

# An Approximation Algorithm for the Solution of Astrophysics Equations Using Rational Scaled Generalized Laguerre Function Collocation Method Based on Transformed Hermite-Gauss Nodes

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## Abstract

In this paper we propose a collocation method for solving Lane-Emden type equation which is nonlinear ordinary differential equation on the semi-infinite domain. This equation is categorized as singular initial value problems. We solve this equation by the generalized Laguerre polynomial collocation method based on Hermite-Gauss nodes. This method solves the problem on the semi-infinite domain without truncating it to a finite domain and transforming domain of the problem to a finite domain. In addition, this method reduces solution of the problem to solution of a system of algebraic equations.

**Keywords:** Lane-Emden Equation, Generalized Laguerre Functions, Collocation Method, Hermite-Gauss Nodes, Nonlinear ODE, Semi-Infinite

## 1. Introduction

There are many problems in science and engineering arising in unbounded domains.

Spectral methods are famous ways to solve these kinds of problems. The most common approach on spectral methods, that is used in this paper too, is through the use of functions that are orthogonal over unbounded domains, such as the Hermite and the Laguerre functions [1-8].

The second approach is reformulating the original problem in the semi-infinite domain to a singular problem in a bounded domain by variable transformation and then using the Jacobi polynomials to approximate the resulting singular problem [9-11]. A third approach of spectral method is based on rational orthogonal functions, for example, Christov [12] and Boyd [13,14] developed some spectral methods on unbounded intervals by using mutually orthogonal systems of rational functions. Boyd [14] defined a new spectral basis, named rational Chebyshev functions on the semi-infinite interval, by mapping it to the Chebyshev polynomials. Guo *et al.* [15] proposed and analyzed a set of Legendre rational func-

tions which are mutually orthogonal in  $L^2_\chi(0, \infty)$  with a non-uniform weight function  $\chi(x) = (x+1)^{-2}$ . A forth approach is replacing the semi-infinite domain with  $[0, L]$  interval by choosing  $L$ , sufficiently large, this method is named as the domain truncation [16].

In this paper, we investigate the Generalized Laguerre-collocation method based on Hermite-Gauss Nodes which is another approach for solving ODEs on the half line. In [7] proposed spectral methods using Laguerre functions and analyzed for model elliptic equations on regular unbounded domains. It is shown that spectral-Galerkin approximations based on Laguerre functions are stable and convergent with spectral accuracy in the Sobolev spaces. Siyyam [8] applied two numerical methods for solving initial value problem differential equations using the Laguerre Tau method. He generated linear systems and solved them. Maday, *et al.* [6] proposed a Laguerre type spectral method for solving partial differential equations. They introduced a general presentation of the method and a description of the derivation discretization matrices and then determined the optimum estimations in the adapted Hilbert norms.

## 2. The Lane-Emden Equation

This equation is one of the basic equations in the theory of stellar structure and has been the focus of many studies [17-21]. This equation describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. The polytropic theory of stars essentially follows out of thermodynamic considerations, that deal with the issue of energy transport, through the transfer of material between different levels of the star. We simply begin with the Poisson equation and the condition for hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}, \quad (1)$$

$$\frac{dM(r)}{dr} = 4\pi\rho r^2, \quad (2)$$

where  $G$  is the gravitational constant,  $P$  is the pressure,  $M(r)$  is the mass of a star at a certain radius  $r$ , and  $\rho$  is the density, at a distance  $r$  from the center of a spherical star. Combination of these equations yields the following equation, which as should be noted, is an equivalent form of the Poisson Equation.

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dP}{dr} \right) = -4\pi G \rho. \quad (3)$$

From these equations one can obtain the Lane-Emden equation through the simple supposition that the density is simply related to the pressure, while remaining independent of the temperature. We already know that in the case of a degenerate electron gas that the pressure and density are  $\rho \sim P^{\frac{3}{5}}$ , assuming that such a relation exists for other states of the star we are led to consider a relation of the following form:

$$P = K \rho^{1+\frac{1}{m}}, \quad (4)$$

where  $K$  and  $m$  are constants, at this point it is important to note that  $m$  is the polytropic index which is related to the ratio of specific heats of the gas comprising the star. Based upon these assumptions we can insert this relation into our first equation for the hydrostatic equilibrium condition and from this rewrite equation to:

$$\left[ \frac{K(m+1)}{4\pi G} \lambda^{\frac{1}{m-1}} \right] \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dy}{dr} \right) = -y^m, \quad (5)$$

where the additional alteration to the expression for density has been inserted with  $\lambda$  representing the central density of the star and  $y$  that of a related dimensionless quantity that are both related to  $\rho$  through the following relation

$$\rho = \lambda y^m. \quad (6)$$

Additionally, if place this result into the Poisson equation, we obtain a differential equation for the mass, with a dependance upon the polytropic index  $m$ . Though the differential equation is seemingly difficult to solve, this problem can be partially alleviated by the introduction of an additional dimensionless variable  $x$ , given by the following:

$$r = ax, \quad (7)$$

$$a = \left[ \frac{K(m+1)}{4\pi G} \lambda^{\frac{1}{m-1}} \right]^{\frac{1}{2}}. \quad (8)$$

Inserting these relations into our previous relations we obtain the famous form of the Lane-Emden equation, given below:

$$\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) = -y^m. \quad (9)$$

Taking this simple relation we will have the Lane-Emden equation:

$$y'' + \frac{2}{x} y' + y^m = 0, \quad x > 0. \quad (10)$$

At this point it is also important to introduce the boundary conditions, which are based upon the following boundary conditions for hydrostatic equilibrium, and normalization consideration of the newly introduced quantities  $x$  and  $y$ . What follows for  $r=0$  is:

$$r=0 \rightarrow x=0, \quad \rho = \lambda \rightarrow y(0) = 1 \quad (11)$$

As a result an additional condition must be introduced in order to maintain the condition of Equation (11) simultaneously:

$$y'(0) = 0. \quad (12)$$

In other words, the boundary conditions are as follow:

$$y(0) = 1, \quad y'(0) = 0. \quad (13)$$

Physically interesting value of  $m$  lie in the interval  $[0, 5]$ . Exact solution for Equation (??) are known only for  $m=0, 1$  and  $5$ . For other value of  $m$  the Lane-Emden equation is to be integrated numerically. In this paper, we solve it for  $m=1.5, 2, 2.5, 3$  and  $4$ .

This paper is arranged as follows: in Section 3, we explain the formulation of rational scaled generalized Laguerre polynomials and Hermite functions required for our subsequent development. In Section 4, we summarize the application of the method for solving Lane-Emden equation and compare it with the existing methods in the literature. Finally we give a brief conclusion in the last section.

### 3. Rational Scaled Generalized Laguerre Polynomials and Hermite Functions Properties

This section is devoted to the introduction of the basic notions and working tools concerning orthogonal rational scaled generalized Laguerre polynomials and later we present some properties of Hermite function and Hermite-Gauss nodes.

#### 3.1. Properties of Rational Scaled Generalized Laguerre Polynomials

$L_n^\alpha(x)$  (generalized Laguerre polynomial) is the  $n$ th eigenfunction of the Sturm-Liouville problem [2,22,23]:

$$x \frac{d^2}{dx^2} L_n^\alpha(x) + (\alpha + 1 - x) \frac{d}{dx} L_n^\alpha(x) + n L_n^\alpha(x) = 0, \quad x \in [0, \infty), \quad n = 0, 1, 2, \dots \tag{14}$$

The generalized Laguerre polynomials are defined with the following recurrence formula:

$$\begin{aligned} L_0^\alpha(x) &= 1, \quad L_1^\alpha(x) = (1 + \alpha - x), \\ n L_n^\alpha(x) &= (2n - 1 + \alpha - x) L_{n-1}^\alpha(x) - (n + \alpha - 1) L_{n-2}^\alpha(x), \\ & \quad n \geq 2, \alpha > -1 \end{aligned} \tag{15}$$

With the normalizing condition:

$$L_n^\alpha(0) = \binom{n + \alpha}{n}. \tag{16}$$

Let  $w(x)$  denotes a non-negative, integrable, real-valued function over the interval  $(0, \infty)$ , we define

$$L_w^2 = \left\{ v : (0, \infty) \rightarrow R \mid v \text{ is measurable and } \|v\|_w < \infty \right\}, \tag{17}$$

where

$$\|v\|_w = \left( \int_0^\infty |v(x)|^2 w(x) dx \right)^{\frac{1}{2}}, \tag{18}$$

Is the norm induced by the inner product of the space  $L_w^2$ ,

$$\langle u, v \rangle_w = \int_0^\infty u(x)v(x)w(x)dx. \tag{19}$$

These are orthogonal polynomials for the weight function  $w_\alpha(x) = x^\alpha e^{-x}$  where  $\delta_{nm}$  is the Kronecker delta function.

$$\int_0^\infty L_n^\alpha(x) L_m^\alpha(x) w_\alpha(x) dx = \left( \frac{\Gamma(n + \alpha + 1)}{n!} \right) \delta_{nm}.$$

Let  $N \geq 1$  be an integer and we define  $x_{j,N}^\alpha, j = 0, \dots, N-1$  to be zeroes of  $\frac{d}{dx} L_N^\alpha$  and the point  $x = 0$ . It

can be shown that  $x_{j,N}^\alpha > 0, j = 0, \dots, N-1$  and the corresponding weights are:

$$\begin{aligned} w_{0,N}^\alpha &= \frac{(\alpha + 1)\Gamma^2(\alpha + 1)(N-1)!}{\Gamma(N + \alpha + 1)} \\ w_{j,N}^\alpha &= \frac{\Gamma(\alpha + N)}{N!} \left( L_N^\alpha(x_{j,N}^\alpha) \frac{d}{dx} L_{N-1}^\alpha(x_{j,N}^\alpha) \right)^{-1}, \\ & \quad j = 1, 2, \dots, N-1. \end{aligned}$$

The following quadrature formula is known:

$$\int_0^{+\infty} f(x) w_N(x) dx = \sum_{j=0}^{N-1} f(x_{j,N}^\alpha) w_{j,N}^\alpha + \left( \frac{\Gamma(N + \alpha + 1)}{(N)!(2N)!} \right) f^{2N-1}(\xi), \quad 0 < \xi < \infty \tag{20}$$

In particular, the second term on the right hand side vanishes when  $f$  is a polynomial of degree at most  $2N - 2$ . For convenience, we shall set  $x_{j,N}^\alpha = x_j$  and  $w_{j,N}^\alpha = w_j$ .

Pseudospectral approximations in unbounded domains by Laguerre polynomials lead to ill-conditioned algorithms and [24] introduced a scaling function and appropriate numerical procedures in order to limit these unpleasant phenomena.

We define scaled Laguerre functions  $\{\ell_n\}$  as follows:

$$\begin{aligned} \ell_0^\alpha(x; k) &= 1, \quad \ell_n^\alpha(x; k) = S_n(x/k) L_n^\alpha(x/k), \\ & \quad n = 1, 2, \dots \end{aligned} \tag{21}$$

where  $k > 0$  is a constant and  $L_n^1(x)$  is generalized Laguerre polynomials for  $\alpha = 1$  and  $S_n(x)$  is defined as follows:

$$S_0(x) = 1, \quad S_n(x) = \left( n + 1 \prod_{t=1}^n (1 + x/(4t)) \right)^{-1}, \tag{22}$$

$n = 1, 2, \dots$

We denote scaled laguerre functions with (SLF).

Boyd [26] offered guidelines for optimizing the map parameter  $k$  where  $k > 0$  is the scaling parameter. From Equations (21) and (22), for  $\alpha = 1$ , we obtain the following formula:

$$\begin{aligned} \ell_0^1(x; k) &= 1, \quad \ell_1^1(x; k) = \frac{4(2k - x)}{2(x + 4k)}, \\ \ell_n^1(x; k) &= \frac{4n}{(n+1)(4n + x/k)} [(2n - x/k) \ell_{n-1}^1(x; k) \\ & \quad - \frac{4(n-1)^2}{4n + x/k - 4} \ell_{n-2}^1(x; k)], \quad n \geq 2, \end{aligned} \tag{23}$$

This system is an orthogonal basis with weight func-

tion  $w(x) = \frac{xe^{-x/k}}{kS_N(x/k)^2}$  [24].

Some of the relations of scaled Laguerre functions and generalized Laguerre-Gauss-type interpolation were introduced by [24,25]. It is convenient to define the weights of  $\ell_n(x; k)$  as follows:

$$w_{0,N} = \frac{w_0}{S_N(0)^{-2}} = \frac{2\Gamma(N+2)}{N(N!)}$$

$$w_{j,N} = \frac{w_j}{S_N(x_j)^{-2}} =$$

$$\frac{\Gamma(N+2)}{4N^2(N!)} \frac{4N+x_j}{\ell_N(x_j)} \left( \frac{d}{dx} \ell_{N-1}(x_j) + \ell_{N-1}(x_j) \sum_{m=1}^{N-1} \frac{1}{4m+x_j} \right)^{-1},$$

$$j = 1, 2, \dots, N-1. \quad (24)$$

where we noted that, for  $n \geq 2$

$$\frac{d}{dx} [S_{N-1}(x)]^{-1} = [S_{N-1}(x)]^{-1} \sum_{m=1}^{N-1} \frac{1}{4m+x_j}, \quad x \in I = [0, \infty)$$

### 3.2. Properties of Hermite Functions

The Hermite function is defined for all  $x \in \mathbb{R}$  and can be written in recursive formula as follows [27-29]:

$$H_{n+1}(x) = x\sqrt{\frac{2}{n+1}}H_n(x) - \sqrt{\frac{n}{n+1}}H_{n-1}(x), \quad n \geq 1$$

$$H_0(x) = e^{-\frac{x^2}{2}}, \quad H_1(x) = \sqrt{2}xe^{-\frac{x^2}{2}} \quad (25)$$

$\{H_n\}$  is an orthogonal system in  $L^2(\mathbb{R})$ , i.e.,

$$\int_{-\infty}^{+\infty} H_n(x)H_m(x)dx = \sqrt{\pi}\delta_{n,m}, \quad (26)$$

where  $\delta_{n,m}$  is the Kronecker delta function. Using the recurrence relation of Hermite polynomials and the above formula leads to

$$H'_n(x) = \sqrt{2n}H_{n-1}(x) - xH_n(x)$$

$$\sqrt{\frac{n}{2}}H_{n-1}(x) - \sqrt{\frac{n+1}{2}}H_{n+1}(x)$$

Let us define

$$P_N = \left\{ u : u = e^{-\frac{x^2}{2}} v, \forall v \in P_N \right\} \quad (27)$$

where  $P_N$  is the set of all Hermite polynomials of degree at most  $N$ . We now introduce the Gauss quadrature associated with the Hermite functions approach. Let  $\{x_j\}_{j=0}^N$  be the Hermite-Gauss nodes which can be  $N+1$  roots of  $H_{N+1}$  and define the weights

$$w_j = \frac{\sqrt{\pi}}{(n+1)H_n^2(x_j)}, \quad 0 \leq j \leq N. \quad (28)$$

Then we have:

$$\int_{\mathbb{R}} p(x)dx = \sum_{j=0}^N p(x_j)w_j, \quad \forall p \in P_{2N+1}.$$

### 4. Solving the Lane-Emden Equation

To apply rational scaled generalized Laguerre collocation method to the standard Lane-Emden Equation introduced in Equation (10) with boundary conditions Equation (13), we define

$$\zeta_N y(x) = \sum_{j=0}^N c_j \ell_j^\alpha(x; k). \quad (29)$$

And we construct the residual function by substituting  $y(x)$  by  $\zeta_N y(x)$  in the Lane-Emden Equation (10):

$$Res(x) = x \frac{d^2}{dx^2} \zeta_N y(x) + 2 \frac{d}{dx} \zeta_N y(x) + x(\zeta_N y(x))^m. \quad (30)$$

To find the unknown coefficients  $\{c_j\}_{j=0}^N$ 's, we equalize  $Res(x)$  to zero at  $N-1$  Hermite-Gauss Nodes  $\{x_j\}_{j=0}^{N-2}$  and we equalize boundary conditions in Equation (13) too, therefore we have:

$$Res(x_j) = 0, \quad j = 0 \dots N-2$$

$$\zeta_N y(0) = 1, \quad \frac{d}{dx} \zeta_N y(0) = 0. \quad (31)$$

But as mentioned before Lane-Emden equations are defined on the interval  $(0, \infty)$  and we know properties of Hermite functions are derived in the infinite domain  $(-\infty, \infty)$ . Also we know approximations can be constructed for infinite, semi-infinite and finite intervals. One of the approaches to construct Hermite-Gauss nodes on the interval  $(0, \infty)$  which is used in the current paper, is to use a mapping, that is a change of variable of the form:

$$z_j = \phi(x_j) = \ln\left(e^x + \sqrt{1 + e^{2x_j}}\right) \quad (32)$$

Where is the inverse map of

$$x_j = \phi^{-1}(z_j) = \ln(\sinh(z_j)) \quad (33)$$

Finally to find the unknown coefficients  $\{c_j\}_{j=0}^N$ 's, we have:

$$Res(z_j) = 0, \quad j = 0 \dots N-2 \quad (34)$$

$$\zeta_N y(0) = 1, \quad \frac{d}{dx} \zeta_N y(0) = 0 \quad (35)$$

where  $z_j$  is transformed root of  $H_{N-1}(x)$ . Equations (34)

and (35) gives  $N + 1$  nonlinear algebraic equations which can be solved for the unknown coefficients  $c_j$  by using the well known Newton's method by Maple programming and we use

$$c_j = 0, \quad j = 0 \dots N + 1$$

as starting points to obtain convergence of the method, consequently,  $y(x)$  given in Equation (10) can be calculated.

The resulting graph of Lane-Emden equation obtained by present method for  $m = 1.5, 2, 2.5, 3$  and 4 is shown in **Figure 1**.

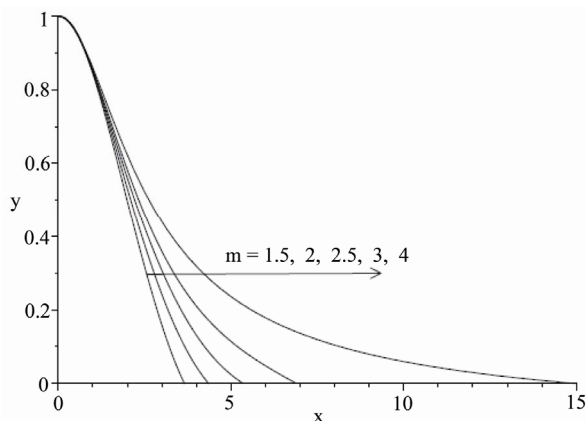
**Tables 1** shows the comparison of the first zero of  $\zeta_N y(x)$ , between Padé approximation used by [17] and the present method for  $\alpha = 1$  and  $m = 1.5, 2, 2.5, 3$  and 4 respectively.

**Tables 2** and **3** show the approximations of  $\zeta_N y(x)$  for standard Lane-Emden with  $m = 2.5, 3$  obtained by the rational scaled generalized Laguerre collocation method for  $\alpha = 1$  and those obtained by Horedt [20].

**Table 4** shows the coefficients of rational scaled generalized Laguerre functions obtained by present method for the Lane-Emden equation with various  $m$ .

### 5. Summary and Conclusions

A set of rational scaled generalized laguerre orthogonal functions are proposed to solve Lane-Emden equation which is defined in the semi-infinite interval and has singularity at  $x = 0$ , by collocation method. But to approximate unknown function in Lane-Emden equation by collocation method we must equalize  $Res(x)$  to zero at suitable points in  $(0, \infty)$  interval. Since Hermite functions are derived in the infinite domain  $(-\infty, \infty)$ , we can't apply Hermite-Gauss nodes for equalizing  $Res(x)$  to zero, therefore we transform this nodes from  $(-\infty, \infty)$  to  $(0, \infty)$  interval by a mapping  $\varphi(x) = \ln(e^x + \sqrt{1 + e^{2x}})$ .



**Figure 1.** Lane-Emden equation graph obtained by present method for various  $m$ .

**Table 1.** Comparison the first zero of  $y(x)$ , between Padé approximation used by [17] and the rational scaled generalized Laguerre collocation method for various  $m$  and  $a = 1$ .

$m$	$N$	$k$	Present method	Bender[17]	Exact value [20]
1.5	4	0.83200	3.653710928	-	3.65375374
2	4	0.52990	4.352761605	4.3603	4.35287460
2.5	4	0.39101	5.355236655	-	5.35527546
3	6	0.29354	6.896951110	7.0521	6.89684862
4	5	0.10022	1497174961	17.967	14.9715463

**Table 2.** Approximation of  $y(x)$  obtained by present method and solutions of Horedt [20] for  $m = 2.5$  and  $a = 1, N = 4$ .

$x$	Present method	Solutions of Horedt [20]
0.000	1.000000	1.000000
0.100	0.998749	0.998335
0.500	0.960372	0.959978
1.000	0.851467	0.851944
5.000	0.024267	0.029019
5.355	0.000005	0.000021

**Table 3.** Approximation of  $y(x)$  obtained by present method and solutions of Horedt [20] for  $m = 3$  and  $a = 1, N = 6$ .

$x$	Present method	Solutions of Horedt [20]
0.000	1.000000	1.000000
0.100	0.998293	0.998336
0.500	0.959837	0.959839
1.000	0.854839	0.855058
5.000	0.110415	0.110820
6.000	0.048132	0.043738
6.800	0.004952	0.004168
6.896	0.000048	0.000036

**Table 4.** Coefficients of the rational scaled generalized Laguerre functions of the Lane-Emden equation for various  $m$ .

l	$c_l$				
	$m = 1.5$	$m = 2$	$m = 2.5$	$m = 3$	$m = 4$
0	0.5552687	0.7618023	0.8553158	0.9108075	0.9786309
1	0.7098163	0.5800179	0.3986864	0.2657800	0.0483911
2	0.5796016	0.2517954	0.2748482	0.2308814	0.0525097
3	1.4222221	0.2455873	0.0468485	0.0164809	0.0455987
4	0.5775354	0.1555624	0.0676945	0.0448774	0.0152142
5	-	-	-	0.0032424	0.0048968
6	-	-	-	-	0.0103069

Through the comparisons among the exact solutions of Horedt and the approximate solutions of Bender and the current work, it has been shown that the present work has provided more exact solutions for Lane-Emden equations.

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