean inclination with fourth-order theory		
		Min. (deg)	Max.(deg)	Min.(deg)	Max.(deg)	Difference (deg)	Min.(deg)	Max.(deg)	Difference (deg)
0.05	5.0	4.998630	5.004135	5.001358	5.001557	0.000199	5.001358	5.001557	0.000199
0.10		4.998543	5.004365	5.001450	5.001672	0.000222	5.001457	5.001665	0.000208
0.15		4.998438	5.004650	5.001546	5.001860	0.000314	5.001572	5.001798	0.000226
0.20		4.998310	5.004999	5.001643	5.002138	0.000495	5.001709	5.001972	0.000263
0.05	30.0	29.993254	30.020246	30.006849	30.006992	0.000143	30.006854	30.006981	0.000127
0.10		29.992831	30.021417	30.007345	30.007649	0.000305	30.007394	30.007533	0.000140
0.15		29.992318	30.022857	30.007732	30.008607	0.000875	30.008036	30.008212	0.000176
0.20		29.991697	30.024619	30.007974	30.009979	0.002005	30.008800	30.009096	0.000296
0.05	60.0	59.993284	60.020159	60.006817	60.006898	0.000081	60.006826	60.006890	0.000064
0.10		59.992866	60.021336	60.007258	60.007552	0.000295	60.007370	60.007447	0.000077
0.15		59.992359	60.022778	60.007617	60.008521	0.000904	60.008017	60.008121	0.000105
0.20		59.991747	60.024538	60.007837	60.009913	0.002076	60.008715	60.008999	0.000284
0.05	85.0	84.998654	84.998654	85.001361	85.001384	0.000023	85.001364	85.001383	0.000019
0.10		84.998571	84.998571	85.001448	85.001514	0.000066	85.001472	85.001495	0.000022
0.15		84.998469	84.998469	85.001518	85.001705	0.000187	85.001601	85.001630	0.000029
0.20		84.998346	84.998346	85.001560	85.001984	0.000424	85.001738	85.001800	0.000062



(a)



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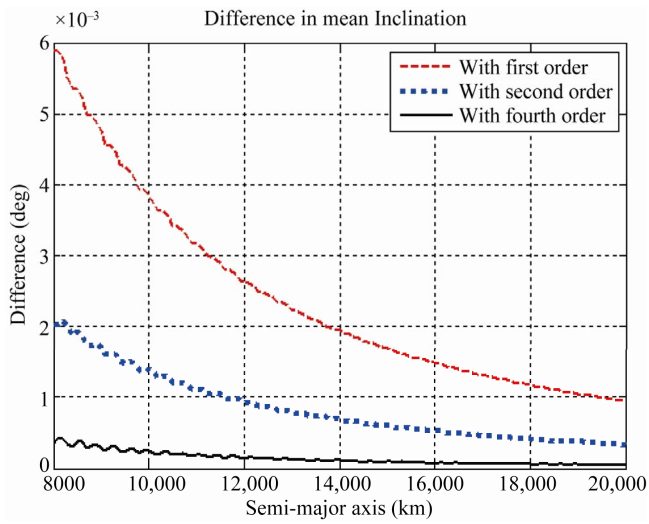


(c)

Figure 1. (a) Deviation in mean semi-major axis with variation in semi-major axis; (b) Deviation in mean eccentricity with variation in semi-major axis; (c) Deviation in mean inclination with variation in semi-major axis.

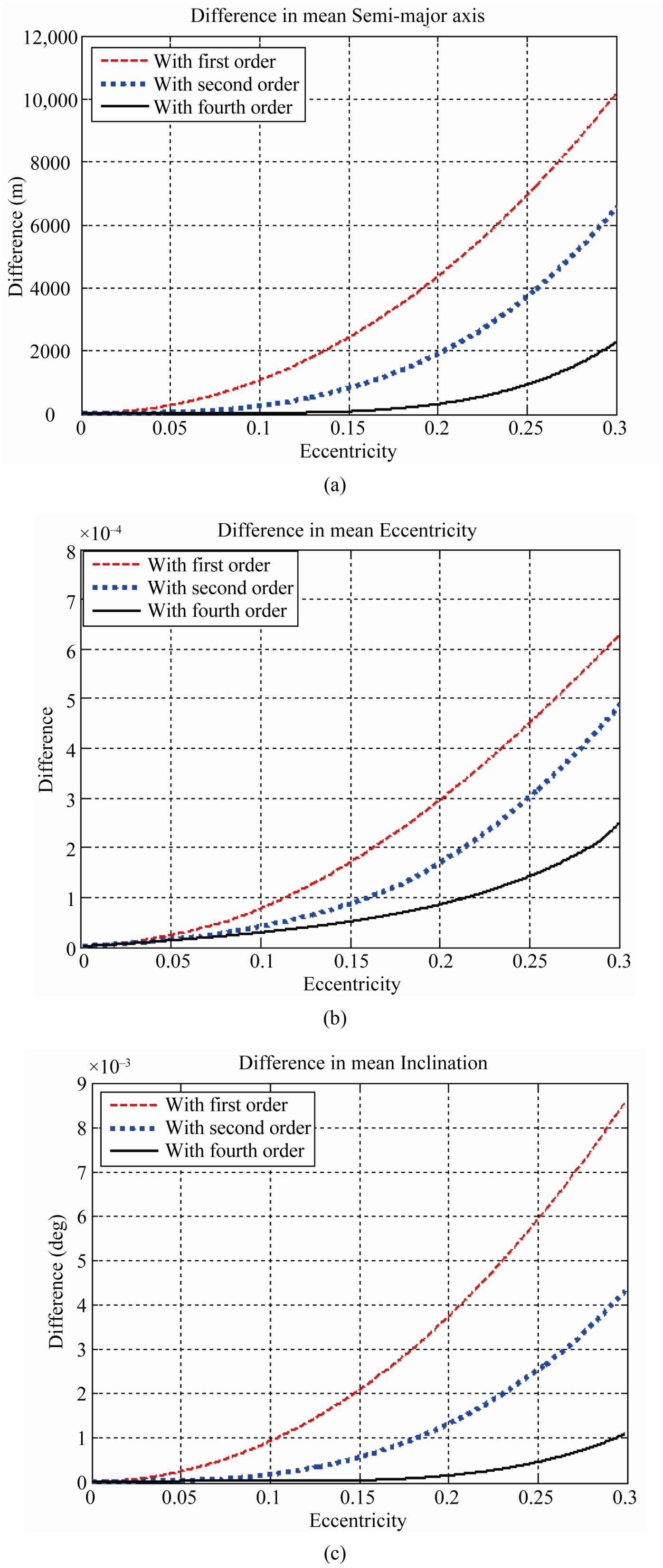
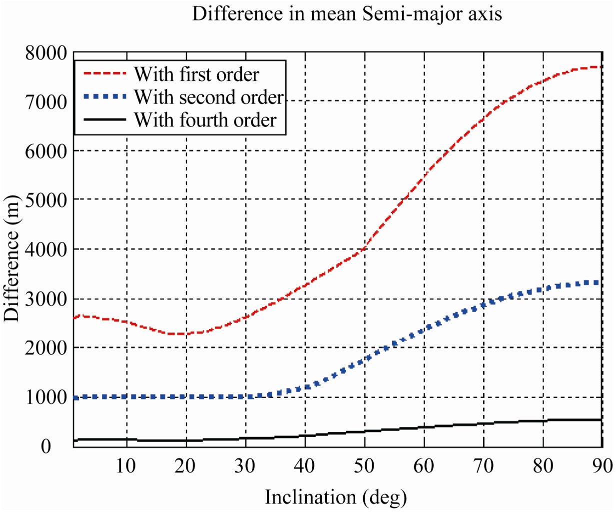


Figure 2. (a) Deviation in mean semi-major axis with variation in eccentricity; (b) Deviation in mean eccentricity with variation in eccentricity; (c) Deviation in mean inclination with variation in eccentricity.

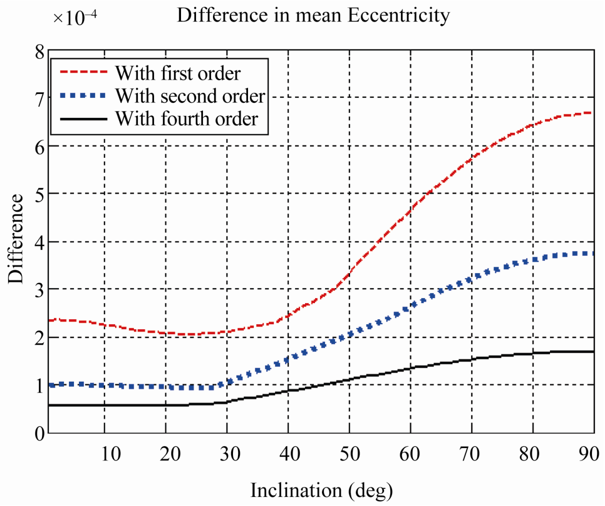
The difference of mean semi-major axis, mean eccentricity and mean inclination over a revolution computed using first-order, second-order and fourth-order theories with inclination varying from 0˚ to 90˚ is plotted in **Figures 3(a)** and **3(b)**, respectively. The initial values for *a*, *e*, Ω, *ω* and M for this case are 8000 km, 0.2, 60˚, 0˚, respectively. It can be observed that the fourth-order theory developed in this paper is better than the first and the second-order theory. The difference in maximum and minimum mean values of semi-major axis and eccentricity increases with inclination, while the difference in mean inclination first increases, attains a maximum value and then decreases.

4. Conclusions

An analytical theory for calculating first-order perturbations of the orbital elements of a satellite to accuracy up to fourth power in eccentricity is derived. The developed fourth-order theory is compared with first-order theory and second-order theory for different values of semi-major axis, eccentricity and inclination. It is observed that there is significant improvement in the important orbital elements: semi-major axis, eccentricity and inclination with the present theory. The theory provides better estimates of the mean orbital elements up to first-order terms of oblateness of Earth.



(a)



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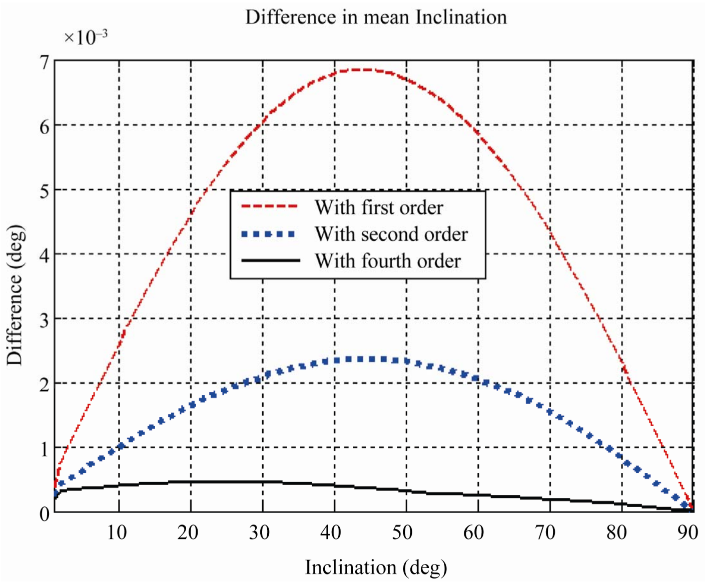

(c)

Figure 3. (a) Deviation in mean semi-major axis with variation in inclination; (b) Deviation in mean eccentricity with variation in inclination; (c) Deviation in mean inclination with variation in inclination.

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Appendix

$$\Delta a=\frac{JR_e^2\left(1-1.5\sin^2(i)\right)}{a^2}\Bigg[\left(2l+\frac{9h^2l}{4}+\frac{9l^3}{4}\right)\cos(\lambda)+\left(2h+\frac{9h^3}{4}+\frac{9hl^2}{4}\right)\sin(\lambda)+\left(-3h^2-\frac{7h^4}{3}+3l^2+\frac{7l^4}{3}\right)\cos(2\lambda)$$

$$+\left(6hl+\frac{14h^3l}{3}+\frac{14hl^3}{3}\right)\sin(2\lambda)+\left(-\frac{53h^2l}{4}+\frac{53l^3}{12}\right)\cos(3\lambda)+\left(-\frac{53h^3}{12}+\frac{53hl^2}{4}\right)\sin(3\lambda)$$

$$+\left(\frac{77h^4}{12}-\frac{77h^2l^2}{2}+\frac{77l^4}{12}\right)\cos(4\lambda)+\left(-\frac{77h^3l}{3}+\frac{77hl^3}{3}\right)\sin(4\lambda)\Bigg]$$

$$+\frac{JR_e^2\sin^2(i)}{a^2}\Bigg[\left(-\frac{l}{2}+\frac{l^3}{12}\right)\cos(\lambda)+\left(\frac{h}{2}-\frac{h^3}{12}\right)\sin(\lambda)+\left(1-\frac{5h^2}{2}+\frac{41h^4}{48}-\frac{5l^2}{2}+\frac{11h^2l^2}{8}+\frac{41l^4}{48}\right)\cos(2\lambda)$$

$$+\left(-\frac{h^3l}{6}+\frac{hl^3}{6}\right)\sin(2\lambda)+\left(\frac{7l}{2}-\frac{123h^2l}{16}-\frac{123l^3}{16}\right)\cos(3\lambda)+\left(\frac{7h}{2}-\frac{123h^3}{16}-\frac{123hl^2}{16}\right)\sin(3\lambda)$$

$$+\left(-\frac{17h^2}{2}+\frac{115h^4}{6}+\frac{17l^2}{2}-\frac{115l^4}{6}\right)\cos(4\lambda)+\left(17hl-\frac{115h^3l}{3}-\frac{115hl^3}{3}\right)\sin(4\lambda)+\left(\frac{845h^2l}{16}+\frac{845l^3}{48}\right)\cos(5\lambda)$$

$$+\left(-\frac{845h^3}{48}+\frac{32525h^5}{768}+\frac{845hl^2}{16}\right)\sin(5\lambda)+\left(\frac{533h^4}{16}-\frac{1599h^2l^2}{8}+\frac{533l^4}{16}\right)\cos(6\lambda)+\left(-\frac{533h^3l}{4}+\frac{533hl^3}{4}\right)\sin(6\lambda)\Bigg]$$

$$\Delta h=\frac{JR_e^2\left(1-1.5\sin^2(i)\right)}{a^2}\Bigg[\left(l+2h^2l+2l^3\right)nt+\left(-\frac{13hl}{4}-\frac{71h^3l}{16}-\frac{71hl^3}{16}\right)\cos(\lambda)$$

$$+\left(1-\frac{3h^2}{8}+\frac{35h^4}{64}+\frac{23l^2}{8}+\frac{177h^2l^2}{32}+\frac{319l^4}{64}\right)\sin(\lambda)+\left(-\frac{3h}{2}+\frac{13h^3}{12}-\frac{17hl^2}{4}\right)\cos(2\lambda)$$

$$+\left(\frac{3l}{2}-\frac{15h^2l}{4}+\frac{19l^3}{12}\right)\sin(2\lambda)+\left(-\frac{53hl}{12}+\frac{239h^3l}{32}-\frac{131hl^3}{32}\right)\cos(3\lambda)$$

$$+\left(-\frac{53h^2}{24}+\frac{293h^4}{128}+\frac{53l^2}{24}-\frac{555h^2l^2}{64}+\frac{77l^4}{128}\right)\sin(3\lambda)+\left(\frac{77h^3}{24}-\frac{77hl^2}{8}\right)\cos(4\lambda)$$

$$+\left(-\frac{77h^2l}{8}+\frac{77l^3}{24}\right)\sin(4\lambda)+\left(\frac{591h^3l}{32}-\frac{591hl^3}{32}\right)\cos(5\lambda)+\left(\frac{591h^4}{128}-\frac{1773h^2l^2}{64}+\frac{591l^4}{128}\right)\sin(5\lambda)\Bigg]$$

$$+\frac{JR_e^2\cos^2(i)}{a^2}\Bigg[\left(l+2h^2l+2l^3\right)nt+\left(-\frac{5hl}{2}-\frac{113h^3l}{24}-\frac{37hl^3}{8}\right)\cos(\lambda)+\left(\frac{7l^2}{2}+\frac{41h^2l^2}{8}+\frac{121l^4}{24}\right)\sin(\lambda)$$

$$+\left(-\frac{9hl^2}{2}\right)\cos(2\lambda)+\left(-\frac{l}{2}-\frac{5h^2l}{4}+\frac{13l^3}{4}\right)sin(2\lambda)+\left(-\frac{9hl^2}{2}\right)\cos(2\lambda)+\left(-\frac{l}{2}-\frac{5h^2l}{4}+\frac{13l^3}{4}\right)\sin(2\lambda)$$

$$+\left(\frac{7hl}{6}+\frac{11h^3l}{48}-\frac{413hl^3}{48}\right)\cos(3\lambda)+\left(-\frac{7l^2}{6}-\frac{223h^2l^2}{48}+\frac{67l^4}{16}\right)\sin(3\lambda)+\left(\frac{17hl^2}{4}\right)\cos(4\lambda)$$

$$+\left(\frac{17h^2l}{8}-\frac{17l^3}{8}\right)\sin(4\lambda)+\left(-\frac{169h^3l}{48}+\frac{169hl^3}{16}\right)\cos(5\lambda)+\left(\frac{169h^2l^2}{16}-\frac{169l^4}{48}\right)\sin(5\lambda)\Bigg]$$

$$+\frac{JR_e^2\sin^2(i)}{a^2}\Bigg[\left(\frac{hl}{4}-\frac{39h^3l}{128}-\frac{155hl^3}{385}\right)\cos(\lambda)+\left(-\frac{1}{4}-\frac{h^2}{8}+\frac{151h^4}{512}+\frac{l^2}{4}+\frac{43h^2l^2}{256}-\frac{43l^4}{1536}\right)\sin(\lambda)$$

$$+\left(-\frac{h}{2}+\frac{43h^3}{24}+\frac{13hl^2}{8}\right)\cos(2\lambda)+\left(-\frac{5l}{4}+\frac{21h^2l}{16}+\frac{71l^3}{48}\right)\sin(2\lambda)+\left(\frac{13hl}{16}+\frac{253h^3l}{128}+\frac{199hl^3}{128}\right)\cos(3\lambda)$$

$$+\left(\frac{7}{12}-\frac{319h^2}{96}+\frac{10523h^4}{1536}-\frac{397l^2}{96}+\frac{8843h^2l^2}{768}+\frac{7811l^4}{1536}\right)\sin(3\lambda)+\left(-\frac{17h}{8}+\frac{485h^3}{48}+\frac{179hl^2}{16}\right)\cos(4\lambda)$$
$$+\left(\frac{17l}{8}-\frac{153h^2l}{16}-\frac{511l^3}{48}\right)\sin(4\lambda)+\left(-\frac{169hl}{16}+\frac{18673h^3l}{384}+\frac{18163hl^3}{384}\right)\cos(5\lambda)$$
$$+\left(-\frac{169h^2}{32}+\frac{37091h^4}{1536}+\frac{169l^2}{32}-\frac{255h^2l^2}{256}-\frac{36581l^4}{1536}\right)\sin(5\lambda)+\left(\frac{533h^3}{48}-\frac{533hl^2}{16}\right)\cos(6\lambda)$$
$$+\left(-\frac{533h^2l}{16}+\frac{533l^3}{48}\right)\sin(6\lambda)+\left(\frac{32621h^3l}{384}-\frac{32621hl^3}{384}\right)\cos(7\lambda)$$
$$+\left(\frac{32621h^4}{1536}-\frac{32621h^2l^2}{256}+\frac{32621l^4}{1536}\right)\sin(7\lambda)\Bigg]$$

$$\Delta l=\frac{JR_e^2\left(1-1.5\sin^2(i)\right)}{a^2}\Bigg[\left(-h-2h^3-2hl^2\right)nt+\left(1+\frac{23h^2}{8}+\frac{319h^4}{64}-\frac{3l^2}{8}+\frac{177h^2l^2}{32}+\frac{35l^4}{64}\right)\cos(\lambda)$$
$$+\left(-\frac{13hl}{4}-\frac{71h^3l}{16}-\frac{71hl^3}{16}\right)\sin(\lambda)+\left(\frac{3l}{2}+\frac{17h^2l}{4}-\frac{13l^3}{12}\right)\cos(2\lambda)+\left(\frac{3h}{2}+\frac{19h^3}{12}-\frac{15hl^2}{4}\right)\sin(2\lambda)$$
$$+\left(-\frac{53h^2}{24}-\frac{77h^4}{128}+\frac{53l^2}{24}+\frac{555h^2l^2}{64}-\frac{293l^4}{128}\right)\cos(3\lambda)+\left(\frac{53hl}{12}+\frac{131h^3l}{32}-\frac{239hl^3}{32}\right)\sin(3\lambda)$$
$$+\left(-\frac{77h^2l}{8}+\frac{77l^3}{24}\right)\cos(4\lambda)+\left(-\frac{77h^3}{24}+\frac{77hl^2}{8}\right)\sin(4\lambda)+\left(\frac{591h^4}{128}-\frac{1773h^2l^2}{64}+\frac{591l^4}{128}\right)\cos(5\lambda)$$
$$+\left(-\frac{591h^3l}{32}+\frac{591hl^3}{32}\right)\sin(5\lambda)\Bigg]$$
$$+\frac{JR_e^2\sin^2(i)}{a^2}\Bigg[\left(\frac{1}{4}-\frac{h^2}{4}+\frac{43h^4}{1536}+\frac{l^2}{8}-\frac{43h^2l^2}{256}-\frac{151l^4}{512}\right)\cos(\lambda)+\left(-\frac{hl}{4}+\frac{155h^3l}{384}+\frac{39hl^3}{128}\right)\sin(\lambda)$$
$$+\left(-\frac{l}{2}+\frac{13h^2l}{8}+\frac{43l^3}{24}\right)\cos(2\lambda)+\left(\frac{5h}{4}-\frac{71h^3}{48}-\frac{21hl^2}{16}\right)\sin(2\lambda)$$
$$\left(\frac{7}{12}-\frac{397h^2}{96}+\frac{7811h^4}{1536}-\frac{319l^2}{96}+\frac{8843h^2l^2}{768}+\frac{10523l^4}{1536}\right)\cos(3\lambda)$$
$$+\left(\frac{13hl}{16}+\frac{199h^3l}{128}+\frac{253hl^3}{128}\right)\sin(3\lambda)+\left(\frac{17l}{8}-\frac{179h^2l}{16}-\frac{485l^3}{48}\right)\cos(4\lambda)+\left(\frac{17h}{8}-\frac{511h^3}{48}-\frac{153hl^2}{16}\right)\sin(4\lambda)$$
$$+\left(-\frac{169h^2}{32}+\frac{36581h^4}{1536}+\frac{169l^2}{32}+\frac{255h^2l^2}{256}-\frac{37091l^4}{1536}\right)\cos(5\lambda)+\left(\frac{169hl}{16}-\frac{18163h^3l}{384}-\frac{18673hl^3}{384}\right)\sin(5\lambda)$$
$$+\left(-\frac{533h^2l}{16}+\frac{533l^3}{48}\right)\cos(6\lambda)+\left(-\frac{533h^3}{48}+\frac{533hl^2}{16}\right)\sin(6\lambda)+\left(\frac{32621h^4}{1536}-\frac{32621h^2l^2}{256}+\frac{32621l^4}{1536}\right)\cos(7\lambda)$$
$$+\left(-\frac{32621h^3l}{384}+\frac{32621hl^3}{384}\right)\sin(7\lambda)\Bigg]$$
$$+\frac{JR_e^2\cos^2(i)}{a^2}\Bigg[\left(-h-2h^3-2hl^2\right)nt+\left(\frac{5h^2}{2}+\frac{113h^4}{24}+\frac{37h^2l^2}{8}\right)\cos(\lambda)+\left(-\frac{7hl}{2}-\frac{41h^3l}{8}-\frac{121hl^3}{24}\right)\sin(\lambda)$$
$$+\frac{9h^2l}{2}\cos(2\lambda)+\left(\frac{h}{2}+\frac{5h^3}{4}-\frac{13hl^2}{4}\right)\sin(2\lambda)+\left(-\frac{7h^2}{6}-\frac{11h^4}{48}+\frac{413h^2l^2}{48}\right)\cos(3\lambda)$$

$$+\left(\frac{7hl}{6}+\frac{223h^3l}{48}-\frac{67hl^3}{16}\right)\sin(3\lambda)-\frac{17h^2l}{4}\cos(4\lambda)+\left(-\frac{17h^3}{8}+\frac{17hl^2}{8}\right)\sin(4\lambda)$$
$$+\left(\frac{169h^4}{48}-\frac{169h^2l^2}{16}\right)\cos(5\lambda)+\left(-\frac{169h^3l}{16}+\frac{169hl^3}{48}\right)\sin(5\lambda)\Bigg]$$

$$\Delta\Omega=\frac{JR_e^2\cos(i)}{a^2}\Bigg[\left(-1-2h^2-3h^4-2l^2-6h^2l^2-3l^4\right)nt+\left(\frac{5h}{2}+\frac{113h^3}{24}+\frac{37hl^2}{8}\right)\cos(\lambda)$$
$$+\left(-\frac{7l}{2}-\frac{41h^2l}{8}-\frac{121l^3}{24}\right)\sin(\lambda)+\left(\frac{9hl}{2}+\frac{35h^3l}{6}+\frac{17hl^3}{3}\right)\cos(2\lambda)$$
$$+\left(\frac{1}{2}+\frac{5h^2}{4}+\frac{275h^4}{96}-\frac{13l^2}{4}-\frac{3h^2l^2}{16}-\frac{277l^4}{96}\right)\sin(2\lambda)+\left(-\frac{7h}{6}-\frac{11h^3}{48}+\frac{413hl^2}{48}\right)\cos(3\lambda)$$
$$+\left(\frac{7l}{6}+\frac{223h^2l}{48}-\frac{67l^3}{16}\right)\sin(3\lambda)+\left(-\frac{17hl}{4}-\frac{13h^3l}{6}+\frac{205hl^3}{12}\right)\cos(4\lambda)$$
$$+\left(-\frac{17h^2}{8}+\frac{127h^4}{96}+\frac{17l^2}{8}+\frac{231h^2l^2}{16}-\frac{589l^4}{96}\right)\sin(4\lambda)+\left(\frac{169h^3}{48}-\frac{169hl^2}{16}\right)\cos(5\lambda)$$
$$+\left(-\frac{169h^2l}{16}+\frac{169l^3}{48}\right)\sin(5\lambda)+\left(\frac{533h^3l}{24}-\frac{533hl^3}{24}\right)\cos(6\lambda)+\left(\frac{533h^4}{96}-\frac{533h^2l^2}{16}+\frac{533l^4}{96}\right)\sin(6\lambda)\Bigg]$$

$$\Delta i=\frac{JR_e^2\sin(i)\cos(i)}{a^2}\Bigg[\left(-\frac{l}{2}-\frac{h^2l}{8}-\frac{5l^3}{24}\right)\cos(\lambda)+\left(\frac{h}{2}+\frac{5h^3}{24}+\frac{hl^2}{8}\right)\sin(\lambda)+\left(\frac{1}{2}-h^2-\frac{5h^4}{96}-l^2+\frac{h^2l^2}{16}-\frac{5l^4}{96}\right)\cos(2\lambda)$$
$$+\left(\frac{h^3l}{12}-\frac{hl^3}{12}\right)\sin(2\lambda)+\left(\frac{7l}{6}-\frac{95h^2l}{48}-\frac{95l^3}{48}\right)\cos(3\lambda)+\left(\frac{7h}{6}-\frac{95h^3}{48}-\frac{95hl^2}{48}\right)\sin(3\lambda)$$
$$+\left(-\frac{17h^2}{8}+\frac{179h^4}{48}+\frac{17l^2}{8}-\frac{179l^4}{48}\right)\cos(4\lambda)+\left(\frac{17hl}{4}-\frac{179h^3l}{24}-\frac{179hl^3}{24}\right)\sin(4\lambda)+\left(-\frac{169h^2l}{16}+\frac{169l^3}{48}\right)\cos(5\lambda)$$
$$+\left(-\frac{169h^3}{48}+\frac{169hl^2}{16}\right)\sin(5\lambda)+\left(\frac{533h^4}{96}-\frac{533h^2l^2}{16}+\frac{533l^4}{96}\right)\cos(6\lambda)+\left(-\frac{533h^3l}{24}+\frac{533hl^3}{24}\right)\sin(6\lambda)\Bigg]$$

$$\Delta\lambda=\frac{JR_e^2\cos(i)}{a^2}\Bigg[\left(1+2h^2+3h^4+2l^2+6h^2l^2+3l^4\right)nt+\left(-\frac{5h}{2}-\frac{113h^3}{24}-\frac{37hl^2}{8}\right)\cos(\lambda)$$
$$+\left(\frac{7l}{2}+\frac{41h^2l}{8}+\frac{121l^3}{24}\right)\sin(\lambda)+\left(-\frac{9hl}{2}-\frac{35h^3l}{6}-\frac{17hl^3}{3}\right)\cos(2\lambda)$$
$$+\left(-\frac{1}{2}-\frac{5h^2}{4}-\frac{275h^4}{96}+\frac{3h^2l^2}{16}+\frac{277l^4}{96}\right)\sin(2\lambda)+\left(\frac{7h}{6}+\frac{11h^3}{48}-\frac{413hl^2}{48}\right)\cos(3\lambda)$$
$$+\left(-\frac{7l}{6}-\frac{223h^2l}{48}+\frac{67l^3}{16}\right)\sin(3\lambda)+\left(\frac{17hl}{4}+\frac{13h^3l}{6}-\frac{205hl^3}{12}\right)\cos(4\lambda)$$
$$+\left(\frac{17h^2}{8}-\frac{127h^4}{96}-\frac{17l^2}{8}-\frac{231h^2l^2}{16}+\frac{589l^4}{96}\right)\sin(4\lambda)+\left(-\frac{169h^3}{48}+\frac{169hl^2}{16}\right)\cos(5\lambda)$$
$$+\left(\frac{169h^2l}{16}+\frac{169h^4l}{32}-\frac{169l^3}{48}\right)\sin(5\lambda)+\left(-\frac{533h^3l}{24}\right)\cos(6\lambda)+\left(-\frac{533h^4}{96}+\frac{533h^2l^2}{16}-\frac{533l^4}{96}\right)\sin(6\lambda)\Bigg]$$

$$+\frac{JR_e^2\left(1-1.5\sin^2(i)\right)}{a^2}\Bigg[\left(2+4h^2+\frac{21h^4}{4}+4l^2+\frac{21h^2l^2}{2}-\frac{15h^4l^2}{2}+\frac{21l^4}{4}-\frac{15h^2l^4}{2}\right)nt$$

$$+\left(-7h-\frac{73h^3}{8}-\frac{73hl^2}{8}\right)\cos(\lambda)+\left(7l+\frac{73h^2l}{8}+\frac{73l^3}{8}\right)\sin(\lambda)+\left(-12hl-\frac{26h^3l}{3}-\frac{26hl^3}{3}\right)\cos(2\lambda)$$

$$+\left(-6h^2-\frac{13h^4}{3}+6l^2+\frac{13l^4}{3}\right)\sin(2\lambda)+\left(\frac{53h^3}{8}-\frac{159hl^2}{8}\right)\cos(3\lambda)+\left(-\frac{159h^2l}{8}+\frac{53l^3}{8}\right)\sin(3\lambda)$$

$$+\left(\frac{385h^3l}{12}-\frac{385hl^3}{12}\right)\cos(4\lambda)+\left(\frac{385h^4}{48}-\frac{385h^2l^2}{8}+\frac{385l^4}{48}\right)\sin(4\lambda)\Bigg]$$

$$+\frac{JR_e^2\cos^2(i)}{a^2}\Bigg[\left(-\frac{7h}{4}+\frac{5h^3}{8}+\frac{hl^2}{4}\right)\cos(\lambda)+\left(-\frac{7l}{4}+\frac{h^2l}{4}+\frac{5l^3}{8}\right)\sin(\lambda)+\left(\frac{5h^3l}{12}-\frac{5hl^3}{12}\right)\cos(2\lambda)$$

$$+\left(\frac{3}{2}-5h^2+\frac{325h^4}{96}-5l^2+\frac{95h^2l^2}{16}+\frac{325l^4}{96}\right)\sin(2\lambda)+\left(-\frac{49h}{12}+\frac{1163h^3}{96}+\frac{1163hl^2}{96}\right)\cos(3\lambda)$$

$$+\left(\frac{49l}{12}-\frac{1163h^2l}{96}-\frac{1163l^3}{96}\right)\sin(3\lambda)+\left(-17hl+\frac{313h^3l}{6}+\frac{313hl^3}{6}\right)\cos(4\lambda)$$

$$+\left(-\frac{17h^2}{2}+\frac{313h^4}{12}+\frac{17l^2}{2}-\frac{313l^4}{12}\right)sin(4\lambda)+\left(\frac{507h^3}{32}-\frac{152hl^2}{32}\right)\cos(5\lambda)+\left(-\frac{1521h^2l}{32}+\frac{507l^3}{32}\right)\sin(5\lambda)$$

$$+\left(\frac{2665h^3l}{24}-\frac{2665hl^3}{24}\right)\cos(6\lambda)+\left(\frac{2665h^4}{96}-\frac{2665h^2l^2}{16}+\frac{2665l^4}{96}\right)\sin(6\lambda)\Bigg]$$